PRODUCTS OF HYPERGROUPOIDS ASSOCIATED TO BINARY RELATIONS

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Abstract. We study closure properties with respect to products for hypergroupoids, semihypergroups and hypergroups associated to binary relations. Using some basic category theory tools, from a certain point, the investigation turns into studying closure properties with respect to direct products for some classes of monounary multialgebras.

1. Introduction

In a natural way, one can associate to each binary homogeneous relation with full domain R on a set H a hypergroupoid H_R. This construction is presented and studied in [10]. In [10], Rosenberg also determined necessary and sufficient conditions on R such that the hypergroupoid H_R is a semihypergroup (or a hypergroup), and necessary and sufficient conditions for a (semi)hypergroup to be the (semi)hypergroup determined by a binary relation. Later, in [3], Corsini investigated when some constructions in the above class of relational structures provides a hypergroup through the association rule introduced by Rosenberg. But the study of hypergroups obtained from binary relations started before Rosenberg's paper and we must mention Chvalina's categorical approach of the subject from [1]. In Chvalina's paper, the considered binary relations were partial ordered relations. The work [4] can complete the background for the existing results on the subject we approach. The basic categorical tools we will use here can be found in [6] and [9].

Our paper is related with Chvalina's paper [1] since we begin by using similar categorical tools in our investigation. In this paper, we will continue our investigation on constructions of hypergroupoids associated to binary relations started in [8] with direct limits of direct systems. We mention that the relational systems with one binary relation with full domain can be seen as multialgebras with one unary multioperation, thus we work in fact with hypergroupoids associated to monounary multialgebras. In our opinion, the results obtained from this identification have a "nicer" form and we hope our paper will prove this. In the case of products the situation is not as "good" as in the case of the direct limits. Yet we obtained interesting results, even from a categorical point of view. For instance, we will show that the hypergroupoids determined by monounary multialgebras form a subcategory $\mathbf{Malg}'(2)$ in the category $\mathbf{Malg}(2)$ of hypergroupoids which is not closed under products. Yet, this subcategory is a category with products and the product of a family of hypergroupoids from $\mathbf{Malg}'(2)$ in $\mathbf{Malg}'(2)$ and the product of these hypergroupoids in $\mathbf{Malg}(2)$ have the same support set.

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2. Preliminaries

Let R be a binary relation on a set H. For $x \in H$, $X \subseteq H$ we denote

$$R\langle x\rangle = \{y \in H \mid xRy\} \text{ and } R(X) = \{y \in H \mid \exists x \in X : xRy\}.$$

Denote by $\overset{-1}{R}$ the inverse of the relation R. The domain of R is the set

$$D(R) = \{x \in H \mid \exists y \in H : xRy\} = \overset{-1}{R}(H).$$

As in [10], one can associate to the binary relation $R \subseteq H \times H$ the partial hypergroupoid $H_R = (H, \circ)$ defined by

$$x \circ y = R(\{x, y\}). \tag{1}$$

It is obvious that

$$x^2 = x \circ x = R\langle x\rangle = \{y \in H \mid xRy\} \text{ and } x \circ y = x^2 \cup y^2.$$

The partial hypergroupoid $H_R = (H, \circ)$ is a hypergroupoid if and only if the domain of R is H (see [10, Lemma 1]). But then, we can identify (H, R) with the multialgebra (H, f) with one unary multioperation $f : H \to P^*(H)$ defined by

$$xRy \Leftrightarrow y \in f(x). \tag{2}$$

If $X \subseteq H$ then $f(X) = \bigcup_{x\in X} f(x)$ and the equality (1) becomes

$$x \circ y = f(\{x, y\}) = f(x) \cup f(y)(= x^2 \cup y^2).$$

From [10, Lemma 1] we have:

Lemma 1. [8, Lemma 2.6] *For any multialgebra (H, f) with one unary multioperation, the equality*

$$x \circ y = f(\{x, y\})$$

defines a hypergroupoid $H_f = (H, \circ)$.

So, we can talk about hypergroupoids associated to monounary multialgebras instead of hypergroupoids associated to binary relations with full domain and we can translate the results from [10] in this new language.

An element $x \in H$ is an *outer element* (of (H, f)) if there exists $h \in H$ such that $x \notin f(f(h))$. An element $x \in H$ is an *inner element* if it is not an outer element (i.e. if $x \in f(f(h))$ for any $h \in H$).

From [10, Proposition 2 and Proposition 3] we have:

Proposition 2. [8, Propositions 2.7, 2.8] *Let (H, f) be a multialgebra with one unary multioperation. The hypergroupoid H_f is a semihypergroup if and only if the following conditions hold:*

a) *$f(x) \subseteq f(f(x))$, $\forall x \in H$;*
b) *whenever x is an outer element we have*

$$x \in f(f(a)) \Rightarrow x \in f(a).$$

The hypergroupoid H_f is a hypergroup if and only if $H \neq \emptyset$ and the conditions a), b) and

c) *$f(H) = H$*

hold.

Proposition 3. [8, Proposition 3.2] *Let $(H, *)$ be a hypergroupoid. There exists a unary multioperation f on H such that $(H, *) = H_f$ if and only if*

$$x * y = x^2 \cup y^2, \ \forall x, y \in H. \tag{3}$$

A hypergroupoid $(H,*)$ *which satisfies the condition* (3) *is a semihypergroup if and only if for any* $x,y\in H$ *we have*

$$x^2\subseteq (x^2)^2 \text{ and } (x^2)^2\cap (H\setminus (y^2)^2)\subseteq x^2.$$

Remark 4. A hypergroupoid $(H,*)$ which satisfies the condition (3) is a hypergroup if and only if it verifies the above conditions and $\bigcup_{x\in H} x^2 = H$.

Remark 5. In the terms of our discussion, a hypergroupoid (or semihypergroup, or hypergroup) $(H,*)$ is determined by a unary multioperation f on H if and only if $(H,*)$ satisfies the condition (3).

Let (H,R), (H',R') be relational systems with binary relations and $h:H\to H'$. One says that h is a *homomorphism* of relational systems if

$$xRy\Rightarrow h(x)R'h(y).$$

Let (H,R) and (H',R') be relational systems with one binary relation with full domain. Take the multialgebras (H,f) and (H',f') obtained from (H,R) and (H',R') using (2). A mapping $h:H\to H'$ is a relational homomorphisms between (H,R) and (H',R') if and only if

$$h(f(x))\subseteq f'(h(x)),\ \forall x\in H,$$

i.e. h is a homomorphism between the multialgebras (H,f) and (H',f').

Remark 6. Let $\mathcal{R}_2$ be the category of the relational systems with one binary relation (having as morphisms the homomorphisms of relational systems and as product the usual composition of homomorphisms) and let us denote by $\mathcal{R}'_2$ the full subcategory of $\mathcal{R}_2$ whose objects are the relational systems (H,R) for which $D(R)=H$. The identification we made by using (2) gives in fact a categorical isomorphism between $\mathcal{R}'_2$ and the category $\mathbf{Malg}(1)$ of the monounary multialgebras (i.e. the multialgebras of type (1)), where the morphisms are the multialgebra homomorphisms and the product of two morphisms is the usual composition of homomorphisms.

Let $(H,\circ)$, $(H',\circ')$ be hypergroupoids. Remember that a mapping $h:H\to H'$ is called *homomorphism* (of hypergroupoids) if

$$h(x\circ y)\subseteq h(x)\circ' h(y),\ \forall x,y\in H.$$

Besides $\mathcal{R}_2$, $\mathcal{R}'_2$ and $\mathbf{Malg}(1)$, the following categories drew our attention:

• the category $\mathbf{Malg}(2)$ of hypergroupoids: the morphisms are the hypergroupoid homomorphisms and the product of two morphisms is the usual composition of homomorphisms;

• the full subcategory of $\mathbf{Malg}(2)$ whose object are the hypergroupoids which satisfy (3), denoted by $\mathbf{Malg}'(2)$;

• the full subcategory of $\mathbf{Malg}(2)$ whose object are the semihypergroups, denoted by $\mathbf{SHG}$;

• the full subcategory of $\mathbf{SHG}$ whose object are the semihypergroups which satisfy (3), denoted by $\mathbf{SHG}'$;

• the category $\mathbf{HG}$ of hypergroups with hypergroup homomorphisms and the usual composition;

• the full subcategory of $\mathbf{HG}$ whose object are the hypergroups which satisfy (3), denoted by $\mathbf{HG}'$;

• the full subcategory $\mathbf{Malg}'(1)$ of $\mathbf{Malg}(1)$ whose objects are the monounary multialgebras (H,f) which satisfy the conditions a),b) from Proposition 2;

• the full subcategory $\mathbf{Malg}''(1)$ of $\mathbf{Malg}(1)$ whose objects are the nonempty monounary multialgebras (H,f) which satisfy the conditions a), b), c) from Proposition 2.

If $(H, f) \in \mathbf{Malg}(1)$ and h is a morphism from $\mathbf{Malg}(1)$, the correspondences

$$(H, f) \mapsto H_f \text{ and } h \mapsto h$$

define a covariant functor

$$F : \mathbf{Malg}(1) \to \mathbf{Malg}'(2)$$

(see [8, Corollary 3.6]). If we take (H, f) and h from $\mathbf{Malg}'(1)$ or from $\mathbf{Malg}''(1)$ we obtain another two covariant functors

$$F' : \mathbf{Malg}'(1) \to \mathbf{SHG}' \text{ and } F'' : \mathbf{Malg}''(1) \to \mathbf{HG}',$$

respectively (see [8, Remark 3.7]).

If $(H, *)$ is a hypergroupoid and consider the multioperation

$$f_* : H \to P^*(H),\ f_*(x) = x * x,$$

then (H, f_*) is in $\mathbf{Malg}(1)$, and, the correspondences

$$(H, *) \mapsto (H, f_*),\ h \mapsto h$$

define a covariant functor $\mathbf{Malg}(2) \to \mathbf{Malg}(1)$ (see [8, Remark 3.8]). We compose this functor with the inclusion functor $\mathbf{Malg}'(2) \to \mathbf{Malg}(2)$ and we obtain a covariant functor

$$G : \mathbf{Malg}'(2) \to \mathbf{Malg}(1).$$

Lemma 7. [8, Lemma 3.9] *The covariant functor F is an isomorphism between the categories $\mathbf{Malg}(1)$ and $\mathbf{Malg}'(2)$, and G is the inverse of F.*

Corollary 8. [8, Corollary 3.11] *The covariant functor F' is an isomorphism between $\mathbf{Malg}'(1)$ and $\mathbf{SHG}'$, and its inverse is the covariant functor $G' : \mathbf{SHG}' \to \mathbf{Malg}'(1)$ given by*

$$G'(H, *) = (H, f_*),\ G'(h) = h.$$

Corollary 9. [8, Corollary 3.12] *The covariant functor F'' is an isomorphism between $\mathbf{Malg}''(1)$ and $\mathbf{HG}'$, and its inverse is the covariant functor $G'' : \mathbf{HG}' \to \mathbf{Malg}''(1)$ given by*

$$G''(H, *) = (H, f_*),\ G''(h) = h.$$

The above result is a generalization of Theorem 1 from [1].

Remark 10. The classes of objects of $\mathbf{Malg}'(2)$, $\mathbf{SHG}'$ and $\mathbf{HG}'$ are closed under isomorphic images, hence so there are the classes of objects of $\mathbf{Malg}'(1)$, $\mathbf{Malg}''(1)$.

3. Products of hypergroupoids associated to monounary multialgebras

Let $((H_i, f_i) \mid i \in I)$ be a family of multialgebras of type (1). The direct product of the multialgebras (H_i, f_i) is the multialgebra $\left(\prod_{i\in I} H_i, f\right)$ with

$$f((x_i)_{i\in I}) = \prod_{i\in I} f_i(x_i).$$

This multialgebra, with the canonical projections of the product

$$e_j^I : \prod_{j\in I} H_i \to H_j,\ e_j^I((x_i)_{i\in I}) = x_j,\ j \in I,$$

is the product of the multialgebras (H_i, f_i) in the category $\mathbf{Malg}(1)$.

Similarly, if $((H_i, \circ_i) \mid i \in I)$ is a family of hypergroupoids then the direct product is the hypergroupoid $\left(\prod_{i\in I} H_i, \circ\right)$ with

$$(x_i)_{i\in I} \circ (y_i)_{i\in I} = \prod_{i\in I} (x_i \circ_i y_i).$$

This multialgebra, with the canonical projections of the product $(e_j^I \mid j \in I)$ is the product of the multialgebras $(H_i, \circ_i)$ in the category $\mathbf{Malg}(2)$.

The first main result of this paper is:

Theorem 11. *The category* $\mathbf{Malg}'(2)$ *is a category with products, but, if we see it as a subcategory in* $\mathbf{Malg}(2)$*,* $\mathbf{Malg}'(2)$ *is not closed under the products of* $\mathbf{Malg}(2)$*.*

Proof. Since F is an isomorphism from $\mathbf{Malg}(1)$ into $\mathbf{Malg}'(2)$ and $\mathbf{Malg}(1)$ is a category with products, it follows that $\mathbf{Malg}'(2)$ is also a category with products. Given a family $((H_i, \circ_i) \mid i \in I)$ of hypergroupoids from $\mathbf{Malg}'(2)$, the product of this family in $\mathbf{Malg}'(2)$ is the hypergroupoid $\left(\prod_{i\in I} H_i, *\right)$ determined by the direct product $\prod_{i\in I} G(H_i, \circ_i)$.

To show that the subcategory $\mathbf{Malg}'(2)$ of $\mathbf{Malg}(2)$ is not closed under products, consider two monounary multialgebras (H_1, f_1) and (H_2, f_2), and the hypergroupoids $(H_1)_{f_1} = (H_1, \circ_1)$ and $(H_2)_{f_2} = (H_2, \circ_2)$. The product of $(H_1)_{f_1}$ and $(H_2)_{f_2}$ in $\mathbf{Malg}(2)$ is the direct product $(H_1 \times H_2, \circ)$, thus we have:

$$\begin{aligned}(x_1, x_2) \circ (y_1, y_2) &= (x_1 \circ_1 y_1) \times (x_2 \circ_2 y_2) = (f_1(x_1) \cup f_1(y_1)) \times (f_2(x_2) \cup f_2(y_2)) \\ &= (f_1(x_1) \times f_2(x_2)) \cup (f_1(x_1) \times f_2(y_2)) \cup (f_1(y_1) \times f_2(x_2)) \cup (f_1(y_1) \times f_2(y_2)).\end{aligned}$$

It is easy to give an example to show that, in general, the direct product hypergroupoid $(H_1 \times H_2, \circ)$ does not satisfy (3), hence it is not in $\mathbf{Malg}'(2)$.

For instance, if we take

$$H_1 = H_2 = \{1, 2\} \text{ and } f_1 = f_2 = 1_{\{1,2\}} : \{1, 2\} \to \{1, 2\} \subseteq P^*(\{1, 2\})$$

then

$$(1, 1) \circ (2, 2) = \{1, 2\} \times \{1, 2\},\ (1, 1) \circ (1, 1) = (1, 1) \text{ and } (2, 2) \circ (2, 2) = (2, 2),$$

hence

$$(1, 1) \circ (2, 2) \neq ((1, 1) \circ (1, 1)) \cup ((2, 2) \circ (2, 2)),$$

which proves that $(H_1 \times H_2, \circ)$ does not satisfy (3). □

Corollary 12. *Let* $((H_i, f_i) \mid i \in I)$ *be a family of monounary multialgebras and let* $((H_i)_{f_i} = (H_i, \circ_i) \mid i \in I)$ *be the corresponding family of hypergroupoids from* $\mathbf{Malg}'(2)$*. The binary multioperation of the product* $\left(\prod_{i\in I} H_i, *\right)$ *of* $((H_i)_{f_i} \mid i \in I)$ *in the category* $\mathbf{Malg}'(2)$ *is given by*

$$(x_i)_{i\in I} * (y_i)_{i\in I} = \prod_{i\in I} f_i(x_i) \cup \prod_{i\in I} f_i(y_i).$$

Indeed, $\left(\prod_{i\in I} H_i, *\right)$ is the hypergroupoid determined by the direct product multialgebra $\left(\prod_{i\in I} H_i, f\right)$ of $((H_i, f_i) \mid i \in I)$. Hence

$$(x_i)_{i\in I} * (y_i)_{i\in I} = f((x_i)_{i\in I}) \cup f((y_i)_{i\in I}) = \prod_{i\in I} f_i(x_i) \cup \prod_{i\in I} f_i(y_i).$$

Remark 13. Since for each $i \in I$ and any $x_i \in H_i$,

$$x_i \circ_i x_i = f_i(x_i),$$

we also have

$$(x_i)_{i\in I} * (y_i)_{i\in I} = \prod_{i\in I}(x_i \circ_i x_i) \cup \prod_{i\in I}(y_i \circ_i y_i).$$

Thus, taking in Corollary 12 monounary multialgebras obtained from preordered relations using (2), the hypergroupoid $\left(\prod_{i\in I} H_i, *\right)$ is the *relational product* introduced by Chvalina in [2].

Let $I = \{1,2\}$ and (H_1, f_1), (H_2, f_2) two monounary multialgebras. The product $(H_1 \times H_2, *)$ of $(H_1)_{f_1}$ and $(H_2)_{f_2}$ in $\mathbf{Malg}'(2)$ is given by

$$(x_1, x_2) * (y_1, y_2) = f(x_1, x_2) \cup f(y_1, y_2) = (f_1(x_1) \times f_2(x_2)) \cup (f_1(y_1) \times f_2(y_2)).$$

Using the notations from the proof of Theorem 11 we have:

$$(x_1, x_2) * (y_1, y_2) \subseteq (x_1, x_2) \circ (y_1, y_2),\ \forall x_1, y_1 \in H_1,\ \forall x_2, y_2 \in H_2. \tag{4}$$

Remark 14. In general, the property (4) is not valid if we replace the inclusion with equality, and the example from the proof of Theorem 11 shows this fact, since

$$(1,1) \circ (2,2) = \{1,2\} \times \{1,2\} \neq \{(1,1),(2,2)\} = (1,1) * (2,2).$$

As a matter of fact, the cases when the property (4) is true if we replace the inclusion with equality are the cases when $(H_1 \times H_2, \circ)$ is in $\mathbf{Malg}'(2)$. These cases are quite rare. One is the trivial case when (at least) one of the given multialgebras is empty and the other cases are presented in the following theorem.

Theorem 15. *Let (H_1, f_1), (H_2, f_2) be nonempty monounary multialgebras. The direct product $(H_1)_{f_1} \times (H_2)_{f_2}$ is in $\mathbf{Malg}'(2)$ if and only if at least one of the unary multioperations f_1 or f_2 is a constant function.*

Proof. The direct product of $(H_1)_{f_1}$ and $(H_2)_{f_2}$ is in $\mathbf{Malg}'(2)$ if and only if

$$(x_1, x_2) \circ (y_1, y_2) = (x_1, x_2) * (y_1, y_2),\ \forall x_1, y_1 \in H_1,\ \forall x_2, y_2 \in H_2.$$

Indeed, $(H_1 \times H_2, \circ) \in \mathbf{Malg}'(2)$ if and only if for any $x_1, y_1 \in H_1$ and $x_2, y_2 \in H_2$,

$$\begin{aligned}(x_1, x_2) \circ (y_1, y_2) &= ((x_1, x_2) \circ (x_1, x_2)) \cup ((y_1, y_2) \circ (y_1, y_2))\\ &= ((x_1 \circ x_1) \times (x_2 \circ x_2)) \cup ((y_1 \circ y_1) \times (y_2 \circ y_2))\\ &= (x_1, x_2) * (y_1, y_2).\end{aligned}$$

Suppose that $x_1, y_1 \in H_1$ and there exists $t \in f_1(x_1) \setminus f_1(y_1)$. Then

$$\begin{aligned}\{t\} \times f_2(y_2) \in (x_1, x_2) \circ (y_1, y_2) &= (x_1, x_2) * (y_1, y_2)\\ &= (f_1(x_1) \times f_2(x_2)) \cup (f_1(y_1) \times f_2(y_2)),\end{aligned}$$

for any $x_2, y_2 \in H_2$, hence

$$\{t\} \times f_2(y_2) \subseteq f_1(x_1) \times f_2(x_2).$$

It follows that $f_2(y_2) \subseteq f_2(x_2)$, for any $x_2, y_2 \in H_2$, which is equivalent to

$$f_2(y_2) = f_2(x_2),\ \forall x_2, y_2 \in H_2.$$

The converse implication is obvious. $\square$

Remark 16. Let $((H_i, \circ_i) \mid i \in I)$ be a family of hypergroupoids from $\mathbf{Malg}'(2)$, let $P = \prod_{i \in I} H_i$, let $e_j^I : P \to H_j$ $(j \in I)$ be the canonical projections of the Cartesian product, let $(P, *)$ be the product of the given family of hypergroupoids in $\mathbf{Malg}'(2)$ and let $(P, \circ)$ be the product of this family in $\mathbf{Malg}(2)$. The morphisms $e_j^I : P \to H_j$ are the canonical projections of both products $(P, *)$ and $(P, \circ)$ in $\mathbf{Malg}'(2)$ and $\mathbf{Malg}(2)$ respectively, and the identity function 1_P is the unique function for which the following diagram is commutative

$$\begin{tikzcd}(P,*) \arrow[r, "e_j^I"] & (H_j, \circ_j)\\ (P,\circ) \arrow[u, "1_P"] \arrow[ur, "e_j^I"'] & \end{tikzcd}.$$

Yet, 1_P is neither a multialgebra isomorphism, hence nor an isomorphism in $\mathbf{Malg}(2)$, since it is not an ideal homomorphism. For the case $I = \{1, 2\}$ this fact is very well illustrated by Remark 14.

Another question arises from our previous discussion: *What about* $\mathbf{Malg}''(2)$ *and* $\mathbf{Malg}'''(2)$*? Are these subcategories of* $\mathbf{Malg}'(2)$ *closed under products?*

Since each member of a family $((H_i, \circ_i) \mid i \in I)$ of hypergroupoids from $\mathbf{Malg}''(2)$ (or $\mathbf{Malg}'''(2)$) is determined by the monounary multialgebra

$$(H_i, f_i) = G'(H_i, \circ_i) \text{ (or } (H_i, f_i) = G''(H_i, \circ_i) \text{ respectively)}$$

and the product of $((H_i, \circ_i) \mid i \in I)$ in $\mathbf{Malg}'(2)$ is the hypergroupoid determined by the direct product $\prod_{i\in I} G(H_i, \circ_i)$ our problem is to establish if the subcategories $\mathbf{Malg}'(1)$ and $\mathbf{Malg}''(1)$ of $\mathbf{Malg}(1)$ are closed under products, hence our question is equivalent to the following one:

(Q) Is the direct product of multialgebras from $\mathbf{Malg}'(1)$ *(or* $\mathbf{Malg}''(1)$*) in* $\mathbf{Malg}'(1)$ *(or* $\mathbf{Malg}''(1)$*, respectively)?*

The answer is *no* and the following example proves it.

Example 17. Let $H_1 = H_2 = \{1, 2, 3\}$ and $f_1, f_2 : \{1, 2, 3\} \to P^*(\{1, 2, 3\})$ given by

$$f_1 = 1_{\{1,2,3\}} \text{ and } f_2(1) = \{2, 3\},\ f_2(2) = \{1, 3\},\ f_2(3) = \{1, 2\}.$$

Clearly, the multialgebras (H_1, f_1) and (H_2, f_2) satisfy the conditions a), b) and c) from Proposition 2, thus they are in $\mathbf{Malg}''(1) \subseteq \mathbf{Malg}'(1)$. Yet, the direct product multialgebra $(H_1 \times H_2, f)$ of (H_1, f_1) and (H_2, f_2) is nor in $\mathbf{Malg}''(1)$ neither in $\mathbf{Malg}'(1)$. Indeed, since

$$f_1(f_1(1)) = \{1\},\ f_1(f_1(2)) = \{2\},\ f_1(f_1(3)) = \{3\}$$

and

$$f_2(f_2(1)) = f_2(f_2(2)) = f_2(f_2(3)) = \{1, 2, 3\},$$

we have

$$(1, 1) \notin \{2\} \times \{1, 2, 3\} = f_1(f_1(2)) \times f_2(f_2(2)) = f(f(2)),$$

hence the element $(1, 1)$ of $(H_1 \times H_2, f)$ is outer element. Yet,

$$(1, 1) \in \{1\} \times \{1, 2, 3\} = f_1(f_1(1)) \times f_2(f_2(1)) = f(f(1))$$

and

$$(1, 1) \notin \{1\} \times \{2, 3\} = f_1(1) \times f_2(1) = f(1),$$

which means that $(H_1 \times H_2, f)$ does not satisfy condition b) from Proposition 2.

Remark 18. Notice that for any $x_1 \in H_1$, $x_2 \in H_2$ the equality

$$f(f(x_1, x_2)) = f_1(f_1(x_1)) \times f_2(f_2(x_2))$$

holds. Moreover, if $((H_i, f_i) \mid i \in I)$ is a family of monounary multialgebras and the multialgebra $\left(\prod_{i\in I} H_i, f\right)$ is the their direct product then for any $(x_i)_{i\in I} \in \prod_{i\in I} H_i$,

$$f(f((x_i)_{i\in I})) = \prod_{i\in I} f_i(f_i(x_i)).$$

This follows from the form of the term functions of a direct product of multialgebras (see [7, Lemma 1]).

Even if the answer to Question *(Q)* is negative, it seems possible to determine some (large enough) classes of multialgebras from $\mathbf{Malg}'(1)$ or $\mathbf{Malg}''(1)$ which are closed under the formation of direct products.

We mention that *all the products of multialgebras which will appear herein after will be direct products.* First, we prove the following:

Lemma 19. *Let $((H_i, f_i) \mid i \in I)$ be a family of nonempty multialgebras of type* (1) *and let $\left(\prod_{i\in I} H_i, f\right)$ be their direct product. The following statements are true:*

1) *an element $(x_i)_{i\in I} \in \prod_{i\in I} H_i$ is an inner element of $\left(\prod_{i\in I} H_i, f\right)$ if and only if for all $i \in I$, x_i is an inner element of (H_i, f_i);*
2) *an element $(x_i)_{i\in I} \in \prod_{i\in I} H_i$ is an outer element of $\left(\prod_{i\in I} H_i, f\right)$ if and only if there exists $i \in I$ such that x_i is an outer element of (H_i, f_i);*
3) *if each multialgebra (H_i, f_i) satisfies the condition a) (or c), respectively) from Proposition 2 then the multialgebra $\left(\prod_{i\in I} H_i, f\right)$ satisfies the condition a) (or c), respectively) from Proposition 2;*
4) *if the multialgebra $\left(\prod_{i\in I} H_i, f\right)$ satisfies the condition a) (or c), respectively) from Proposition 2 then each multialgebra (H_i, f_i) satisfies the condition a) (or c), respectively) from Proposition 2;*
5) *the multialgebra $\left(\prod_{i\in I} H_i, f\right)$ consists only in inner elements if and only if each multialgebra (H_i, f_i) consists only in inner elements;*
6) *if for any $i \in I$, any outer element $x_i \in H_i$ satisfies the condition*

$$x_i \notin f(f(H_i)) \tag{5}$$

then any outer element from $\left(\prod_{i\in I} H_i, f\right)$ satisfies the condition (5)*;*

7) *if $\left(\prod_{i\in I} H_i, f\right)$ is in $\mathbf{Malg}'(1)$ (or $\mathbf{Malg}''(1)$) then each (H_i, f_i) is in $\mathbf{Malg}'(1)$ (or $\mathbf{Malg}''(1)$ respectively).*

Proof. 1) follows immediately from inner element definition and Remark 18.
2) is an immediate consequence of 1).
3) If each (H_i, f_i) satisfies the condition a), and $(x_i)_{i\in I} \in \prod_{i\in I} H_i$ then

$$f((x_i)_{i\in I}) = \prod_{i\in I} f_i(x_i) \subseteq \prod_{i\in I} f_i(f_i(x_i)) = f(f((x_i)_{i\in I})).$$

Suppose that each multialgebra (H_i, f_i) satisfies the condition c). If

$$(x_i)_{i\in I} \in \prod_{i\in I} H_i = \prod_{i\in I} f(H_i)$$

and each $x_i \in f_i(h_i)$ for some $h_i \in H_i$ then

$$(x_i)_{i\in I} \in \prod_{i\in I} f_i(h_i) = f((h_i)_{i\in I}) \subseteq f\left(\prod_{i\in I} H_i\right).$$

4) Obviously, the multialgebras $\prod_{i\in I}(H_i, f_i)$ and $(H_j, f_j) \times \prod_{i\in I\setminus\{j\}}(H_i, f_i)$ are isomorphic. So, it is enough to prove the property for two monounary multialgebras (H_1, f_1) and (H_2, f_2).

If $(H_1 \times H_2, f)$ satisfies the the condition a) from Proposition 2 then for any $x_1 \in H_1$ we take an element $x_2 \in H_2$ and we have

$$f_1(x_1) \times f_2(x_2) = f((x_1, x_2)) \subseteq f(f((x_1, x_2))) = f_1(f_1(x_1)) \times f_2(f_2(x_2)),$$

hence $f_1(x_1) \subseteq f_1(f_1(x_1))$.

If $(H_1 \times H_2, f)$ satisfies the the condition c) from Proposition 2 then

$$H_1 \times H_2 = f(H_1 \times H_2) = f_1(H_1) \times f_2(H_2).$$

Applying the first canonical projection of the direct product to this equality, we obtain $H_1 = f_1(H_1)$.
5) is an immediate consequence of 1).
6) Let $(x_i)_{i\in I} \in \prod_{i\in I} H_i$ be an outer element of $\left(\prod_{i\in I} H_i, f\right)$. Suppose that

$$(x_i)_{i\in I} \in f\left(f\left(\prod_{i\in I} H_i\right)\right).$$

Then there exists $(h_i)_{i\in I} \in \prod_{i\in I} H_i$ such that

$$(x_i)_{i\in I} \in f\left(f\left((h_i)_{i\in I}\right)\right) = \prod_{i\in I} f_i(f_i(h_i)).$$

Thus, there exists $j \in I$ such that x_j is outer element in (H_j, f_j) and

$$x_j \in f_j(f_j(h_j)) \in f_j(f_j(H_j)),$$

which contradicts our hypothesis.

7) As we have seen in the proof of 4), it is enough to prove the property for two monounary multialgebras (H_1, f_1) and (H_2, f_2). Let $(H_1, f_1) \times (H_2, f_2)$ be in $\mathbf{Malg}'(1)$ (or in $\mathbf{Malg}''(1)$). Then, according to 4), (H_1, f_1) satisfies the condition a) from Proposition 2 (and the condition c) from Proposition 2, respectively). We only have to prove that H_1 satisfies the condition b) from Proposition 2. If H_1 has not outer elements then $H_1 \in \mathbf{Malg}'(1)$. Let $x_1 \in H_1$ be an outer element. Then for any $x_2 \in H_2$, $x = (x_1, x_2)$ is an outer element of $H_1 \times H_2$. Suppose that $x_1 \notin f_1(h_1)$ for some $h_1 \in H_1$. We take an element $h_2 \in H_2$. Since

$$x = (x_1, x_2) \notin f_1(h_1) \times f_2(h_2) = f((h_1, h_2)),$$

and b) from Proposition 2 holds for $H_1 \times H_2$ we have

$$x \notin f(f((h_1, h_2))) = f_1(f_1(h_1)) \times f_2(f_2(h_2)).$$

It follows that $x_1 \notin f_1(f_1(h_1))$, and the proof is complete. □

Remark 20. There exists only one (trivial) structure of monounary multialgebra $(\emptyset, f)$ on the empty set. This multialgebra has no (outer) elements, thus it trivially satisfies the conditions from the first part of Proposition 2. If for a family $((H_i, f_i) \mid i \in I)$ of monounary multialgebras there exists $i \in I$ such that $H_i = \emptyset$ then $\prod_{i\in I} H_i = \emptyset$ and the multialgebra $\left(\prod_{i\in I} H_i, f\right)$ is in $\mathbf{Malg}'(1)$.

From the statement 5) of the above lemma we obtain:

Corollary 21. *The subclass of* $\mathbf{Malg}'(1)$ *and the subclass of* $\mathbf{Malg}''(1)$ *whose elements are the multialgebras consisting only in inner elements are closed under the formation of the direct products.*

We denote by $\mathcal{K}_1'$ the subclass of $\mathbf{Malg}'(1)$ which consists in those multialgebras for which any outer element satisfies the condition (5) and by $\mathcal{K}_1''$ the subclass of $\mathbf{Malg}''(1)$ which consists in those multialgebras for which any outer element satisfies the condition (5).

From the statement 6) of Lemma 19, we have:

Corollary 22. *The classes* $\mathcal{K}_1'$ *and* $\mathcal{K}_1''$ *are closed under the formation of the direct products.*

The monounary multialgebras consisting only in inner elements trivially satisfy the request that for any outer element (5) holds. So, the elements of $\mathcal{K}_1'$ and $\mathcal{K}_1''$ are the multialgebras of $\mathbf{Malg}'(1)$ and $\mathbf{Malg}''(1)$, respectively, which consist only in inner elements or/and outer elements satisfying (5). Thus, $\mathcal{K}_1'$ and $\mathcal{K}_1''$ include the class of the multialgebras from $\mathbf{Malg}'(1)$ and $\mathbf{Malg}''(1)$, respectively, which consist only in inner elements.

Lemma 23. *A multialgebra* (H, f) *from* $\mathbf{Malg}'(1)$ *is in* $\mathcal{K}_1'$ *if and only if it satisfies the following condition*

$$f(f(x)) = f(f(H)), \ \forall x \in H. \tag{6}$$

Proof. The set of the inner elements of (H, f) is $\bigcap_{h\in H} f(f(h))$. It follows that in (H, f) any outer element satisfies the condition (5) if and only if

$$H \setminus \left(\bigcap_{h\in H} f(f(h)) \right) \subseteq H \setminus f(f(H)),$$

or, equivalently,

$$f(f(H)) \subseteq \bigcap_{h\in H} f(f(h)). \tag{7}$$

Since for any $x \in H$,

$$\bigcap_{h\in H} f(f(h)) \subseteq f(f(x)) \subseteq f(f(H)),$$

the inclusion (7) holds if and only if the above inclusions are equalities, hence

$$f(f(H)) = \bigcap_{h\in H} f(f(h)) = f(f(x)),\ \forall x \in H,$$

and the lemma is proved. □

Since $f(H) = H$ for any multialgebra (H, f) from $\mathbf{Malg}''(1)$ we have:

Corollary 24. *A multialgebra (H, f) from $\mathbf{Malg}''(1)$ is in $\mathcal{K}_1''$ if and only if*

$$f(f(x)) = H,\ \forall x \in H. \tag{6$'$}$$

For the monounary multialgebras (H, f) which satisfy condition (6) the condition b) from Proposition 2 is clearly satisfied since they do not contain any outer element x such that $x \in f(f(a))$ for some $a \in H$.

Corollary 25. *A monounary multialgebra (H, f) is in $\mathcal{K}_1'$ if and only if it satisfies the condition a) from Proposition 2 and* (6).

Corollary 26. *A nonempty monounary multialgebra (H, f) is in $\mathcal{K}_1''$ if and only if it satisfies the condition c) from Proposition 2 and* (6).

Another main result of the paper is:

Theorem 27. *Let $((H_i, f_i) \mid i \in I)$ be a family of nonempty multialgebras from $\mathbf{Malg}'(1)$. The direct product multialgebra $\prod_{i\in I} (H_i, f_i)$ is in $\mathbf{Malg}'(1)$ if and only if either all the multialgebras (H_i, f_i) are in $\mathcal{K}_1'$ or all the multialgebras (H_i, f_i) satisfy the identity*

$$\mathbf{f}(\mathbf{f}(\mathbf{x})) = \mathbf{f}(\mathbf{x}) \tag{8}$$

*(**f** denotes the symbol of the multioperation corresponding to our multialgebra type).*

Proof. If all the multialgebras H_i are in $\mathcal{K}_1'$ then, according to Corollary 22, the multialgebra $\prod_{i\in I} (H_i, f_i)$ is in $\mathcal{K}_1'$, hence in $\mathbf{Malg}'(1)$. If all the multialgebras H_i satisfy the identity (8) then the direct product $\prod_{i\in I} (H_i, f_i)$ also satisfies (8) (see [7, Proposition 4]). Since any monounary multialgebra which satisfies (8) also satisfy the conditions a) and b) from Proposition 2, the multialgebra $\prod_{i\in I} (H_i, f_i)$ is in $\mathbf{Malg}'(1)$.

Conversely, suppose that the multialgebra $\prod_{i\in I} (H_i, f_i)$ is in $\mathbf{Malg}'(1)$. If all the multialgebras from the given family satisfy the condition (6) then all the multialgebras (H_i, f_i) are in $\mathcal{K}_1'$. Assume there exists an element $i_0 \in I$ such that (H_{i_0}, f_{i_0}) is a multialgebra which does not satisfy the condition (6). It results that in (H_{i_0}, f_{i_0}) there exits an element h_0 such that

$$f_{i_0}(f_{i_0}(H_{i_0})) \setminus f_{i_0}(f_{i_0}(h_0)) \neq \emptyset,$$

so in (H_{i_0}, f_{i_0}) there exist an outer element x_0 and an element a_0 such that

$$x_0 \in f_{i_0}(f_{i_0}(a_0)).$$

Let us denote $(H', f') = \prod_{i \in I \setminus \{i_0\}} (H_i, f_i)$. The multialgebras $\prod_{i \in I}(H_i, f_i)$ and

$$(H_{i_0}, f_{i_0}) \times (H', f') = (H_{i_0} \times H', f)$$

are isomorphic. Thus, according to statement 7) of Lemma 19, the multialgebra (H', f') is in $\mathbf{Malg}'(1)$. We prove that

$$f'(f'(a)) = f'(a), \ \forall a \in H'.$$

Since the inclusion $f'(a) \subseteq f'(f'(a))$ already holds, we have to prove only that

$$f'(f'(a)) \subseteq f'(a), \ \forall a \in H'.$$

We take $x \in f'(f'(a))$. There are two possibilities:
• x is an outer element of (H', f'): in this case, according to condition b) from Proposition 2, we have $x \in f'(a)$.
• x is an inner element of (H', f'): since the direct product $(H_{i_0}, f_{i_0}) \times (H', f')$ is in $\mathbf{Malg}'(1)$, it satisfies the condition b) from Proposition 2. According to Lemma 19, 2), the element (x_0, x) is an outer element of $(H_{i_0} \times H', f)$ and we have

$$(x_0, x) \in f_{i_0}(f_{i_0}(a_0)) \times f'(f'(a)) = f(f((a_0, a))).$$

We deduce that

$$(x_0, x) \in f((a_0, a)) = f_{i_0}(a_0) \times f'(a),$$

hence $x \in f'(a)$.

So, for any $i \in I \setminus \{i_0\}$ and any $x_i \in H_i$, we have

$$\prod_{i \in I \setminus \{i_0\}} f_i(f_i(x_i)) = f'(f'((x_i)_{i \in I \setminus \{i_0\}})) = f'((x_i)_{i \in I \setminus \{i_0\}}) = \prod_{i \in I \setminus \{i_0\}} f_i(x_i).$$

Taking $j \in I \setminus \{i_0\}$ and applying $e_j^{I \setminus \{i_0\}}$ to the above equality we obtain

$$f_j(f_j(x_j)) = f_j(x_j).$$

Since this equality holds for all $i \in I \setminus \{i_0\}$ and all $x_i \in H_i$, we conclude that each multialgebra (H_i, f_i) satisfies (8). □

With no major changes in the proof the above result can be also proved for multialgebras from $\mathbf{Malg}''(1)$.

Theorem 28. *Let $(H_i \mid i \in I)$ be a family of nonempty multialgebras from $\mathbf{Malg}''(1)$. The direct product multialgebra $\left(\prod_{i \in I} H_i, f\right)$ is in $\mathbf{Malg}''(1)$ if and only if either all the multialgebras H_i are in $\mathcal{K}_1''$ or all the multialgebras H_i satisfy the identity* (8).

We denote by $\mathcal{K}_2'$ the subclass of $\mathbf{Malg}'(1)$ which consists in multialgebras which satisfies (8), and by $\mathcal{K}_2''$ the subclass of $\mathbf{Malg}''(1)$ which consists in multialgebras which satisfies (8). Since the direct product of a family of multialgebras which satisfy (8) also satisfies (8), the classes $\mathcal{K}_2'$ and $\mathcal{K}_2''$ are closed under the formation of the direct products.

Remark 29. As we have mentioned before, any monounary multialgebra which satisfies (8) also satisfy the conditions a) and b) from Proposition 2, hence a monounary multialgebra (H, f) is in $\mathcal{K}_2'$ if and only if it satisfies the condition (8). Also, a nonempty monounary multialgebra (H, f) is in $\mathcal{K}_2''$ if and only if it satisfies the condition c) from Proposition 2 and (8).

Corollary 30. *Let $\mathcal{K}$ be a class of multialgebras from $\mathbf{Malg}'(1)$ which contains a multialgebra which is not in $\mathcal{K}_1'$. If $\mathcal{K}$ is closed under the formation of (finite) direct products then $\mathcal{K}$ is included in $\mathcal{K}_2'$.*

Indeed, let $(H_0, f_0) \in \mathcal{K} \setminus \mathcal{K}_1'$. For any multialgebra $(H, f) \in \mathcal{K}$, since

$$(H_0, f_0) \times (H, f) \in \mathcal{K} \subseteq \mathbf{Malg}'(1),$$

both multialgebras (H_0, f_0) and (H, f) satisfy (8), hence $(H, f) \in \mathcal{K}_2'$.

From Corollary 30 we immediately deduce:

Corollary 31. *Let $\mathcal{K}$ be a class of monounary multialgebras from $\mathbf{Malg}'(1)$. If $\mathcal{K}$ is closed under the formation of finite direct products then*

$$\mathcal{K} \subseteq \mathcal{K}_1' \text{ or } \mathcal{K} \subseteq \mathcal{K}_2'.$$

A similar result can be proved in the same way for $\mathbf{Malg}''(1)$.

Corollary 32. *Let $\mathcal{K}$ be a class of monounary multialgebras from $\mathbf{Malg}''(1)$. If $\mathcal{K}$ is closed under the formation of finite direct products then*

$$\mathcal{K} \subseteq \mathcal{K}_1'' \text{ or } \mathcal{K} \subseteq \mathcal{K}_2''.$$

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