QUASI-ISOMORPHISMS AND GROUPS OF QUASI-HOMOMORPHISMS

ULRICH ALBRECHT AND SIMION BREAZ

Abstract. This paper investigates to which extent a self-small mixed Abelian group G of finite torsion-free rank is determined by the groups $\mathrm{Hom}(G, C)$ where C is chosen from a suitable class $\mathcal{C}$ of Abelian groups. We show that G is determined up to quasi-isomorphism if $\mathcal{C}$ is the class of all self-small mixed groups C with $r_0(C) \leq r_0(G)$. Several related results are given, and the dual problem of orthogonal classes is investigated.

1. Introduction

Problem 34 in [8] asks whether it is possible to find a set $\mathcal{C}$ of Abelian groups with the property that two Abelian groups A and B are isomorphic provided that $\mathrm{Hom}(A, C) \cong \mathrm{Hom}(B, C)$ for all $C \in \mathcal{C}$. The negative answer given in [3] raised the question to identify properties of Abelian groups which are determined by $\mathrm{Hom}(-, C)$ when C is chosen from a class $\mathcal{C}$. For instance, if A is a reduced p-group, and $\mathcal{C}$ is the class of all p-groups, then the groups $\{\mathrm{Hom}(A, C)|C \in \mathcal{C}\}$ determine the finite Ulm-Kaplansky invariants $f_n(A)$ of A, but do not affect $f_\sigma(A)$ for $\sigma \geq \omega$ [1]. On the other hand, if $\mathcal{C}$ is the class of torsion-free groups of finite rank, then the invariants $\{r_0(\mathrm{Hom}(A, C)|C \in \mathcal{C}\}$ determine a group $A \in \mathcal{C}$ up to quasi-isomorphism, but not necessarily up to isomorphism [3].

The results in [1] also addressed the question if there are classes of mixed groups which are determined up to isomorphism by homomorphism groups. For instance, two groups A and B in $\mathcal{G}$ are isomorphic if $\mathrm{Hom}(A, G) \cong \mathrm{Hom}(B, G)$ for all $G \in \mathcal{G}$. Here, $\mathcal{G}$ is the class of all self-small groups of finite torsion-free rank G such that $G/T(G)$ is divisible where $T(G)$ denotes the torsion subgroup of G. *Self-small groups* were introduced by Arnold and Murley in [6] as the groups A with the property that $\mathrm{Hom}(A, A^{(I)})$ and $\mathrm{Hom}(A, A)^{(I)}$ are canonically isomorphic for all index-sets I. Although the class $\mathcal{S}$ of all self-small groups of finite torsion-free rank is closed with respect direct summands and endomorphic images [6], it is not closed with respect direct sums [2]. Further information on self-small groups can be found in [7].

Theorem 2.8 shows that a group $A \in \mathcal{S}$ is determined up to quasi-isomorphism by the torsion-free rank of the groups $\mathrm{Hom}(A, C)$ where $C \in \mathcal{S}$ (and $r_0(C) \leq r_0(A)$). Moreover, groups $A \in \mathcal{S}$ with $r_0(A) = 1$ are determined up to isomorphism in this way. However, this fails in general (Example 2.10).

Section 3 addresses a question closely related to Problem 34: Given a self-small group A, can we find a set $\mathcal{C}$ containing A and a property P such that A is determined up to isomorphism by $\mathrm{Ker}(\mathbb{Q}\mathrm{Hom}(A, -)) \cap \mathcal{C}$ and P? Annihilator classes like

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2000 *Mathematics Subject Classification.* 20K25(20K21).
S. Breaz is supported by the grant PN2CD-ID489.

these were first considered by Schultz in [11]; and Wickless solved a dual problem in [13]. Theorem 3.4 gives an answer for some natural arising classes of self-small groups.

For the benefit of the reader, we give a short summary of the notation used. If A is a group, then the symbol $T_p(A)$ indicates the p-component of A, and $\overline{A} = A/T(A)$. The endomorphism ring of A is $E = E(A)$. There exists an adjoint pair $(\mathrm{H}_A, \mathrm{T}_A)$ of functors between the category of abelian groups and the category of right E-modules defined by $\mathrm{H}_A(G) = \mathrm{Hom}(A, G)$ and $\mathrm{T}_A(M) = M \otimes_E A$ for all abelian groups A and all right E-modules M. These functors induce natural maps $\theta_G : \mathrm{T}_A\mathrm{H}_A(G) \to G$ and $\Phi_M : M \to \mathrm{H}_A\mathrm{T}_A(M)$ defined by $\theta_G(\alpha \otimes a) = \alpha(a)$ and $[\Phi_M(x)](a) = x \otimes a$ for all $a \in A$, $x \in M$, and $\alpha \in \mathrm{H}_A(G)$. The *$A$-socle of G*, denoted by $S_A(G)$, is the image of θ_G. If $\mathcal{C}$ is a class of groups, then a group G is (finitely) $\mathcal{C}$-generated if it is an epimorphic image of a (finite) direct sum of groups in $\mathcal{C}$. It is easy to see that G is A-generated if and only if $S_A(G) = G$. The symbol $\sim$ denotes quasi-isomorphisms, i.e. isomorphisms in $\mathbb{Q}Ab$. Two subgroups G and H of a group A are *quasi-equal*, denoted by $G \doteq H$, if $G \cap H$ is of finite index in G and in H. Finally, if V and W are sets such that all but finitely many elements of V are contained in W, then V is *quasi-contained in* W. This is denoted by $V \dot\subseteq W$. The sets V and W are *quasi-equal*, denoted by $V \doteq W$, if $V \dot\subseteq W$ and $W \dot\subseteq V$.

2. Self-small Groups Determined by Homomorphism Groups

We begin our discussion on how far the structure of a mixed group A is determined by the groups $\mathrm{Hom}(A, C)$ with the case that the test groups C have rank 1:

Proposition 2.1. *Let A and B be in $\mathcal{S}$ such that* $\mathrm{Hom}(A, C) \cong \mathrm{Hom}(B, C)$ *for all $C \in \mathcal{G}$ with $r_0(C) = 1$. Then,*

i) $T(A) \cong T(B)$.
ii) $r_p(A/T(A)) = r_p(B/T(B))$ *for all primes p.*
iii) $r_0(A) = r_0(B)$.

Proof. Observe that $T_p(A)$ and $T_p(B)$ are finite for all primes p since A and B are in $\mathcal{S}$. Choose $n < \omega$ with $p^nT_p(A) = p^nT_p(B) = 0$, and select a group $C \in \mathcal{G}$ with $T_p(C) = \mathbb{Z}/p^{n+1}\mathbb{Z}$ and $r_0(G) = 1$. If $C = T_p(C) \oplus C^p$ such that multiplication by p is an automorphism of C^p, then $\mathrm{Hom}(A, C) = \mathrm{Hom}(A, T_p(C)) \oplus \mathrm{Hom}(A, C^p)$; and multiplication by p induces an automorphism on $\mathrm{Hom}(A, C^p)$. Therefore, $T_p(\mathrm{Hom}(A, C)) = \mathrm{Hom}(A, T_p(C))$. Since $T_p(A)$ is finite, $A = T_p(A) \oplus A'$, and $\mathrm{Hom}(A', T_p(C)) \cong \mathrm{Hom}(A/T(A), T_p(C)) \cong [\mathbb{Z}/p^{n+1}\mathbb{Z}]^{n_p}$ where $n_p = r_p(A/T(A))$. On the other hand, $\mathrm{Hom}(T_p(A), T_p(C)) \cong T_p(A)$. Thus, $T_p(\mathrm{Hom}(A, C)) \cong T_p(A) \oplus [\mathbb{Z}/p^{n+1}\mathbb{Z}]^{n_p}$. Similarly, $T_p(\mathrm{Hom}(B, C)) \cong T_p(B) \oplus [\mathbb{Z}/p^{n+1}\mathbb{Z}]^{m_p}$ where $m_p = r_p(B/T(B))$. Since $T_p(A)$ and $T_p(B)$ contain no direct summand isomorphic to $\mathbb{Z}/p^{n+1}\mathbb{Z}$, we have $T_p(A) \cong T_p(B)$ and $n_p = m_p$.

Furthermore, $\mathbb{Q} \in \mathcal{G}$ yields $\mathrm{Hom}(A, \mathbb{Q}) \cong \mathbb{Q}^{r_0(A)}$ and $\mathrm{Hom}(B, \mathbb{Q}) \cong \mathbb{Q}^{r_0(B)}$. Thus, $r_0(A) = r_0(B)$. □

The converse of this proposition is not valid.

Example 2.2. *There exist two groups $A, B \in \mathcal{S}$ and a group $G \in \mathcal{G}$ such that A and B verify i), ii) and iii) in Proposition 2.1 but* $\mathrm{Hom}(A, G) \not\cong \mathrm{Hom}(B, G)$.

Proof. Let $A = \mathbb{Z}$ and B a rank 1 torsion-free group of type $(1, \dots, 1, \dots)$. If $G \in \mathcal{G}$ is a group such that $T_p(G) \neq 0$ for infinitely many primes p, it is not hard to see that $\mathrm{Hom}(A, G) \cong G$ is countable, while $\mathrm{Hom}(B, G)$ is uncountable since it contains a subgroup isomorphic to $\mathrm{Hom}(\oplus_p \mathbb{Z}/p\mathbb{Z}, T(G))$. $\square$

Corollary 2.3. *Let A and B be self-small groups with $r_0(A) = r_0(B) = 1$. Then, $A \cong B$ if and only if $\mathrm{Hom}(A, C) \cong \mathrm{Hom}(B, C)$ for all $C \in \mathcal{S}$ with $r_0(C) = 1$.*

Proof. Since $0 \neq \mathrm{Hom}(A, \overline{A}) \cong \mathrm{Hom}(B, \overline{A})$, $type(\overline{B}) \leq type(\overline{A})$. By symmetry, $\overline{A}$ and $\overline{B}$ have the same type. Moreover, $T(A) \cong T(B)$ by Proposition 2.1 . Hence, [2, Corollary 4.5] yields $A \cong B$. $\square$

Lemma 2.4. *Let $A \in \mathcal{S}$ with $r_0(A) > 0$. If N is the two-sided ideal of E containing $T(E)$ such that $N/T(E) = N(E/T(E))$, then A/NA is not torsion.*

Proof. Suppose that A/NA is torsion. We show that A/N^kA is torsion for all $k < \omega$ by induction on k. Suppose that A/N^kA is a torsion group. The exact sequence $0 \to N^k/N^{k+1} \to E/N^{k+1} \to E/N^k \to 0$ induces $\mathrm{T}_A(N^k/N^{k+1}) \to \mathrm{T}_A(E/N^{k+1}) \to \mathrm{T}_A(E/N^k) \to 0$ in which $\mathrm{T}_A(E/N^k) \cong A/N^kA$ is torsion. Moreover, N^k/N^{k+1} is an E/N-module, and there is an exact sequence $\oplus_I E/N \to N^k/N^{k+1} \to 0$ from which we obtain that $\mathrm{T}_A(N^k/N^{k+1})$ is torsion as an epimorphic image of the torsion group $\oplus_I \mathrm{T}_A(E/N) \cong \oplus_I A/NA$. Then $A/N^{k+1}A$ is torsion.

Since $N(E/T(E))$ is nilpotent, there is $\ell < \omega$ with $N^\ell \subseteq T(E)$. Hence, $N^\ell A \subseteq T(E(A)) \subseteq T(A)$ is torsion and the same holds for A, a contradiction. $\square$

If $A \in \mathcal{S}$, then there are a non-zero integer k and essentially strongly indecomposable subgroups $A_1, \dots, A_n$ of A such that $r_0(A_i) > 0$ for all $i = 1, \dots, n$ and $kA \subseteq A_1 \oplus \ldots \oplus A_n \subseteq A$. These essentially strongly indecomposable groups are unique up to quasi-isomorphisms by [7, Theorem 2.9]. Let $\delta_i : A_i \to A$ be the inclusion map. For the projection maps $\pi_i : A_1 \oplus \ldots \oplus A_n \to A_i$, define $\sigma_i : A \to A_i$ by $\sigma_i(a) = \pi_i(ka)$. We obtain $k1_A = \Sigma_{i=1}^n \delta_i \sigma_i$, and $k1_{A_i} = \sigma_i \delta_i$. If necessary, a superscript referring to A will be added to these maps.

Lemma 2.5. *Let A and B be quasi-isomorphic, essentially strongly indecomposable groups in $\mathcal{S}$. If $\alpha : A \to B$ is a homomorphism then $\mathrm{Ker}(\alpha)$ is bounded if and only if α is a quasi-isomorphism.*

Proof. Suppose that Ker α is bounded. If $\beta : B \to A$ is a quasi-isomorphism, then $\beta\alpha : A \to A$ has a bounded kernel. Thus, $\beta\alpha$ is a quasi-monomorphism, which cannot be nilpotent. Since the quasi-endomorphism ring of A is a local finite-dimensional $\mathbb{Q}$-algebra, $\beta\alpha$ is a quasi-isomorphism, and the same holds for α. The converse implication is obvious (see [12]). $\square$

Lemma 2.6. *For $A \in \mathcal{S}$, let N be the ideal of E containing $T(E)$ with $N/T(E) = N(E/T(E))$. If $\alpha \in \mathrm{Hom}(A_i, A_j)$, then the following hold:*

a) *If α is not a quasi-isomorphism, then $\delta_j \alpha \sigma_i \in N$.*
b) *If A_i and A_j are quasi-isomorphic, and if $\mathrm{Ker}(\alpha)$ is bounded, then α is a quasi-isomorphism.*

Proof. a) By [5, Proposition 9.1], $N/T(E)$ is pure in $E/T(E)$, and $\mathbb{Q}N/T(E) = N(\mathbb{Q}E/T(E))$. Moreover, $N/T(E) = N(\mathbb{Q}E/T(E)) \cap E/T(E)$. In particular, E/N is torsion-free as an abelian group.

If $\delta_j\alpha\sigma_i \notin N$, then $\delta_j\alpha\sigma_i$ has infinite order. Suppose there is $\sigma \in E$ such that $\overline{\sigma_i\sigma\delta_j\alpha} \notin N(\overline{E(A_i)})$. Since $\mathbb{Q}\overline{E(A_i)}$ is a local ring, $\overline{\sigma_i\sigma\delta_j\alpha}$ is a unit of $\mathbb{Q}\overline{E(A_i)}$. Thus, there exist a non-zero integer k and a map $\beta \in E(A_i)$ such that $\beta\sigma_i\sigma\delta_j\alpha - k1_{A_i} \in T(E(A_i))$. Furthermore, we can find an integer ℓ with $\ell\beta\sigma_i\sigma\delta_j\alpha = \ell k1_{A_i}$. Thus, α is a quasi-splitting quasi-monomorphism. Write $A_j \doteq \alpha(A_i) \oplus U$. Since $\alpha(A_i)$ is not bounded, U is bounded. Thus, α is a quasi-isomorphism, which is not possible.

Consequently, $\overline{\sigma_i\sigma\delta_j\alpha} \in N(\overline{E(A_i)})$ for all $\sigma \in E$. For every $\sigma \in E$, we can find $m < \omega$ with $(\sigma_i\sigma\delta_j\alpha)^m \in T(E(A_i))$. Then, $(\sigma\delta_j\alpha\sigma_i)^{m+1} = \sigma\delta_j\alpha(\sigma_i\sigma\delta_j\alpha)^m\sigma_i \in T(E)$. Let $I = E\delta_j\alpha\sigma_i$, and observe that we have just shown that $\overline{I} = (I + T(E))/T(E)$ is a left nilideal of $E/T(E)$. Hence, $\mathbb{Q}\overline{I}$ is a left nil ideal of $\mathbb{Q}(E/T(E))$. Since the latter ring is Artinian, $\mathbb{Q}\overline{I}$ is nilpotent, and the same holds for $\overline{I}$. Therefore, $\overline{I} \subseteq N/T(E)$, and $I \subseteq N$.

b) is a consequence Lemma 2.5. $\square$

Lemma 2.7. *Let A and B be in $\mathcal{S}$ such that $S_B(A) \doteq A$ and $S_A(B) \doteq B$ and $r_0(A), r_0(B) > 0$. Then, there exists an essentially strongly indecomposable direct quasi-summand of A which is not torsion and is quasi-isomorphic to a direct quasi-summand of B.*

Proof. Without loss of generality, we may assume $B = S_A(B)$ and $A = S_B(A)$.

Choose non-zero integers s and t such that $sA \subseteq A_1 \oplus \cdots \oplus A_n \subseteq A$ and $tB \subseteq B_1 \oplus \cdots \oplus B_m \subseteq B$ for essentially strongly indecomposable groups $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$. Choose an ideal N_B of $E(B)$ containing $T(E(B))$ such that $N_B/T(E(B)) = N(\overline{E(B)})$. By Lemma 2.4, B/N_BB is not torsion. Select $b \in B$ such that $b + N_BB$ has infinite order. We shall modify the choice of b to find an element suitable for our purposes.

Since $B = S_A(B)$, we can find $x_1, \ldots, x_\ell \in A$ and $\phi_1, \ldots, \phi_\ell \in \mathrm{H}_A(B)$ with $b = \phi_1(x_1) + \cdots + \phi_\ell(x_\ell)$. At least one of the elements $\phi_i(x_i) + N_BB$ has to have infinite order, and we may assume that $b = \phi(a)$ for some $a \in A$ and $\phi \in \mathrm{H}_A(B)$. There exist $a_i \in A_i$ for $i = 1, \ldots, n$ with $sa = a_1 + \cdots + a_n$. Since $sb + N_BB$ has infinite order, and $sb = \phi(sa) = \phi(a_1) + \cdots + \phi(a_n)$, at least one of the cosets $\phi(a_j) + N_BB$ has to have infinite order. Without loss of generality, this occurs for $j = 1$. Hence, we may assume $b = \phi(a_1)$ for some $a_1 \in A_1$.

Because of $A_1 \doteq S_B(A_1)$, we can find a non-zero integer k, $y_1, \ldots, y_r \in B$ and $\psi_1, \ldots, \psi_r \in \mathrm{H}_B(A_1)$ with $ka_1 = \psi_1(y_1) + \cdots + \psi_r(y_r)$. Since $kb + N_BB$ has infinite order, the same holds for at least one of the cosets $\psi_j(y_j) + N_BB$. Therefore, we may assume $b = \phi\psi(y)$ for some $y \in B$ and $\psi \in \mathrm{H}_B(A)$. There exist $b_j \in B_j$ for $j = 1, \ldots, m$ with $ty = b_1 + \cdots + b_m$. Arguing as before, we may assume that $b = \phi\psi(b_1)$ for some $b_1 \in B_1$. Let $\gamma = \psi_{|B_1} \in \mathrm{Hom}(B_1, A_1)$.

By the remarks preceding Lemma 2.5, the quasi-projections σ_j^B and embeddings δ_j^B satisfy $t1_B = \Sigma_{j=1}^m \delta_j^B\sigma_j^B$. Since $t\phi\gamma(b_1) = \Sigma_{j=1}^m \delta_j^B\sigma_j^B\phi\gamma(b_1)$ has infinite order modulo N_BB, the same has to hold for $\delta_j^B\sigma_j^B\phi\gamma(b_1)$ for some $j \in \{1, \ldots, m\}$. We make this elements our final choice for b.

If $B_1 \not\sim B_j$, then $\delta_j^B\sigma_j^B\phi\gamma\sigma_1^B \in N_B$ by Lemma 2.6a). But then, $\sigma_1^B(b_1) = tb_1$ yields $tb = t\delta_j^B\sigma_j^B\phi\gamma(b_1) = \delta_j^B\sigma_j^B\phi\gamma(tb_1) = \delta_j^B\sigma_j^B\phi\gamma\sigma_1(b_1) \in N_BB$. Thus, b has finite order modulo N_BB, a contradiction.

Therefore, B_1 and B_j are quasi-isomorphic. But then, $\alpha = \sigma_j^B\phi\gamma$ has to be a quasi-isomorphism. Otherwise, we can use Lemma 2.6a) to obtain $\delta_j^B\sigma_j^B\phi\gamma\sigma_1^B \in$

N_B as before. Arguing as in the last paragraph, we obtain a contradiction. Therefore, $\sigma_j^B \phi \gamma$ is a quasi-isomorphism by Lemma 2.6b); and γ is a quasi-splitting quasi-monomorphism. Since A_1 is essentially strongly indecomposable, this is only possible if γ is a quasi-isomorphism. □

Theorem 2.8. *The following are equivalent for groups $A, B \in \mathcal{S}$:*
  a) $A \sim B$.
  b) $r_0(\mathrm{Hom}(A, C)) = r_0(\mathrm{Hom}(B, C))$ *for all groups* $C \in \mathcal{S}$.

Proof. It remains to show $b) \Rightarrow a)$: Observe that

$$r_0(E(B)) = r_0(\mathrm{Hom}(B, B)) = r_0(\mathrm{Hom}(A, B)) = r_0(\mathrm{Hom}(A, S_A(B))) < \infty, \quad \text{and}$$
$$r_0(E(A)) = r_0(\mathrm{Hom}(A, A)) = r_0(\mathrm{Hom}(B, A)) = r_0(\mathrm{Hom}(A, S_B(A))) < \infty.$$

In particular, $\mathrm{Hom}(A, T(B))$ and $\mathrm{Hom}(B, T(A))$ are torsion since otherwise their torsion-free ranks would be infinite. Hence, $A \oplus B$ is self-small by [4, Theorem 2.4]. It follows that every finitely A-generated subgroup of B is self-small since it is an endomorphic image of a finite power of $A \oplus B$.

The group $S_A(B)$ may not be self-small. Hence, choose a finitely A-generated subgroup U of B with $r_0(\mathrm{Hom}(A, U)) = r_0(\mathrm{Hom}(A, S_A(B)))$. Because U is self-small, $r_0(\mathrm{Hom}(A, U)) = r_0(\mathrm{Hom}(B, U))$. Thus, $E(B)/\mathrm{Hom}(B, U)$ is torsion as an abelian group, and there is a non-zero integer d such that $d1_B \in \mathrm{Hom}(B, U)$. Therefore, $dB \subseteq U \subseteq S_A(B) \subseteq B$. Hence, $B \doteq S_A(B)$ and $A \doteq S_B(A)$.

Let $A \doteq A_1 \oplus \ldots \oplus A_n$ and $B \doteq B_1 \oplus \ldots \oplus B_m$ for essentially strongly indecomposable groups $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ of positive rank. The proof will proceed by induction on $n + m$. If $n + m = 0$, then A and B are torsion, and hence finite. Clearly, there is nothing to prove. Thus, assume $n + m > 0$. Without loss of generality, $m > 0$.

By Lemma 2.7, we may assume $A_1 \sim B_1$. In particular, $r_0(\mathrm{Hom}(A_1, C)) = r_0(\mathrm{Hom}(B_1, C))$ whenever $C \in \mathcal{S}$. Moreover,

$$r_0(\mathrm{Hom}(A_2 \oplus \ldots \oplus A_n, C)) = r_0(\mathrm{Hom}(A_1 \oplus \ldots \oplus A_n, C)) - r_0(\mathrm{Hom}(A_1, C)) =$$
$$r_0(\mathrm{Hom}(A, C)) - r_0(\mathrm{Hom}(A_1, C)) = r_0(\mathrm{Hom}(B, C)) - r_0(\mathrm{Hom}(B_1, C)) =$$
$$r_0(\mathrm{Hom}(B_1 \oplus \ldots \oplus B_m, C)) - r_0(\mathrm{Hom}(B_1, C)) = r_0(\mathrm{Hom}(B_2 \oplus \ldots \oplus B_m, C))$$

for all self-small groups C. By induction hypothesis, $B_2 \oplus \ldots \oplus B_m \sim A_2 \oplus \ldots \oplus A_n$. Consequently, $A \sim B$. □

The proof of the last result actually shows that the rank of the test groups C need not exceed the common rank of A and B:

Corollary 2.9. *The following are equivalent for groups $A, B \in \mathcal{S}$:*
  a) $A \sim B$.
  b) $r_0(\mathrm{Hom}(A, C)) = r_0(\mathrm{Hom}(B, C))$ *for all self-small groups C with* $r_0(C) \leq \max(r_0(A), r_0(B))$. □

Corollary 2.9 cannot be extended to determine A and B up to isomorphism:

Example 2.10. *There exist non-isomorphic groups A and B in $\mathcal{S}$ with $r_0(A) = r_0(B) = 2$ and $\mathrm{Hom}(A, C) \cong \mathrm{Hom}(B, C)$ for all $C \in \mathcal{S}$ with $r_0(C) \leq 2$.*

Proof. By [3, Example 3.5], there exist quasi-isomorphic torsion-free groups A and B of rank 2 such that $\mathrm{Hom}(A, D) \cong \mathrm{Hom}(B, D)$ for all torsion-free groups D of rank at most 2, but $A \not\cong B$. Furthermore, $E(A) = E(B) = \mathbb{Z}_{pq}$ for two primes $p \neq q$. By [5, Theorem 0.2], we have $r_p(A) = r_p(B)$ and $r_q(A) = r_q(B)$.

Suppose that $C \in \mathcal{S}$ with $r_0(C) \leq 2$. Write $C = T_p(C) \oplus T_q(C) \oplus C'$. For $s = p, q$, we obtain $\mathrm{Hom}(A, T_s(C)) \cong \oplus_{r_s(A)} T_s(C) \cong \oplus_{r_s(B)} T_s(C) \cong \mathrm{Hom}(B, T_s(C))$. Thus, we may assume $C_p = C_q = 0$.

Since $T_s(C)$ is finite for all primes s, we have $\mathrm{Hom}(A, T(C)) = \mathrm{Hom}(B, T(C)) = 0$, and hence $S_B(C) \cap T(C) = S_A(C) \cap T(C) = 0$. Let $U = S_A(C) + S_B(C)$. We show that U is torsion-free. Suppose $u \in U$ satisfies $nu = 0$ for some non-zero $n \in \mathbb{Z}$, coprime with p and q. Write $u = \phi_1(a_1) + \cdots + \phi_k(a_k) + \psi_1(b_1) + \cdots + \psi_m(b_m)$ for $\phi_1, \ldots, \phi_k \in \mathrm{H}_A(C)$, $\psi_1, \ldots, \psi_m \in \mathrm{H}_B(C)$, $a_1, \ldots a_k \in A$, and $b_1, \ldots b_m \in B$. Since A and B are quasi-isomorphic, we may assume $tA \subseteq B \subseteq A$ where $t = p^i q^j$. Then, $tu = \phi_1(ta_1) + \cdots + \phi_k(ta_k) + \psi_1(tb_1) + \cdots + \psi_m(tb_m)$. Since $\phi_1|B, \ldots, \phi_k|B \in \mathrm{H}_B(C)$, we obtain $tu \in S_B(C)$. Since $(n, t) = 1$, there are $x, y \in \mathbb{Z}$ with $1 = xt + yn$. Then, $u = xtu + ynu = xtu \in S_B(C) \cap T(C) = 0$. Hence, U is torsion-free; and $\mathrm{Hom}(A, C) = \mathrm{Hom}(A, U) \cong \mathrm{Hom}(B, U) = \mathrm{Hom}(B, C)$ as desired. $\square$

A finite torsion-free rank group A is self-small if and only if every $T_p(A)$ of A is finite for all primes p and there exists (for every) a full free subgroup $F \leq A$ such that A/F is p-divisible for almost all p with $T_p(A) \neq 0$. For $A \in \mathcal{S}$, define *the support of* A to be $S(A) = \{p \in \mathbb{P} \mid T_p(A) \neq 0\}$, and *the divisibility of* A the quasi-equality class $D(A)$ of all primes p such that A/F is p-divisible, where F is a fixed full free subgroup of A. For a set W of primes, consider the class $\mathcal{S}(W) = \{A \in \mathcal{S} \mid S(A) \dot\subseteq W \dot\subseteq D(A)\}$, which is closed with respect direct summands, finite direct sums, finitely $\mathcal{S}(W)$-generated subgroups, and quasi-isomorphisms. Observe that, modulo finite direct summands, $\mathcal{S}(\varnothing)$ is the class of all finite rank torsion free groups, and $\mathcal{S}(\mathbb{P})$ is the class of quotient divisible groups. Arguing as in the proof of Theorem 2.8, we obtain

Corollary 2.11. *The following are equivalent for groups A and B in $\mathcal{S}(W)$:*

a) $A \sim B$.
b) $r_0(\mathrm{Hom}(A, C)) = r_0(\mathrm{Hom}(B, C))$ *for all groups* $C \in \mathcal{S}(W)$ *with* $r_0(C) \leq \max(r_0(A), r_0(B))$. $\square$

3. Self-small groups Determined by Their Right Orthogonal Classes

Let W be a set of primes, and consider $A \in \mathcal{S}(W)$. Suppose that $C \in \mathcal{S}(W)$ is a subgroup of A. The symbol $C^\sharp$ denotes the subgroup of A containing C such that $C^\sharp/C$ is the sum of the torsion part of the maximal divisible subgroup of A/C and the p-components of A/C for all primes p not in $S(A)$.

Lemma 3.1. *With the above notations, $C^\sharp$ and $A/C^\sharp$ are in $\mathcal{S}(W)$.*

Proof. It is obvious that $S(A/C^\sharp) \subseteq S(A)$. Let F_1 be a full free subgroup of C. If F_2 is a free subgroup of A such that $F = F_1 \oplus F_2$ is a full free subgroup of A, then $(F_2 + C^\sharp)/C^\sharp$ is a full free subgroup of $A/C^\sharp$. Hence, $[A/C^\sharp]/[(F_2 + C^\sharp)/C^\sharp)] \cong A/(F_2 + C^\sharp)$ is an epimorphic image of A/F, and $A/C^\sharp \in \mathcal{S}(W)$.

Moreover, the middle term in the exact sequence $0 \to C/F_1 \to C^\sharp/F_1 \to C^\sharp/C \to 0$ is torsion and divisible for almost all $p \in W$ since the groups C/F_1 and $C^\sharp/C$ are p-divisible groups for almost all $p \in W$. Therefore $C^\sharp \in \mathcal{S}(W)$. $\square$

If $A, X \in \mathcal{S}(W)$ then $A \oplus X \in \mathcal{S}(W)$, hence $T(\mathrm{Hom}(A, X)) = \mathrm{Hom}(A, T(X))$. Therefore every homomorphism $A \to T(X)$ has finite image. Let

$$A^{\perp} = \{X \in \mathcal{S}(W) \mid \mathbb{Q}\mathrm{Hom}(A, X) = 0\} = \{X \in \mathcal{S}(W) \mid \mathrm{Hom}(A, X) \text{ is torsion}\}.$$

The next results outlines the basic properties of the classes $A^{\perp}$.

Lemma 3.2. *Let W be a set of primes.*

a) *If $A, B \in \mathcal{S}(W)$, then $B \in A^{\perp}$ if and only if the image $S_A(B)$ is torsion.*
b) *If $A_1, \ldots, A_n \in \mathcal{S}(W)$ then $(\oplus_{i=1}^n A_i)^{\perp} = \bigcap_{i=1}^n A_i^{\perp}$.*
c) *$A^{\perp} = (A \oplus \mathbb{Q})^{\perp}$ for all $A \in \mathcal{S}(W)$.*
d) *If A, $B \in \mathcal{S}(W)$ such that there exists a homomorphism $\alpha : A \to B$ with $B/\alpha(A)$ a torsion group, then $A^{\perp} \subseteq B^{\perp}$.*
e) *If A and B are quasi-isomorphic groups in $\mathcal{S}(W)$ then $A^{\perp} = B^{\perp}$.* □

The next result is based on ideas from [13, Section 4] and from [10, Proposition 5.6].

Proposition 3.3. *If $A \in \mathcal{S}(W)$ then there exists a group $C \in \mathcal{S}(W)$ such that:*

i) *C is an epimorphic image of A,*
ii) *C has no non-zero nilpotent quasi-endomorphisms, and*
iii) *$A^{\perp} = C^{\perp}$.*

Proof. If there exists an endomorphism α of A such that $\alpha(A)$ is not torsion, but $\alpha^2 = 0$, then $\mathrm{Im}(\alpha) \subseteq \mathrm{Ker}(\alpha)$ and $A/\mathrm{Ker}(\alpha) \cong \mathrm{Im}(\alpha)$ are in $\mathcal{S}(W)$. Moreover, $[A/\mathrm{Im}(\alpha)]/[\mathrm{Ker}(\alpha)/\mathrm{Im}(\alpha)] \cong A/\mathrm{Ker}(\alpha) \in \mathcal{S}(W)$; and the support of the group $[A/\mathrm{Im}(\alpha)]/[\mathrm{Ker}(\alpha)/\mathrm{Im}(\alpha)]$ is contained in $S(A)$. Consequently, $\mathrm{Ker}(\alpha)/\mathrm{Im}(\alpha)$ contains the torsion part of the maximal divisible subgroup of $A/\mathrm{Im}(\alpha)$ and the p-components of $A/\mathrm{Im}(\alpha)$ for all primes p not in $S(A)$. Moreover, $\mathrm{Im}(\alpha)^{\sharp} \subseteq \mathrm{Ker}(\alpha)$. Let $A' = A/\mathrm{Im}(\alpha)^{\sharp}$ and $\pi : A \to A'$ be the canonical epimorphism.

Consider a group X in $\mathcal{S}(W)$. If we can find a homomorphism $f : A \to X$ such that $\mathrm{Im}(f)$ is not torsion, then $A/\mathrm{Ker}(f) \cong \mathrm{Im}(f) \in \mathcal{S}(W)$. The canonical epimorphism $\rho : A \to A/\mathrm{Ker}(f)$ satisfies $\mathrm{Im}(\rho\alpha) = [\mathrm{Im}(\alpha) + \mathrm{Ker}(f)]/\mathrm{Ker}(f) \in \mathcal{S}(W)$. If the latter is torsion, then it has to be finite. Choose a non-zero integer k with $\mathrm{Im}(k\alpha) \subseteq \mathrm{Ker}(f)$. We observe that the torsion part of $A/\mathrm{Ker}(f)$ is reduced and $S(A/\mathrm{Ker}(f)) \subseteq S(A)$. Then, $\mathrm{Im}(k\alpha)^{\sharp} \subseteq \mathrm{Ker}(f)$, and f induces a homomorphism $\overline{f} : A/\mathrm{Im}(k\alpha)^{\sharp} \to X$ defined by $\overline{f}(a + \mathrm{Im}(k\alpha)^{\sharp}) = f(a)$, whose image is not torsion. On the other hand, if $(\mathrm{Im}(\alpha) + \mathrm{Ker}(f))/\mathrm{Ker}(f)$ is not torsion, then consider the homomorphism $\overline{\alpha} : A/\mathrm{Im}(\alpha)^{\sharp} \to \mathrm{Ker}(\alpha)$ defined by $\overline{\alpha}(a + \mathrm{Im}(\alpha)^{\sharp}) = \alpha(a)$. Let $\iota : \mathrm{Ker}(\alpha) \to B'$ be the inclusion map; and consider $f\iota\overline{\alpha} : A/\mathrm{Im}(\alpha)^{\sharp} \to X$. Choose $a' = a + \mathrm{Im}(\alpha)^{\sharp} \in A/\mathrm{Im}(\alpha)^{\sharp}$ in such a way that $f\iota\overline{\alpha}(ka') = 0$ for some non-zero integer k. Then, $k\alpha(a) \in \mathrm{Ker}(f)$. Hence, there exists $x \in A/\mathrm{Im}(\alpha)^{\sharp}$ such that $f\iota\overline{\alpha}(x)$ has infinite order.

Consequently, $\mathrm{Hom}(A/\mathrm{Im}(\alpha)^{\sharp}, X)$ is not torsion; and $(A/\mathrm{Im}(\alpha)^{\sharp})^{\perp} \subseteq A^{\perp}$. Since the other inclusion is obvious, $(A/\mathrm{Im}(\alpha)^{\sharp})^{\perp} = A^{\perp}$ and $r_0(A/\mathrm{Im}(\alpha)^{\sharp}) < r_0(A)$. If $A' = A/\mathrm{Im}(\alpha)^{\sharp}$ has non-zero nilpotent quasi-endomorphisms, we can repeat this procedure to find a group $A'' \in \mathcal{S}(W)$ with $A''^{\perp} = A^{\perp}$ and $r_0(A'') < r_0(A') < r_0(A)$. This process has to stop after a finite number of steps, since $r_0(A)$ is finite. Hence, there exists a group $C \in \mathcal{S}(W)$ which is an epimorphic image of A such that $C^{\perp} = A^{\perp}$ and $N(\overline{E(C)}) = 0$. This C satisfies ii) as a consequence of Lemma 2.6. □

Theorem 3.4. *The following are equivalent for a group $A \in \mathcal{S}(W)$:*

a) *$A \sim B$ whenever $B \in \mathcal{S}(W)$ with $A^{\perp} = B^{\perp}$;*

b) *$A \sim (\oplus_{i=1}^{n} A_i) \oplus \mathbb{Q}^r$, where A_i are essentially strongly indecomposable groups such that*

   i) *$\mathbb{Q}E(A_i)$ is a division ring for all $i = 1, \ldots, n$,*

   ii) *$\mathrm{Hom}(A_i, A_j)$ is torsion for all $i \neq j$,*

   iii) *If a group $C \in \mathcal{S}(W)$ has torsion-free rank at most r and is an epimorphic image of some A_i for some i, then C is a sum of a (torsion-free) divisible group and a finite group.*

Proof. *a)* $\Rightarrow$ *b)* : We may assume that $A = A' \oplus \mathbb{Q}^r$, where A' is reduced and $A' = \oplus_{i=1}^{n} A_i$. If $\mathbb{Q}E(A')$ has a non-zero nilpotent element then, as in the proof of Proposition 3.3, there exists $C \in \mathcal{S}(W)$ such that ${A'}^{\perp} = C^{\perp}$ and $s = r_0(A') - r_0(C) > 0$. Then $A^{\perp} = (C \oplus \mathbb{Q}^{r+s})^{\perp}$, a contradiction. By Lemma 2.6, A satisfies conditions i) and ii). Suppose that $A_i/N = C \in \mathcal{S}(W)$ and $r_0(C) \leq r$. Then $A^{\perp} = A \oplus C \oplus \mathbb{Q}^{r - r_0(C)}$. Since $C \sim \mathbb{Q}^{r_0(C)}$ because of a), we obtain that iii) holds too.

b) $\Rightarrow$ *a)* : We consider direct decompositions $A = A' \oplus \mathbb{Q}^r$ and $B = B' \oplus \mathbb{Q}^s$ with A' and B' reduced groups. We may assume $A' = \oplus_{i=1}^{n} A_i$, where the A_i's are essentially strongly indecomposable groups which verify i), ii) and iii). Then, ${A'}^{\perp} = A^{\perp} = B^{\perp} = {B'}^{\perp}$. Moreover, there exists a group $C \in \mathcal{S}(W)$ as in Proposition 3.3 such that $B^{\perp} = C^{\perp}$.

As a consequence of Lemma 2.6, we may assume that $C = \oplus_{i=1}^{s} C_i$, where the rings $\mathbb{Q}E(C_i)$ are division rings and $\mathrm{Hom}(C_i, C_j)$ is torsion for all $i \neq j$. Let $j \in \{1, \ldots, s\}$. Since $\mathrm{Hom}(C, C_j)$ is not torsion, there exists $f_j : A' \to C_j$ such that $\mathrm{Im}(f_j)$ is not torsion. Because $\mathrm{Im}(f_j) \in \mathcal{S}(W)$, we can find a homomorphism $g : C \to \mathrm{Im}(f_j)$ whose image is not torsion. Since $\mathrm{Im}(f_j) \subseteq C_j$ and $g(\oplus_{i \neq j} C_i)$ is a finite group, $g(C_j)$ is not torsion either; and the restriction of g to C_j represents a non-zero quasi-endomorphism of C_j. Therefore, it has to be a quasi-epimorphism; and $\mathrm{Im}(g)$ has finite index in C_j. Since the same holds for $\mathrm{Im}(f_j)$, we obtain $S_{A'}(C) \doteq C$. In the same way, $S_C(A') \doteq A'$. By Lemma 2.7, we may assume $A_1 \sim C_1$.

If $j \in \{2, \ldots, s\}$, then $f_j(A_1)$ is a finite group. Hence, the restriction $f_{j|\oplus_{i=2}^{n} A_i} : \oplus_{i=2}^{n} A_i \to C_j$ is a quasi-epimorphism. Then, $S_{\oplus_{i=2}^{n} A_i}(\oplus_{j=2}^{s} C_j) \doteq \oplus_{i=2}^{s} C_j$, and in the same way we obtain $S_{\oplus_{i=2}^{s} C_j}(\oplus_{i=2}^{n} A_i) \doteq \oplus_{i=2}^{n} A_i$. Lemma 2.7 once more allows us to assume $A_2 \sim C_2$. We repeat these arguments for all j, and obtain $A' \sim C$ after a finite number of steps.

Suppose that B' has an endomorphism α such that $\alpha \neq \alpha^2 = 0$ in $\mathbb{Q}E(B')$. In view of the proof of Proposition 3.3, we may assume that C is an epimorphic image of $B'/\mathrm{Im}(\alpha)^{\sharp}$. Then $r_0(\mathrm{Im}(\alpha)^{\sharp}) \leq r_0(B') - r_0(C) \leq r_0(A) - r_0(C) = r$. Since we can view α as a map from B' to $\mathrm{Im}(\alpha)^{\sharp}$, there exists a homomorphism $\beta : A' \to \mathrm{Im}(\alpha)^{\sharp}$ which has a non-torsion image. Using iii), $\mathrm{Im}(\beta) \leq B'$ is divisible. But this is not possible since B' is reduced. Therefore $N_{B'}(E(B')/T(E(B'))) = 0$. Consequently, $B' \sim C \sim A'$, and $B \sim A$. □

References

[1] Albrecht, U., *Fuchs' Problem 34 for Mixed Abelian Groups*, Proc. AMS 131 (2002), 1021-1029.

[2] Albrecht, U., Breaz, S. and Wickless, W., *Self-small abelian groups*, preprint.

[3] Albrecht, U. F., and Goeters, H. P., *Fuchs' Problem 34*, Proc. AMS 124 (1996), 1319–1328.
[4] Albrecht, U., Goeters, P. and Wickless, W.: The Flat Dimension of Abelian Groups as E-modules, Rocky Mount. J. of Math., 25(2) (1995), 569–590.
[5] Arnold, D. M., Finite Rank Torsion Free Abelian Groups and Rings, Lect. Notes in Math. Springer-Verlag, 931, (1982).
[6] Arnold, D. M., Murley, C. E., *Abelian groups, A, such that* $\mathrm{Hom}(A, -)$ *preserves direct sums of copies of A*, Pacific J. Math., 56, (1975), 7-21.
[7] Breaz, S., *Quasi-decompositions for self-small abelian groups*, Comm. Algebra 32 (2004), 1373–1384.
[8] Fuchs, L., *Infinite abelian groups I,II*, Academic Press (1970/1973).
[9] Goeters, H. P., *Fuchs' problem 43*, Rocky Mount. J. of Math., 22(1) (1992), 197–201.
[10] Kerner, O., Trlifaj, J., *Tilting classes over wild hereditary algebras*, J. Algebra 290 (2005), 538-556.
[11] Schultz, P., *Annihilator classes of torsion-free abelian groups*, in Lecture Notes in Mathematics, vol. 697, Springer-Verlag, 1978.
[12] Walker, E.: Quotient categories and quasi-isomorphisms of abelian groups, Proc. Colloq. Abelian Groups, Budapest, (1964), 147-162.
[13] Wickless, W. J., *An equivalence relation for torsion-free abelian groups of finite rank*, J. Algebra, 153 (1992), 1–12.

Department of Mathematics, Auburn University, Auburn, AL 36849, U.S.A.
E-mail address: albreuf@mail.auburn.edu

"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, Str. Mihail Kogălniceanu 1, 400084 Cluj-Napoca, Romania
E-mail address: bodo@math.ubbcluj.ro