QUASI-ISOMORPHISMS AND GROUPS OF QUASI-HOMOMORPHISMS

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Abstract. This paper investigates to which extent a self-small mixed Abelian group $G$ of finite torsion-free rank is determined by the groups $\text{Hom}(G, C)$ where $C$ is chosen from a suitable class $\mathcal{C}$ of Abelian groups. We show that $G$ is determined up to quasi-isomorphism if $\mathcal{C}$ is the class of all self-small mixed groups with $r_0(C) \leq r_0(G)$. Several related results are given, and the dual problem of orthogonal classes is investigated.

1. Introduction

Problem 34 in [8] asks whether it is possible to find a set $\mathcal{C}$ of Abelian groups with the property that two Abelian groups $A$ and $B$ are isomorphic provided that $\text{Hom}(A, C) \cong \text{Hom}(B, C)$ for all $C \in \mathcal{C}$. The negative answer given in [3] raised the question to identify properties of Abelian groups which are determined by $\text{Hom}(\cdot, \mathcal{C})$ when $\mathcal{C}$ is chosen from a class $\mathcal{C}$. For instance, if $A$ is a reduced $p$-group, and $\mathcal{C}$ is the class of all $p$-groups, then the groups $\{\text{Hom}(A, C) | C \in \mathcal{C}\}$ determine the finite Ulm-Kaplansky invariants $f_\sigma(A)$ for $\sigma \geq \omega$ [1]. On the other hand, if $\mathcal{C}$ is the class of torsion-free groups of finite rank, then the invariants $\{r_0(\text{Hom}(A, C)) | C \in \mathcal{C}\}$ determine a group $A \in \mathcal{C}$ up to quasi-isomorphism, but not necessarily up to isomorphism [3].

The results in [1] also addressed the question if there are classes of mixed groups which are determined up to isomorphism by homomorphism groups. For instance, two groups $A$ and $B$ in $\mathcal{G}$ are isomorphic if $\text{Hom}(A, G) \cong \text{Hom}(B, G)$ for all $G \in \mathcal{G}$. Here, $\mathcal{G}$ is the class of all self-small groups of finite torsion-free rank $G$ such that $G/T(G)$ is divisible where $T(G)$ denotes the torsion subgroup of $G$. Self-small groups were introduced by Arnold and Murley in [6] as the groups $A$ with the property that $\text{Hom}(A, A^{(I)})$ and $\text{Hom}(A, A)^{(I)}$ are canonically isomorphic for all index-sets $I$. Although the class $\mathcal{S}$ of all self-small groups of finite torsion-free rank is closed with respect direct summands and endomorphic images [6], it is not closed with respect direct sums [2]. Further information on self-small groups can be found in [7].

Theorem 2.8 shows that a group $A \in \mathcal{S}$ is determined up to quasi-isomorphism by the torsion-free rank of the groups $\text{Hom}(A, C)$ where $C \in \mathcal{S}$ (and $r_0(C) \leq r_0(A)$). Moreover, groups $A \in \mathcal{S}$ with $r_0(A) = 1$ are determined up to isomorphism in this way. However, this fails in general (Example 2.10).

Section 3 addresses a question closely related to Problem 34: Given a self-small group $A$, can we find a set $\mathcal{C}$ containing $A$ and a property $P$ such that $A$ is determined up to isomorphism by $\text{Ker}(\mathbb{Q}\text{Hom}(A, \cdot)) \cap \mathcal{C}$ and $P$? Annihilator classes like...
these were first considered by Schultz in [11]; and Wickless solved a dual problem in [13]. Theorem 3.4 gives an answer for some natural arising classes of self-small groups.

For the benefit of the reader, we give a short summary of the notation used. If $A$ is a group, then the symbol $T_p(A)$ indicates the $p$-component of $A$, and $\overline{A} = A/T(A)$. The endomorphism ring of $A$ is $E = E(A)$. There exists an adjoint pair $(H_A, T_A)$ of functors between the category of abelian groups and the category of right $E$-modules defined by $H_A(G) = \text{Hom}(A,G)$ and $T_A(M) = M \otimes_E A$ for all abelian groups $A$ and all right $E$-modules $M$. These functors induce natural maps $\theta_G : T_AH_A(G) \to G$ and $\Phi_M : M \to H_A T_A(M)$ defined by $\theta_G(\alpha \otimes a) = \alpha(a)$ and $[\Phi_M(x)](a) = x \otimes a$ for all $a \in A$, $x \in M$, and $\alpha \in H_A(G)$. The $A$-socle of $G$, denoted by $S_A(G)$, is the image of $\theta_G$. If $C$ is a class of groups, then a group $G$ is (finitely) $C$-generated if it is an epimorphic image of a (finite) direct sum of groups in $C$. It is easy to see that $G$ is $A$-generated if and only if $S_A(G) = G$. The symbol $\sim$ denotes quasi-isomorphisms, i.e. isomorphisms in $\mathbb{Q}Ab$. Two subgroups $G$ and $H$ of a group $A$ are quasi-equal, denoted by $G \simeq H$, if $G \cap H$ is of finite index in $G$ and in $H$. Finally, if $V$ and $W$ are sets such that all but finitely many elements of $V$ are contained in $W$, then $V$ is quasi-contained in $W$. This is denoted by $V \varsubsetneq W$. The sets $V$ and $W$ are quasi-equal, denoted by $V \doteq W$, if $V \subseteq W$ and $W \subseteq V$.

2. Self-small Groups Determined by Homomorphism Groups

We begin our discussion on how far the structure of a mixed group $A$ is determined by the groups $\text{Hom}(A,C)$ with the case that the test groups $C$ have rank 1:

**Proposition 2.1.** Let $A$ and $B$ be in $S$ such that $\text{Hom}(A,C) \cong \text{Hom}(B,C)$ for all $C \in G$ with $r_0(C) = 1$. Then,

i) $T(A) \cong T(B)$,

ii) $r_p(A/T(A)) = r_p(B/T(B))$ for all primes $p$,

iii) $r_0(A) = r_0(B)$.

**Proof.** Observe that $T_p(A)$ and $T_p(B)$ are finite for all primes $p$ since $A$ and $B$ are in $S$. Choose $n < \omega$ with $p^n T_p(A) = p^n T_p(B) = 0$, and select a group $C \in G$ with $T_p(C) = \mathbb{Z}/p^{n+1} \mathbb{Z}$ and $r_0(G) = 1$. If $C = T_p(C) \oplus C^p$ such that multiplication by $p$ is an automorphism of $C^p$, then $\text{Hom}(A,C) = \text{Hom}(A,T_p(C)) \cong \text{Hom}(A,C^p)$, and multiplication by $p$ induces an automorphism on $\text{Hom}(A,C^p)$. Therefore, $T_p(\text{Hom}(A,C)) = T_p(A)T_p(C))$. Since $T_p(A)$ is finite, $A = T_p(A) \oplus A'$, and $\text{Hom}(A', T_p(C)) \cong \text{Hom}(A/T(A), T_p(C)) \cong [\mathbb{Z}/p^{n+1} \mathbb{Z}]^{n_p}$ where $n_p = r_p(A/T(A))$. On the other hand, $\text{Hom}(T_p(A), T_p(C)) \cong T_p(A)$. Thus, $T_p(\text{Hom}(A,C)) \cong T_p(A) \oplus [\mathbb{Z}/p^{n+1} \mathbb{Z}]^{n_p}$. Similarly, $T_p(\text{Hom}(B,C)) \cong T_p(B) \oplus [\mathbb{Z}/p^{n+1} \mathbb{Z}]^{n_p}$ where $m_p = r_p(B/T(B))$. Since $T_p(A)$ and $T_p(B)$ contain no direct summand isomorphic to $\mathbb{Z}/p^{n+1} \mathbb{Z}$, we have $T_p(A) \cong T_p(B)$ and $n_p = m_p$.

Furthermore, $Q \in G$ yields $\text{Hom}(A,Q) \cong Q^{r_0(A)}$ and $\text{Hom}(B,Q) \cong Q^{r_0(B)}$. Thus, $r_0(A) = r_0(B)$. $\square$

The converse of this proposition is not valid.

**Example 2.2.** There exist two groups $A, B \in S$ and a group $G \in G$ such that $A$ and $B$ verify i), ii) and iii) in Proposition 2.1 but $\text{Hom}(A,G) \ncong \text{Hom}(B,G)$. 
Proof. Let $A = \mathbb{Z}$ and $B$ a rank 1 torsion-free group of type $(1, \ldots, 1, \ldots)$. If $G \in \mathcal{G}$ is a group such that $T_p(G) \neq 0$ for infinitely many primes $p$, it is not hard to see that $\text{Hom}(A, G) \cong G$ is countable, while $\text{Hom}(B, G)$ is uncountable since it contains a subgroup isomorphic to $\text{Hom}(\oplus_p \mathbb{Z}/p\mathbb{Z}, T(G))$. \hfill $\square$

**Corollary 2.3.** Let $A$ and $B$ be self-small groups with $r_0(A) = r_0(B) = 1$. Then, $A \cong B$ if and only if $\text{Hom}(A, C) \cong \text{Hom}(B, C)$ for all $C \in \mathcal{S}$ with $r_0(C) = 1$.

**Proof.** Since $0 \neq \text{Hom}(A, \mathfrak{A}) \cong \text{Hom}(B, \mathfrak{A})$, $\text{type}(\mathfrak{B}) \leq \text{type}(\mathfrak{A})$. By symmetry, $\mathfrak{A}$ and $\mathfrak{B}$ have the same type. Moreover, $T(A) \cong T(B)$ by Proposition 2.1. Hence, [2, Corollary 4.5] yields $A \cong B$. \hfill $\square$

**Lemma 2.4.** Let $A \in \mathcal{S}$ with $r_0(A) > 0$. If $N$ is the two-sided ideal of $E$ containing $T(E)$ such that $N/T(E) = N(E/T(E))$, then $A/NA$ is not torsion.

**Proof.** Suppose that $A/NA$ is torsion. We show that $A/N^kA$ is torsion for all $k < \omega$ by induction on $k$. Suppose that $A/N^kA$ is a torsion group. The exact sequence $0 \rightarrow N^{k+1}/N^{k+1} \rightarrow E/N^{k+1} \rightarrow E/N^{k} \rightarrow 0$ induces $T_A(N^{k+1}/N^k) \rightarrow T_A(E/N^k) \rightarrow 0$ in which $T_A(E/N^k) \cong A/N^kA$ is torsion. Moreover, $N^{k+1}/N^k$ is an $E/N$-module, and there is an exact sequence $\oplus_i E/N \rightarrow N^{k+1}/N^k \rightarrow 0$ from which we obtain that $T_A(N^{k+1}/N^k)$ is torsion as an epimorphic image of the torsion group $\oplus_i T_A(E/N) \cong \oplus_i A/NA$. Then $A/N^{k+1}A$ is torsion.

Since $N(E/T(E)) = N_f(E) = 0$, there is $\ell < \omega$ with $N_f \subseteq T(E)$. Hence, $N^\ell A \subseteq T(E(A)) \subseteq T(A)$ is torsion and the same holds for $A$, a contradiction. \hfill $\square$

If $A \in \mathcal{S}$, then there are a non-zero integer $k$ and essentially strongly indecomposable subgroups $A_1, \ldots, A_n$ of $A$ such that $r_0(A_i) > 0$ for all $i = 1, \ldots, n$ and $kA \subseteq A_1 \oplus \cdots \oplus A_n \subseteq A$. These essentially strongly indecomposable groups are unique up to quasi-isomorphisms by [7, Theorem 2.9]. Let $\delta_i : A_i \rightarrow A$ be the inclusion map. For the projection maps $\pi_i : A_1 \oplus \cdots \oplus A_n \rightarrow A_i$, define $\sigma_i : A \rightarrow A_i$ by $\sigma_i(a) = \pi_i(ka)$. We obtain $k1_A = \sum_{i=1}^n \delta_i \sigma_i$, and $k1_{A_i} = \sigma_i \delta_i$. If necessary, a superscript referring to $A$ will be added to these maps.

**Lemma 2.5.** Let $A$ and $B$ be quasi-isomorphic, essentially strongly indecomposable groups in $\mathcal{S}$. If $\alpha : A \rightarrow B$ is a homomorphism then $\text{Ker}(\alpha)$ is bounded if and only if $\alpha$ is a quasi-isomorphism.

**Proof.** Suppose that $\text{Ker}(\alpha)$ is bounded. If $\beta : B \rightarrow A$ is a quasi-isomorphism, then $\beta \alpha : A \rightarrow A$ has a bounded kernel. Thus, $\beta \alpha$ is a quasi-monomorphism, which cannot be nilpotent. Since the quasi-endomorphism ring of $A$ is a local finite-dimensional $\mathbb{Q}$-algebra, $\beta \alpha$ is a quasi-isomorphism, and the same holds for $\alpha$. The converse implication is obvious (see [12]). \hfill $\square$

**Lemma 2.6.** For $A \in \mathcal{S}$, let $N$ be the ideal of $E$ containing $T(E)$ with $N/T(E) = N(E/T(E))$. If $\alpha \in \text{Hom}(A_1, A_2)$, then the following hold:

a) If $\alpha$ is not a quasi-isomorphism, then $\delta \alpha \sigma_1 \in N$.

b) If $A_i$ and $A_j$ are quasi-isomorphic, and if $\text{Ker}(\alpha)$ is bounded, then $\alpha$ is a quasi-isomorphism.

**Proof.** a) By [5, Proposition 9.1], $N/T(E)$ is pure in $E/T(E)$, and $QN/T(E) = N(QE/T(E))$. Moreover, $N/T(E) = N(QE/T(E)) \cap E/T(E)$. In particular, $E/N$ is torsion-free as an abelian group.
If $\delta_i\alpha_i \notin N$, then $\delta_i\alpha_i$ has infinite order. Suppose there is $\sigma \in E$ such that $\sigma\delta_i\alpha_i \notin N(E(A_i))$. Since $QE(A_i)$ is a local ring, $\sigma\delta_i\alpha_i$ is a unit of $QE(A_i)$. Thus, there exist a non-zero integer $k$ and a map $\beta \in E(A_i)$ such that $\beta\sigma\delta_i\alpha_i = k1_{A_i} \in T(E(A_i))$. Furthermore, we can find an integer $t$ with $t\beta\sigma\delta_i\alpha_i = tk1_{A_i}$. Thus, $\alpha$ is a quasi-splitting quasi-monomorphism. Write $A_j \cong \alpha(A_i) \oplus U$. Since $\alpha(A_i)$ is not bounded, $U$ is bounded. Thus, $\alpha$ is a quasi-isomorphism, which is not possible.

Consequently, $\sigma\delta_i\alpha_i \in N(E(A_i))$ for all $\sigma \in E$. For every $\sigma \in E$, we can find $m < \omega$ with $(\sigma\delta_i\alpha_i)^m \in T(E(A_i))$. Then, $(\sigma\delta_i\alpha_i)^{m+1} = \sigma\delta_i\alpha(\sigma\delta_i\alpha)^m \sigma_i \in T(E)$. Let $I = E\delta_i\alpha_i$, and observe that we have just shown that $T = (I + T(E))/T(E)$ is a left nilideal of $E/T(E)$. Hence, $QT$ is a left nil ideal of $Q(E/T(E))$. Since the latter ring is Artinian, $QT$ is nilpotent, and the same holds for $T$. Therefore, $T \subseteq N(T(E))$, and $I \subseteq N$.

b) is a consequence Lemma 2.5.

Lemma 2.7. Let $A$ and $B$ be in $S$ such that $S_B(A) \cong A$ and $S_A(B) \cong B$ and $r_0(A), r_0(B) > 0$. Then, there exists an essentially strongly indecomposable direct quasi-summand of $A$ which is not torsion and is quasi-isomorphic to a direct quasi-summand of $B$.

Proof. Without loss of generality, we may assume $B = S_A(B)$ and $A = S_B(A)$.

Choose non-zero integers $s$ and $t$ such that $sA \subseteq A_1 \oplus \cdots \oplus A_n \subseteq A$ and $tB \subseteq B_1 \oplus \cdots \oplus B_m \subseteq B$ for essentially strongly indecomposable groups $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$. Choose an ideal $N_B$ of $E(B)$ containing $T(E(B))$ such that $N_B/T(E(B)) = N(E(B))$. By Lemma 2.4, $B/N_B B$ is not torsion. Select $b \in B$ such that $b + N_B B$ has infinite order. We shall modify the choice of $b$ to find an element suitable for our purposes.

Since $B = S_A(B)$, we can find $x_1, \ldots, x_{\ell} \in A$ and $\phi_1, \ldots, \phi_{\ell} \in H_A(B)$ with $b = \phi_1(x_1) + \cdots + \phi_{\ell}(x_{\ell})$. At least one of the elements $\phi_i(x_i) + N_B B$ has to have infinite order, and we may assume that $b = \phi(a)$ for some $a \in A$ and $\phi \in H_A(B)$. There exist $a_i \in A_i$ for $i = 1, \ldots, n$ with $sa = a_1 + \cdots + a_n$. Since $sb + N_B B$ has infinite order, and $sb = \phi(sa) = \phi(a_1) + \cdots + \phi(a_n)$, at least one of the cosets $\phi(a_j) + N_B B$ has to have infinite order. Without loss of generality, this occurs for $j = 1$. Hence, we may assume $b = \phi(a_1)$ for some $a_1 \in A_1$.

Because of $A_1 \cong S_B(A_1)$, we can find a non-zero integer $k, y_1, \ldots, y_{m} \in B$ and $\psi_1, \ldots, \psi_{m} \in H_B(A_1)$ with $ka_1 = \psi_1(y_1) + \cdots + \psi_{m}(y_{m})$. Since $k + N_B B$ has infinite order, the same holds for at least one of the cosets $\psi_j(y_j) + N_B B$. Therefore, we may assume $b = \phi\psi(y)$ for some $y \in B$ and $\psi \in H_B(A)$. There exist $b_j \in B_j$ for $j = 1, \ldots, m$ with $t_{b_j} = b_1 + \cdots + b_m$. Arguing as before, we may assume that $b = \phi\psi(b_1)$ for some $b_1 \in B_1$. Let $\gamma = \psi_{b_1} \in \Hom(B_1, A_1)$.

By the remarks preceding Lemma 2.5, the quasi-projections $\sigma^B_j$ and embeddings $\delta^B_j$ satisfy $t_{b_1} = \sum_{j=1}^m \delta^B_j \sigma^B_j \phi \gamma(1)$. Since $t_\phi \gamma(b_1) = \sum_{j=1}^m \delta^B_j \sigma^B_j \phi \gamma(1)$ has infinite order modulo $N_B B$, the same has to hold for $\delta^B_j \sigma^B_j \phi \gamma(1)$ for some $j \in \{1, \ldots, m\}$. We make this elements our final choice for $b$.

If $B_1 \neq B_j$, then $\delta^B_1 \sigma^B_1 \phi \gamma(b_1) \in N_B$ by Lemma 2.6a). But then, $\sigma^B_1(b_1) = t_{b_1}$ yields $t_{b_1} = t\delta^B_1 \sigma^B_1 \phi \gamma(1) = \delta^B_1 \sigma^B_1 \phi \gamma(b_1) = \delta^B_1 \sigma^B_1 \phi \sigma \gamma(1) \in N_B B$. Thus, $b$ has finite order modulo $N_B B$, a contradiction.

Therefore, $B_1$ and $B_j$ are quasi-isomorphic. But then, $\alpha = \sigma^B_j \phi \gamma$ has to be a quasi-isomorphism. Otherwise, we can use Lemma 2.6a) to obtain $\delta^B_j \sigma^B_j \phi \gamma(1) \in N_B B$. This is a contradiction. Therefore, $B_1$ and $B_j$ are quasi-isomorphic. But then, $\alpha = \sigma^B_j \phi \gamma$ has to be a quasi-isomorphism. Otherwise, we can use Lemma 2.6a) to obtain $\delta^B_j \sigma^B_j \phi \gamma(1) \in N_B B$. This is a contradiction. Therefore, $B_1$ and $B_j$ are quasi-isomorphic. But then, $\alpha = \sigma^B_j \phi \gamma$ has to be a quasi-isomorphism.
N_{\alpha} as before. Arguing as in the last paragraph, we obtain a contradiction. Therefore, $\sigma_{\beta}^\# \phi \gamma$ is a quasi-isomorphism by Lemma 2.6b); and $\gamma$ is a quasi-splitting quasi-monomorphism. Since $A_1$ is essentially strongly indecomposable, this is only possible if $\gamma$ is a quasi-isomorphism.

**Theorem 2.8.** The following are equivalent for groups $A, B \in S$:

a) $A \sim B$.

b) $r_0(\text{Hom}(A, C)) = r_0(\text{Hom}(B, C))$ for all groups $C \in S$.

**Proof.** It remains to show b) $\Rightarrow$ a): Observe that

$$r_0(E(B)) = r_0(\text{Hom}(B, B)) = r_0(\text{Hom}(A, B)) = r_0(\text{Hom}(A, S_A(B))) < \infty,$$

and

$$r_0(E(A)) = r_0(\text{Hom}(A, A)) = r_0(\text{Hom}(B, A)) = r_0(\text{Hom}(A, S_B(A))) < \infty.$$

In particular, $\text{Hom}(A, T(B))$ and $\text{Hom}(B, T(A))$ are torsion since otherwise their torsion-free ranks would be infinite. Hence, $A \oplus B$ is self-small by [4, Theorem 2.4]. It follows that every finitely $A$-generated subgroup of $B$ is self-small since it is an endomorphic image of a finite power of $A \oplus B$.

The group $S_A(B)$ may not be self-small. Hence, choose a finitely $A$-generated subgroup $U$ of $B$ with $r_0(\text{Hom}(A, U)) = r_0(\text{Hom}(A, S_A(B)))$. Because $U$ is self-small, $r_0(\text{Hom}(A, U)) = r_0(\text{Hom}(B, U))$. Thus, $E(B)/\text{Hom}(B, U)$ is torsion as an abelian group, and there is a non-zero integer $d$ such that $d1_B \in \text{Hom}(B, U)$. Therefore, $dB \subseteq U \subseteq S_A(B) \subseteq B$. Hence, $B \cong S_A(B)$ and $A \cong S_B(A)$.

Let $A \cong A_1 \oplus \ldots \oplus A_n$ and $B \cong B_1 \oplus \ldots \oplus B_m$ for essentially strongly indecomposable groups $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ of positive rank. The proof will proceed by induction on $n + m$. If $n + m = 0$, then $A$ and $B$ are torsion, and hence finite. Clearly, there is nothing to prove. Thus, assume $n + m > 0$. Without loss of generality, $m > 0$.

By Lemma 2.7, we may assume $A_1 \sim B_1$. In particular, $r_0(\text{Hom}(A_1, C)) = r_0(\text{Hom}(B_1, C))$ whenever $C \in S$. Moreover,

$$r_0(\text{Hom}(A_2 \oplus \ldots \oplus A_n, C)) = r_0(\text{Hom}(A_1 \oplus \ldots \oplus A_n, C)) - r_0(\text{Hom}(A_1, C)) = r_0(\text{Hom}(A, C)) - r_0(\text{Hom}(A_1, C)) = r_0(\text{Hom}(B, C)) - r_0(\text{Hom}(B_1, C)) = r_0(\text{Hom}(B_1 \oplus \ldots \oplus B_m, C)) - r_0(\text{Hom}(B_1, C)) = r_0(\text{Hom}(B_2 \oplus \ldots \oplus B_m, C))$$

for all self-small groups $C$. By induction hypothesis, $B_2 \oplus \ldots \oplus B_m \sim A_2 \oplus \ldots \oplus A_n$. Consequently, $A \sim B$.

The proof of the last result actually shows that the rank of the test groups $C$ need not exceed the common rank of $A$ and $B$:

**Corollary 2.9.** The following are equivalent for groups $A, B \in S$:

a) $A \sim B$.

b) $r_0(\text{Hom}(A, C)) = r_0(\text{Hom}(B, C))$ for all self-small groups $C$ with $r_0(C) \leq \max(r_0(A), r_0(B))$.

**Example 2.10.** There exist non-isomorphic groups $A$ and $B$ in $S$ with $r_0(A) = r_0(B) = 2$ and $\text{Hom}(A, C) \cong \text{Hom}(B, C)$ for all $C \in S$ with $r_0(C) \leq 2$.

**Proof.** By [3, Example 3.5], there exist quasi-isomorphic torsion-free groups $A$ and $B$ of rank 2 such that $\text{Hom}(A, D) \cong \text{Hom}(B, D)$ for all torsion-free groups $D$ of
rank at most 2, but \( A \not\sim B \). Furthermore, \( E(A) = E(B) = \mathbb{Z}_m \) for two primes \( p \neq q \). By [5, Theorem 0.2], we have \( r_p(A) = r_p(B) \) and \( r_q(A) = r_q(B) \).

Suppose that \( C \in S \) with \( r_0(C) \leq 2 \). Write \( C = T_p(C) \oplus T_q(C) \oplus C' \). For \( s = p, q \), we obtain \( \text{Hom}(A, T_s(C)) \cong \oplus_{r_s(A)} T_s(C) \cong \oplus_{r_s(B)} T_s(C) \cong \text{Hom}(B, T_s(C)) \). Thus, we may assume \( C_p = C_q = 0 \).

Since \( T_s(C) \) is finite for all primes \( s \), we have \( \text{Hom}(A, T(C)) = \text{Hom}(B, T(C)) = 0 \), and hence \( S_B(C) \cap T(C) = S_A(C) \cap T(C) = 0 \). Let \( U = S_A(C) + S_B(C) \). We show that \( U \) is torsion-free. Suppose \( u \in U \) satisfies \( nu = 0 \) for some non-zero \( n \in \mathbb{Z} \), coprime with \( p \) and \( q \). Write \( u = \phi_1(a_1) + \cdots + \phi_k(a_k) + \psi_1(b_1) + \cdots + \psi_m(b_m) \) for \( \phi_1, \ldots, \phi_k \in H_A(C), \psi_1, \ldots, \psi_m \in H_B(C), a_1, \ldots, a_k \in A, \) and \( b_1, \ldots, b_m \in B \). Since \( A \) and \( B \) are quasi-isomorphic, we may assume \( tA \subseteq B \subseteq A \) where \( t = p^aq^q \). Then, \( tu = \phi_1(ta_1) + \cdots + \phi_k(ta_k) + \psi_1(tb_1) + \cdots + \psi_m(tb_m) \). Since \( \phi_1|B, \ldots, \phi_k|B \in H_B(C) \), we obtain \( tu \in S_B(C) \). Since \( (n, t) = 1 \), there are \( x, y \in \mathbb{Z} \) with \( 1 = xt + ny \). Then, \( u = xtu + ynu = xtu \in S_B(C) \cap T(C) = 0 \). Hence, \( U \) is torsion-free; and \( \text{Hom}(A, C) = \text{Hom}(A, U) \cong \text{Hom}(B, U) = \text{Hom}(B, C) \) as desired. \( \square \)

A finite torsion-free rank group \( A \) is self-small if and only if every \( T_p(A) \) of \( A \) is finite for all primes \( p \) and there exists (for every) a full free subgroup \( F \leq A \) such that \( A/F \) is \( p \)-divisible for almost all \( p \) with \( T_p(A) \neq 0 \). For \( A \in S \), define the support of \( A \) to be \( S(A) = \{ p \in \mathbb{P} \mid T_p(A) \neq 0 \} \), and the divisibility of \( A \) the quasi-equality class \( D(A) \) of all primes \( p \) such that \( A/F \) is \( p \)-divisible, where \( F \) is a fixed full free subgroup of \( A \). For a set \( W \) of primes, consider the class \( S(W) = \{ A \in S \mid S(A) \subseteq W \subseteq D(A) \} \), which is closed with respect to direct summands, finite direct sums, finitely \( S(W) \)-generated subgroups, and quasi-isomorphisms. Observe that, modulo finite direct summands, \( S(\emptyset) \) is the class of all finite rank torsion free groups, and \( S(\mathbb{P}) \) is the class of quotient divisible groups. Arguing as in the proof of Theorem 2.8, we obtain

**Corollary 2.11.** The following are equivalent for groups \( A \) and \( B \) in \( S(W) \):

a) \( A \sim B \).

b) \( r_0(\text{Hom}(A, C)) = r_0(\text{Hom}(B, C)) \) for all groups \( C \in S(W) \) with \( r_0(C) \leq \max(r_0(A), r_0(B)) \). \( \square \)

3. **Self-small groups Determined by Their Right Orthogonal Classes**

Let \( W \) be a set of primes, and consider \( A \in S(W) \). Suppose that \( C \in S(W) \) is a subgroup of \( A \). The symbol \( C^i \) denotes the subgroup of \( A \) containing \( C \) such that \( C^i/C \) is the sum of the torsion part of the maximal divisible subgroup of \( A/C \) and the \( p \)-components of \( A/C \) for all primes \( p \) not in \( S(A) \).

**Lemma 3.1.** With the above notations, \( C^i \) and \( A/C^i \) are in \( S(W) \).

**Proof.** It is obvious that \( S(A/C^i) \subseteq S(A) \). Let \( F_1 \) be a full free subgroup of \( C \). If \( F_2 \) is a free subgroup of \( A \) such that \( F_2 = F_1 \oplus F_2 \) is a full free subgroup of \( A \), then \( (F_2 + C^i)/C^i \) is a full free subgroup of \( A/C^i \). Hence, \( [A/C^i]/[(F_2 + C^i)/C^i] \] is isomorphic to \( A/F \), and \( A/C^i \in S(W) \).

Moreover, the middle term in the exact sequence \( 0 \to C/F_1 \to C^i/F_1 \to C^i/C \to 0 \) is torsion and divisible for almost all \( p \in W \) since the groups \( C/F_1 \) and \( C^i/C \) are \( p \)-divisible groups for almost all \( p \in W \). Therefore \( C^i \in S(W) \). \( \square \)
If \( A, X \in S(W) \) then \( A \oplus X \in S(W) \), hence \( T(\Hom(A, X)) = \Hom(A, T(X)) \). Therefore every homomorphism \( A \to T(X) \) has finite image. Let
\[
A^\perp = \{ X \in S(W) \mid \mathbb{Q}\Hom(A, X) = 0 \} = \{ X \in S(W) \mid \Hom(A, X) \text{ is torsion} \}.
\]
The next results outlines the basic properties of the classes \( A^\perp \).

**Lemma 3.2.** Let \( W \) be a set of primes.

a) If \( A, B \in S(W) \), then \( B \in A^\perp \) if and only if the image \( S_A(B) \) is torsion.

b) If \( A_1, \ldots, A_n \in S(W) \) then \( (\bigoplus_{i=1}^n A_i)^\perp = \bigcap_{i=1}^n A_i^\perp \).

c) \( A^\perp = (A \oplus \mathbb{Q})^\perp \) for all \( A \in S(W) \).

d) If \( A, B \in S(W) \) such that there exists a homomorphism \( \alpha : A \to B \) with \( B/\alpha(A) \) a torsion group, then \( A^\perp \subseteq B^\perp \).

e) If \( A \) and \( B \) are quasi-isomorphic groups in \( S(W) \) then \( A^\perp = B^\perp \). \( \square \)

The next result is based on ideas from [13, Section 4] and from [10, Proposition 5.6].

**Proposition 3.3.** If \( A \in S(W) \) then there exists a group \( C \in S(W) \) such that:

i) \( C \) is an epimorphic image of \( A \).

ii) \( C \) has no non-zero nilpotent quasi-endomorphisms, and

iii) \( A^\perp = C^\perp \).

**Proof.** If there exists an endomorphism \( \alpha \) of \( A \) such that \( \alpha(A) \) is not torsion, but \( \alpha^2 = 0 \), then \( \text{Im}(\alpha) \subseteq \text{Ker}(\alpha) \) and \( A/\text{Ker}(\alpha) \cong \text{Im}(\alpha) \) are in \( S(W) \). Moreover, \( [A/\text{Im}(\alpha)]/[\text{Ker}(\alpha)/\text{Im}(\alpha)] \cong A/\text{Ker}(\alpha) \in S(W) \); and the support of the group \( [A/\text{Im}(\alpha)]/[\text{Ker}(\alpha)/\text{Im}(\alpha)] \) is contained in \( S(A) \). Consequently, \( \text{Ker}(\alpha)/\text{Im}(\alpha) \) contains the torsion part of the maximal divisible subgroup of \( A/\text{Im}(\alpha) \) and the \( p \)-components of \( A/\text{Im}(\alpha) \) for all primes \( p \) not in \( S(A) \). Moreover, \( \text{Im}(\alpha)^\perp \subseteq \text{Ker}(\alpha) \). Let \( A' = A/\text{Im}(\alpha)^\perp \) and \( \pi : A \to A' \) be the canonical epimorphism.

Consider a group \( X \in S(W) \). If we can find a homomorphism \( f : A \to X \) such that \( \text{Im}(f) \) is not torsion, then \( A/\text{Ker}(f) \cong \text{Im}(f) \in S(W) \). The canonical epimorphism \( \rho : A \to A/\text{Ker}(f) \) satisfies \( \text{Im}(\rho a) = [\text{Im}(\alpha) + \text{Ker}(f)]/\text{Ker}(f) \in S(W) \). If the latter is torsion, then it has to be finite. Choose a non-zero integer \( k \) with \( \text{Im}(ka) \subseteq \text{Ker}(f) \). We observe that the torsion part of \( A/\text{Ker}(f) \) is reduced and \( S(A/\text{Ker}(f)) \subseteq S(A) \). Then, \( \text{Im}(ka)^\perp \subseteq \text{Ker}(f) \), and \( f \) induces a homomorphism \( \overline{f} : A/\text{Im}(ka)^\perp \to X \) defined by \( \overline{f}([a + \text{Im}(ka)^\perp]) = f(a) \), whose image is not torsion. On the other hand, if \( (\text{Im}(\alpha) + \text{Ker}(f))/\text{Ker}(f) \) is not torsion, then consider the homomorphism \( \overline{\alpha} : A/\text{Im}(\alpha)^\perp \to \text{Ker}(A) \) defined by \( \overline{\alpha}(a + \text{Im}(\alpha)^\perp) = \alpha(a) \). Let \( \iota : \text{Ker}(\alpha) \to B' \) be the inclusion map; and consider \( f\overline{\alpha} : A/\text{Im}(\alpha)^\perp \to X \). Choose \( a' = a + \text{Im}(\alpha)^\perp \in A/\text{Im}(\alpha)^\perp \) in such a way that \( f\overline{\alpha}(ka') = 0 \) for some non-zero integer \( k \). Then, \( \text{Ker}(\alpha) \subseteq \text{Ker}(f) \). Hence, there exists \( x \in A/\text{Im}(\alpha)^\perp \) such that \( f\overline{\alpha}(x) \) has infinite order.

Consequently, \( \text{Hom}(A/\text{Im}(\alpha)^\perp, X) \) is not torsion; and \( (A/\text{Im}(\alpha)^\perp)^\perp \subseteq A^\perp \). Since the other inclusion is obvious, \( (A/\text{Im}(\alpha)^\perp)^\perp = A^\perp \) and \( r_0(A/\text{Im}(\alpha)^\perp) < r_0(A) \). If \( A' = A/\text{Im}(\alpha)^\perp \) has non-zero nilpotent quasi-endomorphisms, we can repeat this procedure to find a group \( A'' \in S(W) \) with \( A''^\perp = A^\perp \) and \( r_0(A'') < r_0(A') < r_0(A) \). This process has to stop after a finite number of steps, since \( r_0(A) \) is finite. Hence, there exists a group \( C \in S(W) \) which is an epimorphic image of \( A \) such that \( C^\perp = A^\perp \) and \( N(E(C)) = 0 \). This \( C \) satisfies iii) as a consequence of Lemma 2.6. \( \square \)
Theorem 3.4. The following are equivalent for a group $A \in S(W)$:

a) $A \sim B$ whenever $B \in S(W)$ with $A^\perp = B^\perp$;

b) $A \sim (\oplus_{i=1}^n A_i) \oplus \mathbb{Q}^n$, where $A_i$ are essentially strongly indecomposable groups such that

i) $\mathbb{Q}E(A_i)$ is a division ring for all $i = 1, \ldots, n$,

ii) $\text{Hom}(A_i, A_j)$ is torsion for all $i \neq j$,

iii) If a group $C \in S(W)$ has torsion-free rank at most $r$ and is an epimorphic image of some $A_i$ for some $i$, then $C$ is a sum of a (torsion-free) divisible group and a finite group.

Proof. a) $\Rightarrow$ b) : We may assume that $A = A' \oplus \mathbb{Q}^r$, where $A'$ is reduced and $A' = \oplus_{i=1}^n A_i$. If $\mathbb{Q}E(A')$ has a non-zero nilpotent element then, as in the proof of Proposition 3.3, there exists $C \in S(W)$ such that $A'^\perp = C^\perp$ and $s = r_0(A') - r_0(C) > 0$. Then $A'^\perp = (C \oplus \mathbb{Q}^{r+s})^\perp$, a contradiction. By Lemma 2.6, $A$ satisfies conditions i) and ii). Suppose that $A_i/N = C \in S(W)$ and $r_0(C) \leq r$. Then $A'^\perp = A + C + \mathbb{Q}^{r_0(C)}$. Since $C \sim \mathbb{Q}^{r_0(C)}$ because of a), we obtain that iii) holds too.

b) $\Rightarrow$ a) : We consider direct decompositions $A = A' \oplus \mathbb{Q}^r$ and $B = B' \oplus \mathbb{Q}^s$ with $A'$ and $B'$ reduced groups. We may assume $A' = \oplus_{i=1}^n A_i$, where the $A_i$'s are essentially strongly indecomposable groups which verify i), ii) and iii). Then, $A'^\perp = A + C + \mathbb{Q}^{r_0(C)}$. Moreover, there exists a group $C \in S(W)$ as in Proposition 3.3 such that $B'^\perp = C^\perp$.

As a consequence of Lemma 2.6, we may assume that $C = \oplus_{i=1}^s C_i$, where the rings $\mathbb{Q}E(C_i)$ are division rings and $\text{Hom}(C_i, C_j)$ is torsion for all $i \neq j$. Let $j \in \{1, \ldots, s\}$. Since $\text{Hom}(C_i, C_j)$ is not torsion, there exists $f_j : A' \to C_j$ such that $\text{Im}(f_j)$ is not torsion. Because $\text{Im}(f_j) \in S(W)$, we can find a homomorphism $g : C \to \text{Im}(f_j)$ whose image is not torsion. Since $\text{Im}(f_j) \subseteq C_j$ and $g(\oplus_{i \neq j} C_i)$ is a finite group, $g(C_j)$ is not torsion either; and the restriction of $g$ to $C_j$ represents a non-zero quasi-endomorphism of $C_j$. Therefore, it has to be a quasi-epimorphism; and $\text{Im}(g)$ has finite index in $C_j$. Since the same holds for $\text{Im}(f_j)$, we obtain $S_{C_j}(C) \sim C$. In the same way, $S_C(A') \sim A'$. By Lemma 2.7, we may assume $A_1 \sim C_1$.

If $j \in \{2, \ldots, s\}$, then $f_j(A_1)$ is a finite group. Hence, the restriction $f_j|_{\oplus_{i=2}^n A_i} : \oplus_{i=2}^n A_i \to C_j$ is a quasi-epimorphism. Then, $S_{\oplus_{i=2}^n A_i}(\oplus_{i=2}^n C_j) \sim \oplus_{i=2}^n C_j$, and in the same way we obtain $S_{\oplus_{i=2}^n C_j}(\oplus_{i=2}^n A_i) \sim \oplus_{i=2}^n A_i$. Lemma 2.7 once more allows us to assume $A_2 \sim C_2$. We repeat these arguments for all $j$, and obtain $A' \sim C$ after a finite number of steps.

Suppose that $B'$ has an endomorphism $\alpha$ such that $\alpha \neq \alpha^2 = 0$ in $\mathbb{Q}E(B')$. In view of the proof of Proposition 3.3, we may assume that $C$ is an epimorphic image of $B'/\text{Im}(\alpha)^\perp$. Then $r_0(\text{Im}(\alpha)^\perp) \leq r_0(B') - r_0(C) \leq r_0(A) - r_0(C) = r$. Since we can view $\alpha$ as a map from $B'$ to $\text{Im}(\alpha)^\perp$, there exists a homomorphism $\beta : A' \to \text{Im}(\alpha)^\perp$ which has a non-torsion image. Using iii), $\text{Im}(\beta) \leq B'$ is divisible. But this is not possible since $B'$ is reduced. Therefore $\text{N}_{B'}(E(B')/T(E(B'))) = 0$. Consequently, $B' \sim C \sim A'$, and $B \sim A$. \( \square \)

References


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