[1]

Conditions under which a lattice is isomorphic to the subgroup lattice of an abelian group

Carolina Conţiu

Abstract. In this paper, we provide necessary and sufficient conditions under which a lattice is isomorphic to the subgroup lattice of an arbitrary abelian group. We also give necessary and sufficient conditions for a lattice L to be isomorphic to the normal subgroup lattice of an arbitrary group.

1. Introduction

The problem of finding necessary and sufficient conditions under which a lattice is isomorphic to the subgroup lattice of an arbitrary group was first raised by Suzuki in [**5**]. Yakovlev was the one who offered a complete solution for this problem, in his work [**6**]. He characterized subgroup lattices of free groups and normal subgroups of such groups, as well. The conclusion was a simple consequence of the fact that every group is the epimorphic image of a convenient free group. However, his result does not provide us information about some (basic) properties of the group. In the same manner, Scoppola tried to characterize subgroup lattices of abelian groups in his papers [**3**] and [**4**]. He only partially answered the question, since groups of torsion-free rank 1 remain outside of his solution. We intend to fill this gap and offer a complete solution to this problem.

Using the same techniques as Yakovlev, we obtain necessary and sufficient conditions under which a lattice is isomorphic to the subgroup lattice of an abelian group.

Our notation is mostly standard. We denote the lattice operations by $\wedge$ (the meet) and $\vee$ (the join). Let L be a complete lattice and 0 its least element. As in [**6**] and [**2**], an element $c \in L$ is said to be *cyclic* if the interval $c/0$ is a distributive lattice satisfying the ascending chain condition. The set of cyclic elements of L is denoted by $C(L)$. Respecting the notations from [**2**], if $a, b \in C(L)$, we recall the following two subsets of $C(L)$

$$a \circ b = \{x \in C(L) \mid x \vee a = x \vee b = a \vee b\}, \tag{1}$$

$$b \uparrow a = \{z \in C(L) \mid z \in (a \circ b) \circ a, z \notin (a \circ a) \circ b, z \circ z \subseteq (a \circ (b \circ b)) \circ a\}. \tag{2}$$

For groups, we will use the multiplicative notation. By 1 we will denote the identity of a group, but the trivial subgroup, formed only from the identity, as well. If G is a group, we denote by $L(G)$ its subgroup lattice.

Our first goal is to identify the commutator subgroup in the subgroup lattice of a free group. As in [**2**], we will work in a more general framework: the one of 2-free groups. A group is said to be 2-*free* if is nonabelian and any two of its elements generate a free group. The Nielsen-Schreier Theorem guarantees that every free

[1]The author is supported by the UEFISCSU-CNCSIS grant ID–489

group is 2-free. We recall the following simple properties of the subgroup generated by the two elements in such a group.

LEMMA 1. *Let G be a 2-free group and let a, b be nontrivial elements of G.*

i) *If $\langle a\rangle \cap \langle b\rangle \neq 1$, the rank of $\langle a, b\rangle$ is 1, hence $ab = ba$.*
ii) *If $\langle a\rangle \cap \langle b\rangle = 1$, $\langle a, b\rangle$ is free on $\{a, b\}$.*
iii) *If there exist $i, j \in \mathbb{Z} \setminus \{0\}$ such that $a^i b^j = b^j a^i$, then $ab = ba$.*

When dealing with the subgroup lattice of a 2-free group, the subsets introduced in (1) and (2) are as in the next Lemmas.

LEMMA 2. [**6**, Lemma 1],[**2**, Lemma 7.1.7] *Let G be a 2-free group and $a, b \in G$ such that $a \neq 1 \neq b$ and $\langle a\rangle \cap \langle b\rangle = 1$. Then*

$$\langle a\rangle \circ \langle b\rangle = \{\langle ab\rangle, \langle a^{-1}b\rangle, \langle ab^{-1}\rangle, \langle a^{-1}b^{-1}\rangle\},$$

and these four groups are distinct.

LEMMA 3. [**2**, Lemma 7.1.15] *Let G be a 2-free group and $a, b \in G$ such that $a \neq 1 \neq b$ and $\langle a\rangle \cap \langle b\rangle = 1$. Then*

$$\langle b\rangle \uparrow \langle a\rangle = \{\langle aba^{-1}\rangle, \langle a^{-1}ba\rangle\}.$$

For the sake of completeness, we shall recall the basic notions and state some of the main results of Yakovlev. For more details you can consult [**6**] or [**2**].

Let L be a complete lattice, 0 its least element, $n \in \mathbb{N}$ and $e_i \in C(L)$, $i \in \{1, \ldots, n\}$. For $0 \neq a \in C$, the n-tuple $\alpha = (A_1, \ldots, A_n)$ is said to be an *a-complex with respect to* the system $E = (e_1, \ldots, e_n)$ if for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$ $A_i \subseteq e_i \circ a$, $|A_i| = 2$ and $a_i \circ a_j \cap e_i \circ e_j \neq \emptyset$, for all $a_i \in A_i$, $a_j \in A_j$.

The set of all a-complexes with respect to E is denoted by $K(a, E)$, while the set of all complexes with respect to E is denoted by $K(E)$. The 0-complex with respect to E, $\varepsilon = (\{e_1\}, \ldots, \{e_n\})$, is called the *trivial complex.*

Multiplication of complexes can be introduced. Let L, $C(L)$, n, E, $K(E)$ be as above and let $\alpha = (A_1, \ldots, A_n)$ and $\beta = (B_1, \ldots, B_n)$ be complexes in $K(E)$. The *product* $\alpha\beta$ of α and β is the set of all complexes $\delta = (D_1, \ldots, D_n) \in K(E)$ for which there exist $a, b, d \in C$ such that $d \in a \circ b$, $\alpha \in K(a, E)$, $\beta \in K(b, E)$, $\delta \in K(d, E)$ and $D_i \circ B_j \cap A_i \circ e_j \neq \emptyset$, for all $i, j, \in \{1, \ldots, n\}$.

The theorem above provides sufficient conditions for a lattice to be isomorphic to the subgroup lattice of a group.

THEOREM 4. [**6**, Theorem 1], [**2**, Theorem 7.1.6] *Let L be a complete lattice in which every element is the join of cyclic elements and suppose there exists a system $E = (e_1, \ldots, e_n)$ of elements $e_i \in C(L)$ with the following properties:*

a) *For each $a \in C \setminus \{0\}$, $|K(a, E)| = 2$.*
b) *If $a \in C$, $\alpha = (A_1, \ldots, A_n)$, $\alpha' = (A'_1, \ldots, A'_n) \in K(a, E)$, $\alpha \neq \alpha'$, then*

$$e_i \circ A'_j \cap A_i \circ e_j \neq \emptyset, \text{ for all } i, j \in \{1, \ldots, n\}.$$

c) *If $a, b \in C$, $\alpha \in K(a, E)$ and $\beta \in K(b, E)$ such that $\alpha = \beta$, then $a = b$.*
d) *For all $\alpha, \beta \in K(E)$, the product $\alpha\beta$ consists of a unique complex $\alpha * \beta$.*
e) *For all $\alpha, \beta, \gamma \in K(E)$, $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.*
f) *Let $a \in C$ and $X \subseteq C$ such that $a \leq \bigvee X$ and let $\alpha \in K(a, E)$. Then there exist finitely many elements $b_i \in X$ and $\beta_i \in K_i(b_i, E)$ such that $\alpha \in ((\ldots(\beta_1\beta_2)\beta_3\ldots)\beta_{m-1})\beta_m$.*

In these conditions $G = K(E)$ with operation $* : G \times G \to G$ given by d),

$$(\alpha, \beta) \mapsto \alpha * \beta, \ \alpha, \beta \in G,$$

is a group whose subgroup lattice is isomorphic to L.

For the subgroup lattice of a 2-free group the conditions from Theorem 4 are necessary, when additional conditions are imposed to the system E. Under these circumstances, the basic systems were introduced.

If L is a complete lattice, a *basic system of* L is a family

$$E = (e_{11}, \dots, e_{1m}, e_{21}, \dots, e_{2m}, \dots, e_{n1}, \dots, e_{nm}),$$

of elements $e_{ij} \in C(L) \setminus \{0\}$ with $n \geq 5$, $m \geq 259$, satisfying

(3) $e_{ij} \neq e_{kl}$, for all $i, k \in \{1, \dots, n\}$ and $j, l \in \{1, \dots, m\}$, when $(i, j) \neq (k, l)$,

(4) there exist $e_1, \dots, e_n \in C(L)$ such that $e_i \wedge e_k = 0$ when $i \neq k$ and $e_{ij} \leq e_i$ for all $i \in \{1, \dots, n\}$ şi $j \in \{1, \dots, m\}$.

In particular, subgroup lattices of free groups (of rank at least 2) will satisfy assumptions of Theorem 4, for an appropriate basic system, E.

THEOREM 5. [**6**, Theorem 5], [**2**, Theorem 7.1.13] *Let $r \geq 2$ be a cardinal number. The lattice L is isomorphic to the subgroup lattice of a free group of rank r if and only if L is complete, any of its elements is the join of cyclic elements, and L has the following properties:*

a) *For each $c \in C(L) \setminus \{0\}$, the interval $c/0$ is infinite.*
b) *If $a, b \in C(L)$ such that $a \vee b \notin C(L)$ and if $d \in a \circ b$, then $d \wedge a = d \wedge b = 0$.*
c) *There exists a basic system E of L and a subset S of $C(L)$ such that $|S| = r$, $\bigvee S = \bigvee L$ and for every finite sequence $b_1, \dots, b_s$, where $b_i \in S$, with $b_i \neq b_{i+1}$ $(i = 1, \dots, s-1)$ and $a_i \in L$ with $0 \neq a_i \leq b_i$ and $\alpha_i \in K(a_i, E)$, the trivial complex ε is not contained in $(\dots((\alpha_1 \alpha_2)\alpha_3)\dots)\alpha_s$.*

where the basic system, E, satisfies a)-f) from 4.

2. The commutator subgroup

In order to obtain the characterization of the commutator subgroup of a 2-free group, we describe the commutator of two nontrivial elements. Note that if G is a group we will denote by G' its commutator subgroup, while if $a, b \in G$, we will write $[a, b]$ for their commutator. In this purpose, we introduce the following subset of the set of all cyclic elements of a complete lattice.

DEFINITION 6. *Let L be a complete lattice. If $x, y \in C(L)$, we define*

$$y \updownarrow x = \{z \in C(L) \,|\, z \in (y \uparrow x) \circ y \text{ and } \exists t_1, t_2 \in C(L), t_1 \neq t_2, \text{ such that } t_1, t_2 \in x \circ y, \\ z \in t_1 \circ t_2, x \circ x \cap t_1 \circ t_2 = \emptyset\}.$$

LEMMA 7. *If G is a 2-free group and $a, b \in G$ such that $a \neq 1 \neq b$ and $\langle a \rangle \cap \langle b \rangle = 1$, then*

$$\langle b \rangle \updownarrow \langle a \rangle = \{\langle [a, b] \rangle, \langle [a^{-1}, b] \rangle, \langle [a, b^{-1}] \rangle, \langle [a^{-1}, b^{-1}] \rangle\}.$$

Proof. Note first that $F = \langle a, b\rangle$ is free on $\{a, b\}$. By the previous remark, we have

$$(\langle b\rangle \uparrow \langle a\rangle) \circ \langle b\rangle = \{\langle aba^{-1}\rangle, \langle a^{-1}ba\rangle\} \circ \langle b\rangle = \langle aba^{-1}\rangle \circ \langle b\rangle \cup \langle a^{-1}ba\rangle \circ \langle b\rangle.$$

Since $b \neq 1$, it follows $aba^{-1} \neq 1 \neq a^{-1}ba$. We also have $\langle aba^{-1}\rangle \cap \langle b\rangle = 1 = \langle a^{-1}ba\rangle \cap \langle b\rangle$. Indeed, if $\langle aba^{-1}\rangle \cap \langle b\rangle \neq 1$, by i) from Lemma 1, it would follow $aba^{-1}b = baba^{-1}$. This identity cannot hold in the free group $F = \langle a, b\rangle$. Therefore, by Lemma 2, we obtain

$$\langle aba^{-1}\rangle \circ \langle b\rangle = \{\langle aba^{-1}b\rangle, \langle aba^{-1}b^{-1}\rangle, \langle ab^{-1}a^{-1}b\rangle, \langle ab^{-1}a^{-1}b^{-1}\rangle\}, \tag{5}$$

$$\langle a^{-1}ba\rangle \circ \langle b\rangle = \{\langle a^{-1}bab\rangle, \langle a^{-1}bab^{-1}\rangle, \langle a^{-1}b^{-1}ab\rangle, \langle a^{-1}b^{-1}ab^{-1}\rangle\} \tag{6}$$

and these eight groups are distinct.

Let t_1, $t_2 \in G$ such that $\langle t_1\rangle \neq \langle t_2\rangle$ and $\langle t_1\rangle, \langle t_2\rangle \in \langle a\rangle \circ \langle b\rangle$. Then $\langle t_1\rangle = \langle a^{\mu_1}b^{\nu_1}\rangle$ and $\langle t_2\rangle = \langle a^{\mu_2}b^{\nu_2}\rangle$, where $\mu_i, \nu_i \in I = \{+1, -1\}$, for $i \in \{1, 2\}$ and $(\mu_1, \nu_1) \neq (\mu_2, \nu_2)$. We have $t_1 = (a^{\mu_1}b^{\nu_1})^{\varepsilon_1}$, while $t_2 = (a^{\mu_2}b^{\nu_2})^{\varepsilon_2}$, where $\varepsilon_1, \varepsilon_2 \in \{+1, -1\}$. We assume $\varepsilon_1 = \varepsilon_2 = 1$. The other three cases are very similar. Note that $b^{-1} \neq a \neq b$ and hence $t_1 \neq 1 \neq t_2$.

In order to prove that $\langle t_1\rangle \cap \langle t_2\rangle = 1$, we consider first case in which $\mu_1 = \mu_2$ or $\nu_1 = \nu_2$. Assume that $\mu_1 = \mu_2$. We must have $\nu_1 \neq \nu_2$. Indeed, if $\langle t_1\rangle \cap \langle t_2\rangle \neq 1$, then $t_1t_2 = t_2t_1$. We would have $a^{\mu_1}b^{\nu_1}a^{\mu_2}b^{\nu_2} = a^{\mu_2}b^{\nu_2}a^{\mu_1}b^{\nu_1}$, hence $b^{\nu_1}a^{\mu_1}b^{-\nu_1} = b^{-\nu_1}a^{\mu_1}b^{\nu_1}$. This means $b^{2\nu_1}a^{\mu_1} = a^{\mu_1}b^{2\nu_1}$. Since G is a 2-free group, by iii) from Lemma 1, we have $ab = ba$, contradicting the choice of a and b.

Let us investigate the case when $\mu_1 \neq \mu_2$ and $\nu_1 \neq \nu_2$. Suppose first $\langle t_1\rangle \cap \langle t_2\rangle \neq 1$. Then, we would get $a^{\mu_1}b^{\nu_1}a^{-\mu_1}b^{-\nu_1} = a^{-\mu_1}b^{-\nu_1}a^{\mu_1}b^{\nu_1}$. Such identity cannot hold in the free group F.

In conclusion, we have $t_1 \neq 1 \neq t_2$, $\langle t_1\rangle \cap \langle t_2\rangle = 1$ and by Lemma 2, we have

$$\langle t_1\rangle \circ \langle t_2\rangle = \{\langle (a^{\mu_1}b^{\nu_1})^{\varepsilon_1}(a^{\mu_2}b^{\nu_2})^{\varepsilon_2}\rangle \mid \varepsilon_i \in I = \{+1, -1\}, i = 1, 2\}.$$

Since each subgroup $H \in (\langle b\rangle \uparrow \langle a\rangle) \circ \langle b\rangle$ has the form $\langle a^{\gamma}b^{\delta_1}a^{-\gamma}b^{\delta_2}\rangle$, where $\gamma, \delta_1, \delta_2 \in \{+1, -1\}$, we can find exactly two subgroups $\langle t_1\rangle$, $\langle t_2\rangle \in \langle a\rangle \circ \langle b\rangle$ such that $H \in \langle t_1\rangle \circ \langle t_2\rangle$, namely $\langle t_1\rangle = \langle a^{\gamma}b^{\delta_1}\rangle$ and $\langle t_2\rangle = \langle a^{-\gamma}b^{\delta_2}\rangle$. Note that $\langle t_1\rangle$, $\langle t_2\rangle$ are distinct. We are interested in the case when $\langle t_1\rangle$, $\langle t_2\rangle \in \langle a\rangle \circ \langle b\rangle$ distinct and verifying $\langle a\rangle \circ \langle a\rangle \cap \langle t_1\rangle \circ \langle t_2\rangle = \emptyset$. Notice first that $\langle a\rangle \circ \langle a\rangle$ represents the interval $\langle a\rangle/0$ in $L(G)$. If $\langle t_1\rangle$, $\langle t_2\rangle$ are as above, we shall prove that the last intersection is empty if and only if $\nu_1 \neq \nu_2$. If $\nu_1 = \nu_2$, $\langle a^{\mu_1-\mu_2}\rangle = \langle a^{\mu_1}b^{\nu_1}b^{-\nu_1}a^{-\mu_2}\rangle \in \langle a^{\mu_1}b^{\nu_1}\rangle \circ \langle a^{\mu_2}b^{\nu_1}\rangle = \langle t_1\rangle \circ \langle t_2\rangle$. On the other hand, if $\langle a^{\mu_1}b^{\nu_1}\rangle \circ \langle a^{\mu_2}b^{\nu_2}\rangle \cap \langle a\rangle \circ \langle a\rangle \neq \emptyset$, at least one of the subgroups $\langle a^{\mu_1}b^{\nu_1}a^{\mu_2}b^{\nu_2}\rangle$, $\langle b^{-\nu_1}a^{\mu_2-\mu_1}b^{\nu_2}\rangle$, $\langle a^{\mu_1}b^{\nu_1-\nu_2}a^{-\mu_2}\rangle$, $\langle b^{-\nu_1}a^{-\mu_1}b^{-\nu_2}a^{-\mu_2}\rangle$ lies in $\langle a\rangle \circ \langle a\rangle$ and hence its generators must be contained in $\langle a\rangle$. In all cases, we must have $\nu_1 = \nu_2$.

We conclude now that the elements of $\langle b\rangle \updownarrow \langle a\rangle$ are $\langle a^{\gamma}b^{\delta}a^{-\gamma}b^{-\delta}\rangle$, where $\gamma, \delta \in \{+1, -1\}$. $\square$

Lemma 8. *Let G be a 2-free group and let $H \leq G$. Then H contains the commutator subgroup of G, if and only if, $\langle b\rangle \updownarrow \langle a\rangle \subseteq H/1$ for all $a, b \in G$, such that $a \neq 1 \neq b$ and $\langle a\rangle \cap \langle b\rangle = 1$.*

Proof. If $G' \leq H$ and $a, b \in G$ such that $a \neq 1 \neq b$ and $\langle a\rangle \cap \langle b\rangle = 1$, then by Lemma 7, $\langle b\rangle \updownarrow \langle a\rangle \subseteq H/1$.

Conversely, suppose this condition holds and let $a, b \in G$. If $\langle a, b\rangle$ is abelian, then it is obvious that $[a, b] = 1 \in H$. If $\langle a, b\rangle$ is not abelian, then $\langle a, b\rangle$ is free on $\{a, b\}$. Moreover, $a \neq 1 \neq b$ and $\langle a\rangle \cap \langle b\rangle = 1$. By hypothesis and Lemma 7, $\langle [a, b]\rangle \in \langle b\rangle \updownarrow \langle a\rangle \subseteq H/1$. Therefore, $\langle [a, b]\rangle \in H/1$ and so $[a, b] \in H$. We proved that for any $a, b \in G$, $[a, b] \in H$, hence $G' \leq H$. □

Lemma 9. *Let G be a 2-free group and let $H \leq G$. Then H is the commutator subgroup of G if and only if $H = \bigvee(\bigcup_{a,b\in G, a\neq 1\neq b, \langle a\rangle\cap\langle b\rangle=1} \langle b\rangle \updownarrow \langle a\rangle)$.*

Proof. It is obvious that a subgroup H is equal to the commutator subgroup of G if and only if $H = \bigvee\{\langle [a, b]\rangle \mid a, b \in G\}$ in $L(G)$. Let $a, b \in G$. Since G is 2-free, the identity $[a, b] = 1$ can hold if and only if $a = 1$ or $b = 1$ or $\langle a\rangle \cap \langle b\rangle \neq 1$. The conclusion is a simple consequence of Lemma 7. □

3. Conditions for the subgroup lattices of abelian groups

Once the commutator subgroup was identified in the subgroup lattice of a free group, we can formulate the characterization of subgroup lattices of free abelian groups.

Theorem 10. *Let $r \geq 2$ be a cardinal number. The lattice L is isomorphic to the subgroup lattice of a free abelian group of rank r if and only if there exist a lattice L^* and an element $d \in L^*$ with the following properties:*

a) *L^* is a complete lattice in which every element is the join of cyclic elements. Furthermore, L^* satisfies a)-c) from Theorem 5, for the cardinal number r, where the basic system E satisfies in addition a)-f) from Theorem 4.*
b) *$d = \bigvee(\bigcup_{a,b\in C(L^*)\setminus\{0\}, a\wedge b=0} b \updownarrow a)$.*
c) *$L \cong 1^*/d$, where 1^* is the greatest element of L^*.*

Proof. Suppose $L \cong L(G)$, where G is a free abelian group of rank r. It is well known that G is isomorphic with F/F', where F is the free group of rank r, $r \geq 2$ and F' its commutator subgroup. Thus, if $L^* = L(F)$, by Theorem 5, condition a) is satisfied. Moreover, taking $d = F'$, the commutator subgroup of F, by Lemma 9, condition b) holds, while the isomorphism $L \cong 1^*/d$ is obvious.

Conversely, suppose there exists L^* and $d \in L^*$ satisfying conditions a)-c). By a) and Theorem 5, there exists a free group of rank r such that $L(F) \cong L^*$. Let N be the image of d under this isomorphism. Moreover, by b) and Lemma 9 we conclude that N is the commutator subgroup of F. Finally, c) implies $L \cong L(F/N)$, where F/N is the free abelian group of rank r. □

Remark 11. The Theorem above characterizes the subgroup lattice of a free abelian group of rank $r \geq 2$, finite or infinite. The subgroup lattice of the free abelian group of rank 1 is well known. This is the T_∞ lattice, with the set of natural numbers as underlying set, ordered by

$$a \leq' b \Leftrightarrow b \text{ divides } a.$$

Of course, if $a, b \in T_\infty$, $a \vee b$ is the greatest common divisor and $a \wedge b$ is the least common multiple of a and b.

We are now able to give the desired characterization.

Theorem 12. *The lattice L is isomorphic to the subgroup lattice of some abelian group if and only if L is isomorphic to a principal filter of the T_∞ lattice or there exists a lattice L^* and two elements $d, e \in L^*$ such that:*

a) *L^* and $d \in L^*$ satisfies a),b) from Theorem 10.*
b) *$e \in 1^*/d$, where 1^* is the greatest element of L^* and $L \cong 1^*/e$.*

Proof. Suppose first $L \cong L(G)$, where G is an abelian group. Then there exists H a free abelian group and N a subgroup of H such that $G \cong H/N$. If H is of rank 1, then $L(G)$ is isomorphic with the principal filter generated by N in T_∞. Otherwise if the rank of H is at least 2, from Theorem 10 we deduce the existence of a lattice L^* and an element $d \in L^*$ such that condition a) from our Theorem is satisfied. Moreover, we have $L(H) \cong 1^*/d$. If e is the image of N under this isomorphism, $e \in 1^*/d$ and one can easily observe that $L(H/N) \cong 1^*/e$.

Conversely, if $1/N$ is a principal filter of T_∞, than it is obvious that $1/N \cong L(\mathbb{Z}/N)$, which is of course an abelian group.

Suppose now there exists L^* and $d, e \in L^*$ satisfying conditions a) and b). By a) and Theorem 10, we conclude that the interval $1^*/d$ of L^* is isomorphic with the subgroup lattice of a free abelian group of rank $r \geq 2$. Let H be this group. We have $L(H) \cong 1^*/d$. Let N be the image of $e \in 1^*/d$ under this isomorphism. From c) it follows that $L \cong L(H/N)$.

$\square$

Another direct consequence of Yakovlev's Theorem is the characterization of the normal subgroup lattice of a group. If G is a group, we shall denote by $\mathrm{Norm}(G)$ its normal subgroup lattice.

Let L be a complete lattice. We say that an element $d \in L$ is *normal* in L and write $d \trianglelefteq L$, every time $b \uparrow a \subseteq d/0$ holds for each $a, b \in C(L) \setminus \{0\}$ such that $a \wedge b = 0$ and $b \leq d$. In [**6**] the author proved that the normal elements in the subgroup lattice of a 2-free group are exactly the normal subgroups of that group.

Theorem 13. *The lattice L is isomorphic to the normal subgroup lattice of a group if and only if there is a lattice L^* and an element $d \in L^*$ such that:*

a) *L^* is a complete lattice in which every element is the join of cyclic elements. Furthermore, L^* satisfies a)-c) from Theorem 5, for some cardinal number $r \geq 2$ where the basic system E satisfies in addition a)-f) from Theorem 4.*
b) *$d \trianglelefteq L^*$.*
c) *$\{d' \in L^* \mid d' \trianglelefteq L^*, d \leq d'\}$ is a complete sublattice of $1^*/d$, isomorphic with L.*

Proof. Assume first that there exists a group G such that $L \cong \mathrm{Norm}(G)$. Hence, there exists a cardinal number $r \geq 2$, a free group F of rank r and a normal subgroup N of F such that $G \cong F/N$. If tacking $L^* = L(F)$ and $d = N$, it is obvious that $L(G) \cong 1^*/d$. Furthermore, conditions a) and b) are satisfied. On the other hand $\{d' \in L^* \mid d' \trianglelefteq L^*, d \leq d'\} = \{H \leq F \mid H \trianglelefteq F, N \subseteq H\}$ forms a complete sublattice of $1^*/d$ and by the correspondence theorem it is isomorphic with $\{H/N \mid H/N \trianglelefteq F/N\} = \mathrm{Norm}(F/N) \cong \mathrm{Norm}(G)$.

Conversely, suppose there exists a lattice L^* and an element $d \in L^*$ satisfying conditions a)-c). By a) and Theorem 5, there exists a free group F of rank r, $r \geq 2$, such that $L^* \cong L(F)$. Let N be the image of d under this isomorphism and hence $N \trianglelefteq F$, by b). But then, $1^*/d \cong L(G)$, where $G = F/N$. On the other hand, by the correspondence theorem, $\{d' \in L^* \mid d' \trianglelefteq L^*, d \leq d'\} \cong \{H \leq F \mid H \trianglelefteq F, N \subseteq H\}$ is isomorphic with $\mathrm{Norm}(F/N) = \mathrm{Norm}(G)$. Finally, $L \cong \mathrm{Norm}(G)$, by c). $\square$

References

[1] Crawley, P., Dilworth, R.P. *Algebraic Theory of Lattices, Prentice-Hall*, Englewood Cliffs, N.J., 1973.

[2] Schmidt, R. *Subgroup lattices of groups, de Gruyter Expositions in Mathematics*, 14, de Gruyter, Berlin, 1994.

[3] Scoppola, Carlo Maria *On the lattice of subgroups of a torsion-free abelian group of rank different from* 1*: a lattice characterization.* (Italian) Rend. Sem. Mat. Univ. Padova 65 (1981), 205-221 (1982).

[4] Scoppola, Carlo Maria *A lattice-theoretic characterization of the lattice of subgroups of an abelian group containing two independent aperiodic elements.* (Italian) Rend. Sem. Mat. Univ. Padova 73 (1985), 191-207.

[5] Suzuki, M. *Structure of a Group and the Structure of its Lattice of Subgroups*, Springer Verlag, 1956.

[6] Yakovlev, B. V. *Conditions under which a lattice is isomorphic to the lattice of subgroups of a group.* (Russian) Algebra i Logika 13 (1974), no. 6, 694-712, 720.

"Babeş-Bolyai" University,
Faculty of Mathematics and Computer Science,
Str. Mihail Kogălniceanu 1,
400084 Cluj-Napoca, Romania
bodo@math.ubbcluj.ro