MODULES DETERMINED BY THEIR ANNIHILATOR CLASSES

SIMION BREAZ AND JAN TRLIFAJ

Abstract. We present a classification of those finite length modules X over a ring A which are isomorphic to every module Y of the same length such that $\mathrm{Ker}(\mathrm{Hom}_A(-,X)) = \mathrm{Ker}(\mathrm{Hom}_A(-,Y))$, i.e. X is determined by its length and the torsion pair cogenerated by X. We also prove the dual result using the torsion pair generated by X. For A right hereditary, we prove an analogous classification using the cotorsion pair generated by X, but show that the dual result is not provable in ZFC.

1. Introduction

Let A be a ring. For a class of (right A-) modules $\mathcal{C}$, we consider the following *annihilator classes*

$$^\circ\mathcal{C} = \{M \in \text{Mod-}A \mid \mathrm{Hom}_A(M,\mathcal{C}) = 0\}, \quad \mathcal{C}^\circ = \{M \in \text{Mod-}A \mid \mathrm{Hom}_A(\mathcal{C},M) = 0\},$$

and

$$^\perp\mathcal{C} = \{M \in \text{Mod-}A \mid \mathrm{Ext}^1_A(M,\mathcal{C}) = 0\}, \quad \mathcal{C}^\perp = \{M \in \text{Mod-}A \mid \mathrm{Ext}^1_A(\mathcal{C},M) = 0\}.$$

The annihilator classes of the form $^\circ\mathcal{C}$ for some $\mathcal{C} \subseteq \text{Mod-}A$ are well-known to coincide with the *torsion classes* of modules, i.e., the classes closed under direct sums, extensions, and homomorphic images. Dually, $\mathcal{C}^\circ$ are the *torsion-free classes*, i.e., the classes closed under direct products, extensions, and submodules, [9, §VI.2].

The annihilator class $^\perp\mathcal{C}$ ($\mathcal{C}^\perp$) is closed under direct summands, extensions, direct sums (direct products), and contain all projective (injective) modules, but it is not characterized by these closure properties in general (see Examples 1 and 2 below). This is the reason why it is hard to compute the annihilator classes of the form $^\perp\mathcal{C}$ and $\mathcal{C}^\perp$ explicitly, and in some cases (e.g., for the class of all Whitehead groups $^\perp\mathbb{Z}$), their structure depends on additional set-theoretic assumptions, cf. [3, Chap.XIII].

In this paper we address the more tractable problem of comparing rather than computing the annihilator classes, and of characterizing modules by their annihilator classes.

Recall that in particular cases, there are close relations among some of the annihilator classes. For example, if $\mathcal{C}$ consists of finitely presented modules of projective dimension ≤ 1, then the classes $\mathcal{C}^\perp$ are exactly the tilting torsion classes of modules, [6, §6.1]. If moreover A is an artin algebra then $\mathcal{C}^\perp$ are exactly the torsion classes closed under direct products, pure submodules, and containing all injective modules, cf. [2, 3.7]. In this case the Auslander-Reiten formula provides a precise

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relation, namely $X^\perp = {}^\circ(\tau X)$ for each finitely presented module X of projective dimension ≤ 1 where τ denotes the Auslander-Reiten translation. Dually, if Y is a finitely presented module of injective dimension ≤ 1, then ${}^\perp Y = (\tau^- Y)^\circ$, see [1, IV.2].

Surprisingly, the conditions ${}^\circ X \subseteq {}^\circ Y$ and $X^\perp \subseteq Y^\perp$ (and the dual ones) are closely related even for general modules. We show this by expressing these conditions in terms of existence of certain chains of submodules. Thus we prove equivalence of the two conditions for certain finite length modules (see Theorem 7 below).

Of course, in general we may have $X \ncong Y$ even if ${}^\circ X = {}^\circ Y$ and $X^\perp = Y^\perp$ (just take $Y = X^2$ where X is any non-zero module of finite length). Moreover, this is possible even if we impose the condition "X and Y have the same length": If X is indecomposable such that $\mathrm{Ext}^1_A(X, X) \neq 0$, there exists $Y \ncong X^2$ such that X embeds in Y and $Y/X \cong X$; it is not hard to see that ${}^\circ X^2 = {}^\circ Y$, and $X^{2\perp} = Y^\perp$ if A is hereditary (see also [8, Example 5.1]).

Developing further some of the ideas from [8] and [12], we characterize in Theorem 21 those modules X of length $\lg(X) < \infty$ which are isomorphic to each finite length module Y such that $\lg(X) = \lg(Y)$ and ${}^\circ X = {}^\circ Y$. The corresponding version for $X^\perp = Y^\perp$ is proved in Theorem 24 assuming that A is a right hereditary ring. The dual of Theorem 24 fails by Example 28. However, Theorem 21 can be dualized; this is proved in Theorem 27.

2. Comparing the annihilator classes

We start by two examples showing that unlike the classes of the form ${}^\circ\mathcal{C}$ and $\mathcal{C}^\circ$, the annihilator classes ${}^\perp\mathcal{C}$ and $\mathcal{C}^\perp$ are not characterized by their basic closure properties in general.

Example 1. This is an example of a class $\mathcal{D}$ of modules closed under direct summands, direct sums, extensions, and containing all projective modules, but such that $\mathcal{D} \neq {}^\perp\mathcal{C}$ for any class of modules $\mathcal{C}$.

We consider the setting of (abelian) groups (= $\mathbb{Z}$-modules), and $\mathcal{D}$ will be the class of all $\aleph_1$-*free* groups (i.e., the groups M such that each countable subgroup of M is free). Clearly $\mathcal{D}$ contains all free groups, and it is closed under direct summands and extensions. The Baer-Specker theorem says that any direct product of copies of $\mathbb{Z}$ is $\aleph_1$–free (cf. [3, IV.2.8]). By [5, Lemma 1.2], if C is a group such that $\mathrm{Ext}^1_{\mathbb{Z}}(P, C) = 0$ for any direct product P of copies of $\mathbb{Z}$, then C is a cotorsion group, so ${}^\perp\{C\}$ contains all torsion-free groups. In particular the group of all rational numbers $\mathbb{Q} \in {}^\perp\{C\}$, but $\mathbb{Q}$ is not $\aleph_1$–free. So there is no class of groups $\mathcal{C}$ such that $\mathcal{D} = {}^\perp\mathcal{C}$.

Example 2. Now we give an example of a class $\mathcal{D}$ of modules closed under direct summands, direct products, extensions, and containing all injective modules, but such that $\mathcal{D} \neq \mathcal{C}^\perp$ for any class of modules $\mathcal{C}$. In this example, we will assume that there are no ω-measurable cardinals (this holds under the Axiom of Constructibility V = L, for example, see [3, VI.3.14]).

We will work in the setting of (right A-) modules where A is a simple non-artinian von Neumann regular ring such that A has countable dimension over its center K. (Note that K is a field by [7, Corollary 1.15]). For a concrete example of such ring, we can take $A = \varinjlim M_{2^n}(K)$, the direct limit of the direct system of full matrix

K-algebras where K is a field and

$$K \overset{f_0}{\hookrightarrow} M_2(K) \overset{f_1}{\hookrightarrow} \dots \overset{f_{n-1}}{\hookrightarrow} M_{2^n}(K) \overset{f_n}{\hookrightarrow} M_{2^{n+1}}(K) \overset{f_{n+1}}{\hookrightarrow} \dots,$$

where $f_n : M_{2^n}(K) \hookrightarrow M_{2^{n+1}}(K)$ is the block-diagonal embedding defined by $f_n(A) = \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$.

$\mathcal{D}$ will be the class of all modules that have no maximal submodules. Clearly $\mathcal{D}$ is closed under direct summands and extensions. Moreover, A is a hereditary ring, and $\mathrm{Ext}^1_A(M, N) \neq 0$ whenever M, N are non-zero finitely generated and M is non-projective by [11, Lemma 3.2 and Proposition 3.3]. Baer's Criterion then yields that $\mathcal{D}$ contains all injective modules.

We claim that $\mathcal{D}$ is closed under direct products. Note that each simple module is slender by [10, Lemma 3.7]. Let κ be a (non-ω-measurable) cardinal and consider a sequence $(D_\alpha \mid \alpha < \kappa)$ of elements of $\mathcal{D}$. By [3, Corollary III.3.4], any non-zero homomorphism from $\prod_{\alpha<\kappa} D_\alpha$ to a simple module S is necessarily non-zero on the direct sum $\bigoplus_{\alpha<\kappa} D_\alpha$, and hence on some D_α, in contradiction with $D_\alpha \in \mathcal{D}$. This proves our claim.

Finally assume that $\mathcal{D} = \mathcal{C}^\perp$ for a class of modules $\mathcal{C}$. Since $\mathcal{D}$ is closed under direct sums and A is hereditary, the class $\mathcal{D}$ is 1-tilting by [6, Corollary 6.1.7], so $\mathcal{D} = \text{Mod-}A$ by [6, Corollary 6.2.4], a contradiction.

In the next few lemmas, we will work with particular chains $(Y_\alpha \mid \alpha \leq \sigma)$ consisting of submodules of a module Y (where σ is an ordinal).

A chain $(Y_\alpha \mid \alpha \leq \sigma)$ is called a *strictly increasing continuous chain* of submodules of Y provided that $Y_0 = 0$, $Y_\alpha \subsetneq Y_{\alpha+1}$ for each $\alpha < \sigma$, $Y_\alpha = \bigcup_{\beta<\alpha} Y_\beta$ for each limit ordinal $\alpha \leq \sigma$, and $Y_\sigma = M$.

Dually, $(Y_\alpha \mid \alpha \leq \sigma)$ is a *strictly decreasing continuous chain* of submodules of Y provided that $Y_0 = Y$, $Y_\alpha \supsetneq Y_{\alpha+1}$ for each $\alpha < \sigma$, $Y_\alpha = \bigcap_{\beta<\alpha} Y_\beta$ for each limit ordinal $\alpha \leq \sigma$, and $Y_\sigma = 0$.

The following result was proved in [12, Theorem 2] for finite rank torsion-free abelian groups. Here we simplify our notation by writing ${}^\circ X$ instead of ${}^\circ\{X\}$ for a module X, and similarly for the other annihilator classes.

Lemma 3. *Let A be a ring, and X, Y be non-zero modules. Then the following conditions are equivalent:*

a) ${}^\circ X \subseteq {}^\circ Y$ *(that is, $Y \in ({}^\circ X)^\circ$).*
b) *There exist a strictly decreasing continuous chain $(Y_\alpha \mid \alpha \leq \sigma)$ of submodules of Y, and A-homomorphisms $\varphi_\alpha : Y_\alpha \to X$ $(\alpha < \sigma)$, such that $Y_{\alpha+1} = \mathrm{Ker}(\varphi_\alpha)$ for all $\alpha < \sigma$.*

Moreover, if $X = \bigoplus_{\gamma<\kappa} X_\gamma$ is any direct sum decomposition of X, then we can choose each homomorphism φ_α so that $\mathrm{Im}(\varphi_\alpha) \subseteq X_\gamma$ for some $\gamma < \kappa$.

Proof. a)⇒b) We construct the chain $(Y_\alpha \mid \alpha \leq \sigma)$ by induction on α. Let $Y_0 = Y$. Assume Y_α is defined and $Y_\alpha \neq 0$. Then $\mathrm{Hom}_A(Y_\alpha, Y) \neq 0$, so there exists a non-zero homomorphism $\varphi_\alpha : Y_\alpha \to X$. Without loss of generality, we can suppose that $\mathrm{Im}(\varphi_\alpha) \subseteq X_\gamma$ for some $\gamma < \kappa$, and put $Y_{\alpha+1} = \mathrm{Ker}(\varphi_\alpha) \subsetneq Y_\alpha$. If α is a limit ordinal, we define $Y_\alpha = \bigcap_{\beta<\alpha} Y_\beta$. Since the Y_αs form a strictly decreasing chain, our construction must stop at some σ, hence $Y_\sigma = 0$.

b)⇒a) Suppose that $\varphi : U \to Y$ is a non-zero homomorphism. Then there is a least index $\alpha < \sigma$ such that $\mathrm{Im}(\varphi) \subseteq Y_\alpha$. So $\mathrm{Im}(\varphi) \nsubseteq Y_{\alpha+1}$, and $0 \neq \varphi_\alpha \varphi : U \to X$. Therefore ${}^\circ X \subseteq {}^\circ Y$. □

Of course, the strictly decreasing chain in condition b) is always finite when Y is artinian. But if Y is only assumed noetherian then an infinite chain is needed in general in condition b) even if X has finite length.

For a simple example, consider the case of the abelian groups $X = \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ (p a prime integer) and $Y = \mathbb{Z}$. Then ${}^\circ X$ is the class of all groups containing no maximal subgroup of index p, and ${}^\circ Y$ the (larger) class of all groups having no non-zero free summands. Notice that there exists exactly one chain satisfying condition b), namely $Y_n = \mathbb{Z}p^n$ for $n < \omega$ and $Y_\omega = 0$.

We could have expressed condition b) of Lemma 3 simply in terms of the existence of a strictly decreasing continuous chain of submodules of Y, $(Y_\alpha \mid \alpha \leq \sigma)$, such that $Y_\alpha/Y_{\alpha+1}$ is isomorphic to a submodule of X for each $\alpha < \sigma$.

The dual result, concerning the annihilator classes of the form X°, then reads as follows:

Lemma 4. *Let A be a ring, and X, Y be non-zero modules. Then the following conditions are equivalent:*

a) $X^\circ \subseteq Y^\circ$ *(that is, $Y \in {}^\circ(X^\circ)$).*

b) *There exist a strictly increasing continuous chain $(Y_\alpha \mid \alpha \leq \sigma)$ of submodules of Y such that $Y_{\alpha+1}/Y_\alpha$ is a homomorphic image of X for each $\alpha < \sigma$.*

Moreover, if $X = \bigoplus_{\gamma<\kappa} X_\gamma$ is any direct sum decomposition of X, then in condition b), we can assume, without loss of generality, that for each $\alpha < \sigma$, the module $Y_{\alpha+1}/Y_\alpha$ is a homomorphic image of X_γ for some $\gamma < \kappa$.

Proof. a)$\Rightarrow$b) We construct the chain $(Y_\alpha \mid \alpha \leq \sigma)$ by induction on α. Let $Y_0 = 0$. Assume Y_α is defined and $Y_\alpha \neq Y$. Then $\mathrm{Hom}_A(Y, Y/Y_\alpha) \neq 0$, so there exists a non-zero homomorphism $f : X \to Y/Y_\alpha$ (If $X = \bigoplus_{\beta<\kappa} X_\beta$, we replace f by its non-zero restriction to some X_β.) We take $Y_{\alpha+1} \subseteq Y$ such that $Y_\alpha \subsetneq Y_{\alpha+1}$ and $Y_{\alpha+1}/Y_\alpha = \mathrm{Im}(f)$. For a limit ordinal α, we define $Y_\alpha = \bigcup_{\beta<\alpha} Y_\beta$. Clearly, the construction stops at some ordinal σ, so $Y_\sigma = Y$.

b)$\Rightarrow$a) If $\varphi : Y \to U$ is a non-zero homomorphism, then there is a least index $\beta < \sigma$ such that $\mathrm{Ker}(\varphi) \not\supseteq Y_\beta$. Then $\beta = \alpha + 1$ is a non-limit ordinal, $\mathrm{Ker}(\varphi) \supseteq Y_\alpha$, and φ has a factorization through the canonical projection $\pi : Y \to Y/Y_\alpha$, $\varphi = \psi\pi$ where $\psi : Y/Y_\alpha \to U$ is non-zero when restricted to Y_β/Y_α. By assumption, there is an epimorhism $f : X \to Y_\beta/Y_\alpha$, hence $0 \neq \psi f : X \to U$. $\square$

The strictly increasing chain in condition b) of Lemma 4 must be finite in case Y is noetherian. However, if Y is only assumed artinian then an infinite chain is needed in general even if X has finite length. Again, consider the case of abelian groups, $X = \mathbb{Z}_p$ (p a prime integer) and $Y = \mathbb{Z}_{p^\infty}$ (the Prüfer group). Then X° is the class of all groups with trivial p-torsion part while Y° is the (larger) class of all groups with no direct summand isomorphic to Y. The only chain satisfying condition b) of Lemma 4 is the (unique) composition series of $\mathbb{Z}_{p^\infty}$.

Now we turn to the annihilator classes of the form $X^\perp$ and generalize [8, Lemma 5.2]. We will call a module M *torsionless* provided that M embeds into a free module.

Lemma 5. *Let A be a ring, and X, Y be non-zero modules. Consider the following conditions:*

a) $X^\perp \subseteq Y^\perp$ *(that is, $Y \in {}^\perp(X^\perp)$).*

b) *Y is a direct summand in a module Z such that Z possesses a strictly increasing continuous chain of submodules $(Z_\alpha \mid \alpha \leq \rho)$ with Z_1 a free module, and $Z_{\alpha+1}/Z_\alpha \cong X$ for each $0 < \alpha < \rho$.*

c) *There exist a strictly increasing continuous chain $(Y_\alpha \mid \alpha \leq \sigma)$ of submodules of Y and A-homomorphisms $\varphi_\alpha : Y_{\alpha+1} \to X$ $(0 < \alpha < \sigma)$ such that $Y_\alpha = \operatorname{Ker}(\varphi_\alpha)$ for all $0 < \alpha < \sigma$. Moreover, either Y_1 is torsionless or Y_1 embeds into X.*

Then a) is equivalent to b), and it implies c).

Moreover, if $X = \bigoplus_{\gamma<\kappa} X_\gamma$ is any direct sum decomposition of X, then we can choose each homomorphism φ_α $(0 < \alpha < \sigma)$ in c) so that $\operatorname{Im}(\varphi_\alpha) \subseteq X_\gamma$ for some $\gamma < \kappa$.

Conversely, c) implies b) if and only if $X^\perp$ consists of modules of injective dimension ≤ 1.

Proof. The equivalence of a) and b) is well-known (see e.g. [6, 3.2.3]).

Assume b). Let $Y_\alpha = Y \cap Z_\alpha$ for each $\alpha \leq \rho$. Then $Y_0 = 0$, Y_1 is torsionless (but possibly zero), $Y_\alpha \subseteq Y_{\alpha+1}$, and $Y_{\alpha+1}/Y_\alpha \cong (Z_\alpha + Y_{\alpha+1})/Z_\alpha \subseteq Z_{\alpha+1}/Z_\alpha \cong X$ for each $0 < \alpha < \rho$. Moreover, $Y_\alpha = Y \cap Z_\alpha = \bigcup_{\beta<\alpha} Y_\beta$ for each limit ordinal $\alpha \leq \rho$, and $Y_\rho = Y$. Removing possible repetitions from the chain $(Y_\alpha \mid \alpha \leq \rho)$, we obtain the required strictly increasing chain as in c).

If $X = \bigoplus_{\gamma<\kappa} X_\gamma$ then we can refine the original chain $(Z_\alpha \mid \alpha \leq \rho)$ so that each consecutive factor $Z_{\alpha+1}/Z_\alpha$ is isomorphic to some X_γ, and then proceed as in the previous paragraph. Then for each $0 < \alpha < \sigma$ there is some $\gamma < \kappa$ such that $\operatorname{Im}(\varphi_\alpha) \subseteq X_\gamma$.

Assume c). For each $\alpha < \sigma$, the A-homomorphism φ_α yields an embedding $Y_{\alpha+1}/Y_\alpha \hookrightarrow X$. If $X^\perp$ consist of modules of injective dimension ≤ 1, then $\operatorname{Ext}^1_A(Y_{\alpha+1}/Y_\alpha, M) = 0$ for each $M \in X^\perp$, and a) follows by the Eklof Lemma [6, 3.1.2].

Conversely, if $X^\perp$ contains a module M of injective dimension > 1, then by Baer's Criterion, there is a right ideal Y of A such that $\operatorname{Ext}^1_A(Y, M) \neq 0$. Since Y is torsionless, condition c) of Lemma 5 holds for $\sigma = 1$, but a) fails since $M \in X^\perp \setminus Y^\perp$. $\square$

Corollary 6. *A ring A is right hereditary if and only the conditions b) and c) in Lemma 5 are equivalent for all right A-modules X and Y.*

Proof. If A is not hereditary and X is a projective right A-module then the class $X^\perp = \text{Mod-}A$ contains an element of injective dimension at least 2. The last part of Lemma 5 shows that, under this hypothesis, the conditions b) and c) are not equivalent. $\square$

The condition b) of Lemma 3 is quite close to condition c) of Lemma 5 in case Y is both artinian and noetherian, that is, when Y has finite length:

Theorem 7. *Let A be a ring, and X, Y be non-zero modules. Consider the following conditions:*

a) *${}^\circ X \subseteq {}^\circ Y$;*

b) *$X^\perp \subseteq Y^\perp$.*

Then a) implies b) in case Y is artinian and $X^\perp$ consists of modules of injective dimension ≤ 1.

Conversely, b) implies a) in case Y is noetherian and Y has no non-zero torsionless submodules.

In particular, a) is equivalent to b) in case A is right hereditary, Y has finite length and contains no non-zero projective submodules.

Clearly, the binary relations $\preceq$ and $\sqsubseteq$ defined on the set of all finite length modules by $X \preceq Y$ if and only if ${}^{\circ}X \subseteq {}^{\circ}Y$, and by $X \sqsubseteq Y$ if and only if $X^{\perp} \subseteq Y^{\perp}$, are preorder relations. We can now characterize the hereditary rings over which these relations coincide:

Theorem 8. *The following are equivalent for a right hereditary ring A:*

a) *Either A is simple artinian, or $Soc(A) = 0$.*
b) *${}^{\circ}X \subseteq {}^{\circ}Y$ if and only if $X^{\perp} \subseteq Y^{\perp}$, for all non-zero finite length modules X and Y.*
c) *${}^{\circ}X = {}^{\circ}Y$ if and only if $X^{\perp} = Y^{\perp}$, for all non-zero finite length modules X and Y.*

Proof. a)$\Rightarrow$b) If A is simple artinian then b) is clear. If $\mathrm{Soc}(A) = 0$, then there are no simple projective modules and b) holds by the final claim of Theorem 7.

b)$\Rightarrow$c) is obvious.

c)$\Rightarrow$a) Suppose that S and T are non-isomorphic simple modules such that S is projective. Then $(S \oplus T)^{\perp} = (T \oplus T)^{\perp}$, hence ${}^{\circ}(S \oplus T) = {}^{\circ}(T \oplus T)$. This gives $\mathrm{Hom}(S, T) \neq 0$, a contradiction. So either all simple modules are isomorphic and projective, or there are no projective simple modules.

If all simple modules are isomorphic and projective, then each maximal right ideal M is a direct summand in A, so the socle of A is not contained in M, hence A is simple artinian.

If there are no projective simple modules then $\mathrm{Soc}(A) = 0$ because A is right hereditary. □

Our next result is a variant of the equivalence a)$\Leftrightarrow$b) of Theorem 7 that restricts condition a) to finite length modules with no projective direct summands:

Proposition 9. *Let A be a right hereditary ring and Y a module of finite length. The following are equivalent:*

a) *$X^{\perp} \subseteq Y^{\perp}$;*
b) *${}^{\circ}X \cap \mathcal{C} \subseteq {}^{\circ}Y \cap \mathcal{C}$, where $\mathcal{C}$ is the class of all finite length modules which have no non-zero projective direct summands.*

Proof. a)$\Rightarrow$b) If Y is projective then clearly ${}^{\circ}X \cap \mathcal{C} \subseteq {}^{\circ}Y \cap \mathcal{C} = \mathcal{C}$.

Suppose that Y is not projective. By Lemma 5 there are a finite strictly increasing continuous chain $(Y_\alpha \mid \alpha \leq n)$ of submodules of Y and A-homomorphisms $\varphi_\alpha : Y_{\alpha+1} \to X$ $(0 < \alpha < n)$ such that $Y_\alpha = \mathrm{Ker}(\varphi_\alpha)$ for all $0 < \alpha < n$ and Y_1 is either projective or embeds into X. In the latter case, let φ_0 denote an embedding of Y_1 into X.

Let $Z \in \mathcal{C}$ and suppose that $Z \notin {}^{\circ}Y$, so there is a non-zero homomorphism $f : Z \to Y$. Observe that $\mathrm{Im}(f) \nsubseteq Y_1$ in case Y_1 is projective. There is an index $0 < \alpha \leq n$ such that $\mathrm{Im}(f) \subseteq Y_\alpha$ and $\mathrm{Im}(f) \nsubseteq Y_{\alpha-1}$. Then $g = \varphi_{\alpha-1} f$ gives a non-zero homomorphism from Z into X, and $Z \notin {}^{\circ}X$.

b)$\Rightarrow$a) There is nothing to prove if Y is projective.

Suppose that Y is not projective. Since Y has finite length, Y has a decomposition $Y = P_0 \oplus C_0$ such that P_0 is projective and $0 \neq C_0 \in \mathcal{C}$. By assumption there exists a non-zero homomorphism $f_0 : C_0 \to X$. Let $Z_0 = \mathrm{Ker}(f_0) \subsetneq C_0$. If Z_0 is not projective, we decompose it as $Z_0 = P_1 \oplus C_1$ where P_1 projective and $0 \neq C_1 \in \mathcal{C}$. Then there exists a non-zero homomorphism $f_1 : C_1 \to X$ and we define $Z_1 = \mathrm{Ker}(f_1) \subsetneq C_1$. Proceeding in this way, we obtain a decreasing chain $Y \supseteq C_0 \supsetneq Z_0 \supseteq C_1 \supsetneq Z_1 \supseteq \dots$ and projective modules $P_0, \dots, P_n$ such that $Z_{n-1} = P_n \oplus C_n$. Since Y has finite length, there is an $n < \omega$ such that the construction stops, i.e., Z_n is projective.

We define an increasing chain of submodules of Y as follows: $Y_0 = 0$, $Y_1 = P_0 \oplus P_1 \oplus \cdots \oplus P_n \oplus Z_n$, $Y_2 = Y_1 + C_n$, $\dots$, $Y_{n+1} = Y_1 + C_1$, $Y_{n+2} = Y_1 + C_0 = Y$. Then Y_1 is projective, and we have

$$Y_{k+1}/Y_k = (Y_1 + C_{n-k+1})/(Y_1 + C_{n-k+2}) \cong$$

$$C_{n-k+1}/((Y_1 + C_{n-k+2}) \cap C_{n-k+1}) = C_{n-k+1}/Z_{n-k+1}$$

because in the construction above, $Y_1 + C_{n-k+2} = P_0 + \cdots + P_{n-k+1} + Z_{n-k+1}$, $Z_{n-k+1} \subsetneq C_{n-k+1}$, and $(P_0 + \cdots + P_{n-k+1}) \cap C_{n-k+1} = 0$ by construction, so Y_{k+1}/Y_k is isomorphic to a submodule of X for each $1 \leq k \leq n+1$. By Lemma 5 we conclude that $X^\perp \subseteq Y^\perp$. $\square$

Concerning the dual classes X° and ${}^\perp X$, the reader might expect a dualization of Lemma 5. There is no such result available. In other words, there is no handy description of the class $({}^\perp X)^\perp$ for an arbitrary module X (see [6, §3.3] for more details).

We will see in Example 12 that the dual of the implication b)⇒a) in Theorem 7 fails in general. However, the dual of the implication a)⇒b) does hold:

Lemma 10. *Let A be a ring, and X, Y be non-zero modules such that Y is noetherian and ${}^\perp X$ consists of modules of projective dimension ≤ 1.*

Then $X^\circ \subseteq Y^\circ$ implies ${}^\perp X \subseteq {}^\perp Y$.

Proof. By Lemma 4, there exist $0 < m < \omega$ and a strictly increasing continuous chain $(Y_n \mid n \leq m)$ of submodules of Y such that Y_{n+1}/Y_n is a homomorphic image of X for each $n < m$. Let $M \in {}^\perp X$. Then M has projective dimension ≤ 1, so $\mathrm{Ext}^1_A(M, Y_{n+1}/Y_n) = 0$ for each $n < m$, and by induction on n, we conclude that $M \in {}^\perp Y$. $\square$

The equivalence a)⇔b) in Theorem 7 holds in particular for all hereditary artin algebras A (and all finitely generated modules Y containing no indecomposable projective submodules). For these algebras, the dual result does hold. In order to prove this we need to recall that if A is a hereditary artin algebra and $\tau = \mathrm{DTr}$ is the Auslander-Reiten translation in the category of all finitely presented modules, then by the Auslander-Reiten formula we get an isomorphism $\mathrm{Hom}_A(X, \tau N) \cong \mathrm{DExt}^1_A(N, X)$ for each finitely presented A-module N and each A-module X (see e.g. [8, Lemma 1.1]).

Proposition 11. *Let A be a hereditary artin algebra. Let X, Y be non-zero modules such that Y has finite length and Y has no indecomposable injective factor-modules. Then $X^\circ \subseteq Y^\circ$ if and only if ${}^\perp X \subseteq {}^\perp Y$.*

Proof. The direct implication holds by Lemma 10.

For the converse, assume that ${}^\perp X \subseteq {}^\perp Y$. We have to show that $X^\circ \subseteq Y^\circ$. Since Y is finitely presented, Y° is a torsion-free class closed under direct limits, so it suffices to prove the implication

$$\mathrm{Hom}_A(X, M) = 0 \Rightarrow \mathrm{Hom}_A(Y, M) = 0$$

for each finitely generated module M.

By the assumption on Y, if this implication fails, then it fails for a finitely generated module M with no indecomposable injective direct summands. So it suffices to prove the implication when M is finitely generated, indecomposable and not injective. Then $M = \tau N$ for a finitely generated indecomposable module N, so $\mathrm{Hom}_A(X, M) = \mathrm{Hom}_A(X, \tau N) = 0$ implies $\mathrm{Ext}_A(N, X) = 0$ by the Auslander-Reiten formula, hence $\mathrm{Ext}_A(N, Y) = 0$ by assumption. Then also $\mathrm{Hom}_A(Y, M) = \mathrm{Hom}_A(Y, \tau N) = 0$, q.e.d. □

The reverse implication of Proposition 11 will clearly fail for all injective simple modules X, Y such that $X \not\cong Y$. We finish this section by an example showing that it is not possible to extend Proposition 11 to general hereditary rings, even if we restrict ourselves only to non-injective simple modules:

Example 12. Let A be a simple countable, but not artinian, von Neumann regular ring (so A is as in Example 2, but we moreover require the field K to be countable). By [11, Proposition 6.3], any representative set simp-A of all simple modules has 2^ω elements, and all of them are non-injective. The condition $X^\circ \subseteq Y^\circ$ clearly fails for all $X \neq Y \in$ simp-A. However, assuming V = L, for *each* module X of finite length, the class ${}^\perp X$ coincides with the class of all projective modules by [11, Corollary 3.19]. In particular, the condition ${}^\perp X \subseteq {}^\perp Y$ holds for all $X, Y \in$ simp-A.

3. Modules of finite length and their annihilator classes

Now we turn to finite length modules that are direct sums of bricks.

Recall that a module X is a *brick* if $\mathrm{End}_A(X)$ is a division ring. Notice that any brick is indecomposable.

Theorem 13. *Let A be a ring, and X, Y be non-zero modules of finite length. Assume that $X = \bigoplus_{i=1}^m X_i$ is a direct sum of bricks such that $\mathrm{Hom}(X_i, X_j) = 0$ if $i \neq j$. Then the following are equivalent:*

a) $X \cong Y$;
b) ${}^\circ X = {}^\circ Y$ *and* $\lg(Y) \leq \lg(X)$;
c) ${}^\circ X = {}^\circ Y$ *and* $\lg(X) = \lg(Y)$;
d) ${}^\circ X = {}^\circ Y$ *and* $Y = \bigoplus_{i=1}^n Y_i$ *is a direct sum of bricks such that* $\mathrm{Hom}(Y_i, Y_j) = 0$ *for all* $i \neq j$.

Proof. The implications a)⇒d) and c)⇒b) are trivial.

b)⇒a) By Lemma 3 there are a chain of submodules $Y = Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_k = 0$ and homomorphisms $\varphi_i : Z_i \to X$ such that $\mathrm{Ker}(\varphi_i) = Z_{i+1}$, and for each $i < k$ there is $j \in \{1, \dots, m\}$ with $\mathrm{Im}(\varphi_i) \subseteq X_j$.

Let $j \in \{1, \dots, m\}$. Since $\mathrm{Hom}(X_j, X) \neq 0$, there exists a non-zero homomorphism $f : X_j \to Y$. Then there is an index $i < k$ such that $\mathrm{Im}(f) \subseteq Z_i$ and $\mathrm{Im}(f) \nsubseteq Z_{i+1}$, hence $\varphi_i f \neq 0$. Since $\mathrm{Hom}(X_j, X_s) = 0$ for all $s \neq j$, we have $0 \neq \mathrm{Im}(\varphi_i f) \subseteq X_j$, and $\mathrm{Im}(\varphi_i) \subseteq X_j$. If $\pi_j : X \to X_j$ is the canonical projection, then $\pi_j \varphi_i f$ is an isomorphism because X_j is a brick. It follows that $f : X_j \to Z_i$ is a split monomorphism, and

$$Z_i = f(X_j) \oplus \mathrm{Ker}(\pi_j \varphi_i) = f(X_j) \oplus \mathrm{Ker}(\varphi_i) \cong X_j \oplus \mathrm{Ker}(\varphi_i).$$

Since $\operatorname{Ker}(\varphi_i) = Z_{i+1}$, there is a subset $I = \{i_1, \dots, i_m\} \subseteq \{0, \dots, k-1\}$ such that Z_{i_j+1} is a direct summand of Z_{i_j} with a complement isomorphic to X_j, for all $j \in \{1, \dots, m\}$. In particular, $\lg(X) \leq \lg(Y)$.

But $\lg(X) \geq \lg(Y)$ by assumption, so $I = \{0, \dots, k-1\}$ and $Y \cong \bigoplus_{i=1}^m X_i = X$.

d)⇒c) As in the proof of b)⇒a), the inclusion ${}^\circ X \subseteq {}^\circ Y$ implies $\lg(X) \leq \lg(Y)$. Swapping the roles of X and Y, we obtain that $\lg(Y) \leq \lg(X)$. □

We also have the dual result:

Theorem 14. *Let A, X and Y be as in Theorem 13. Then the following are equivalent:*

a) *$X \cong Y$;*
b) *$X^\circ = Y^\circ$ and $\lg(Y) \leq \lg(X)$;*
c) *$X^\circ = Y^\circ$ and $\lg(X) = \lg(Y)$;*
d) *$X^\circ = Y^\circ$ and $Y = \bigoplus_{i=1}^n Y_i$ is a direct sum of bricks such that $\operatorname{Hom}(Y_i, Y_j) = 0$ for all $i \neq j$.*

Proof. We only give the proof for the implication b)⇒a), the rest is easy. By Lemma 4 there is a chain of submodules $0 = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_k = Y$ such that Y_{i+1}/Y_i is a homomorphic image of X_j for some $j \in \{1, \dots, m\}$.

Let $j \in \{1, \dots, m\}$. Since $\operatorname{Hom}(X, X_j) \neq 0$, also $\operatorname{Hom}(Y, X_j) \neq 0$, so there exist $i < k$ and a non-zero homomorphism $f : Y_{i+1}/Y_i \to X_j$. Since $\operatorname{Hom}(X_s, X_j) = 0$ for all $s \neq j$, there is an epimorphism $\pi : X_j \to Y_{i+1}/Y_i$. But X_j is a brick, so π is an isomorphism. In particular, $\lg(X) \leq \lg(Y)$. The assumption of $\lg(X) \geq \lg(Y)$ then implies that $k = m$, and there is a permutation $p \in S_m$ such that $Y_{i+1}/Y_i \cong X_{p(i)}$ for all $i < m$. Without loss of generality, we will assume that $p = \mathrm{id}$.

Since $\operatorname{Hom}(X, Y_1) \neq 0$, there exists $0 \neq f \in \operatorname{Hom}(Y, Y_1)$. As $\operatorname{Hom}(X_j, X_0) = 0$ for all $j \neq 0$, we have $f \restriction Y_1 \neq 0$. But Y_1 is a brick, so the inclusion $Y_1 \subseteq Y$ splits, and $Y = Y_1 \oplus Y'$ where Y' possesses a chain $0 = Y'_0 \subsetneq Y'_1 \subsetneq \cdots \subsetneq Y'_{m-1} = Y'$ with $Y'_{i+1}/Y'_i \cong X_{i+1}$ for all $i < m-1$.

In particular, $Y'_1 \cong X_1$, so $\operatorname{Hom}(X, Y'_1) \neq 0$. Then $\operatorname{Hom}(Y, Y'_1) \neq 0$, and also $\operatorname{Hom}(Y', Y'_1) \neq 0$ because $\operatorname{Hom}(X_0, X_1) = 0$. Consider $0 \neq f' \in \operatorname{Hom}(Y', Y'_1)$. Then $f' \restriction Y'_1 \neq 0$. Since Y'_1 is a brick, the inclusion $Y'_1 \subseteq Y'$ splits, giving a decomposition $Y' = Y'_1 \oplus Y''$. Proceeding in this way, we obtain a direct sum decomposition of Y showing that $Y \cong \bigoplus_{i=1}^m X_i = X$. □

There is a similar result for the annihilator classes of the form $X^\perp$ over hereditary rings:

Theorem 15. *Let A be a right hereditary ring, and X, Y be non-zero modules of finite length. Assume that $X = \bigoplus_{i=1}^m X_i$ is a direct sum of non-projective bricks such that $\operatorname{Hom}(X_i, X_j) = 0$ if $i \neq j$. Then the following are equivalent:*

a) *$X \cong Y$;*
b) *$X^\perp = Y^\perp$ and $\lg(Y) \leq \lg(X)$;*
c) *$X^\perp = Y^\perp$ and $\lg(X) = \lg(Y)$;*
d) *$X^\perp = Y^\perp$ and $Y = \bigoplus_{i=1}^n Y_i$ is a direct sum of bricks such that $\operatorname{Hom}(Y_i, Y_j) = 0$ for all $i \neq j$.*

Proof. This is again a variation of the proof of Theorem 13, so we only indicate the necessary changes:

- We use Lemma 5 in place of Lemma 3, so the module Z_k can be non-zero, but it is always projective.
- If $j \in \{1, \dots, m\}$, using $Y^\perp = X^\perp \subseteq X_j^\perp$ and Proposition 9, we deduce that there exists a non-zero homomorphism $f : X_j \to Y$. Since the brick X_j is not projective, $\mathrm{Im}(f) \nsubseteq Z_k$. Then there exists an index $i_j \in \{0, \dots, k-1\}$ such that Z_{i_j+1} is a direct summand of Z_{i_j} with a complement isomorphic to X_j.
- In the last part, from the equalities of lengths we obtain that $Z_k = 0$.

The rest of the proof is the same as for Theorem 13. □

Remark 16. By Example 12, assuming $V = L$, there exist a hereditary ring A and non-isomorphic non-injective simple modules (hence bricks) S and T such that ${}^\perp S = {}^\perp T$. So the dual of Theorem 15 cannot be proved in ZFC.

Next we list without proof several elementary properties of the annihilator classes of the form ${}^\circ X$:

Lemma 17. *Let A be a ring, and X, Y, and Z be modules.*

a) ${}^\circ(X \oplus Y) = {}^\circ X \cap {}^\circ Y$;

b) *If* $0 \to X \to Y \to Z \to 0$ *is a short exact sequence then* ${}^\circ(X \oplus Z) \subseteq {}^\circ Y \subseteq {}^\circ X$;

c) *If* $f \in \mathrm{End}(X)$ *and* $K = \mathrm{Ker}(f)$ *then* ${}^\circ X = {}^\circ(K \oplus X/K)$. *If moreover* $f^2 = 0$ *then also* ${}^\circ X = {}^\circ K$.

The following lemma gives a way of replacing finite length modules by direct sums of bricks without changing the annihilator class:

Lemma 18. *Let A be a ring and Y be a non-zero module of finite length. Then there exists a chain of submodules $Y = Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_k$ such that*

i) ${}^\circ Y = {}^\circ Y_1 = \cdots = {}^\circ Y_k$,

ii) $Y_k = \bigoplus_{i=1}^m Z_i$ *where each* Z_i *is a brick such that* $\mathrm{Hom}(Z_i, Z_j) = 0$ *for all* $i \neq j$,

iii) Y_j/Y_{j+1} *embeds into* Y_j *for all* $j \in \{0, \dots, k-1\}$.

Proof. Suppose that Y has a non-zero nilpotent endomorphism. Let $Y_0 = Y$, and take $0 \neq f_0 \in \mathrm{End}_A(Y_0)$ such that $f_0^2 = 0$. Let $Y_1 = \mathrm{Ker}(f_0) \subsetneq Y_0$. If Y_1 has a non-zero nilpotent endomorphism, take $0 \neq f_1 \in \mathrm{End}_A(Y_1)$ such that $f_1^2 = 0$ and let $Y_2 = \mathrm{Ker}(f_1) \subsetneq Y_1$.

Since Y is of finite length, there is a least k such that the ring $E = \mathrm{End}_A(Y_k)$ has no non-zero nilpotent elements (i.e. it is reduced). Then the chain $Y = Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_k$ satisfies condition iii) by construction, and condition i) by Lemma 17.c). By [13, 54.1] the reduced ring E is completely reducible, hence E is a finite direct product of division rings. It follows that the decomposition of Y_k into the direct sum of indecomposable modules consists of bricks, and there are no non-zero homomorphisms between different members of the decomposition, so ii) holds. □

Using Theorem 7, we obtain a result which generalizes [8, Proposition 5.6].

Corollary 19. *Let A be a right hereditary ring and Y be a non-zero module of finite length. Then there exists a chain of submodules $Y = Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_k$ such that*

i) $Y^\perp = Y_1^\perp = \cdots = Y_k^\perp$,

ii) $Y_k = \bigoplus_{i=1}^m Z_i$ *where each* Z_i *is a brick such that* $\mathrm{Hom}(Z_i, Z_j) = 0$ *for all* $i \neq j$,

iii) Y_j/Y_{j+1} *embeds into* Y_j *for all* $j \in \{0, \dots, k-1\}$.

For the proof of the main results we recall that the endomorphism ring of each indecomposable module of finite length is local [13, 32.4(3)], so by the Krull-Schmidt Theorem, every finite length module has a unique indecomposable decomposition (where uniqueness is understood up to order and isomorphism of the indecomposable factors).

Recall that a module X is *S-filtered* if X possesses a strictly increasing continuous chain $(S_\alpha \mid \alpha \leq \sigma)$ of submodules of X such that $S_{\alpha+1}/S_\alpha \cong S$ for all $\alpha < \sigma$.

By [4], for each module S there exists an S-filtered module X with $\mathrm{Ext}^1_A(S, X) = 0$. The module X can be used to test for vanishing of $\mathrm{Ext}^1_A(S, S)$ in the case when S is simple:

Lemma 20. *Let X be any module, and S be any simple submodule of X such that* $\mathrm{Ext}^1_A(S, X) = 0$.

Then $\mathrm{Ext}^1_A(S, S) = 0$ *if and only if* $N \cong S^2$ *for each S-filtered submodule N of X with* $\lg(N) = 2$.

Proof. Assume that N is an S-filtered submodule of X such that $\lg(N) = 2$ and $N \not\cong S^2$. Then the exact sequence $0 \to \mathrm{Soc}(N) \to N \to S \to 0$ is non-split, so $\mathrm{Ext}^1_A(S, S) \neq 0$.

Conversely, assume $\mathrm{Ext}^1_A(S, S) \neq 0$, and consider a non-split exact sequence $0 \to S \xrightarrow{\mu} T \to S \to 0$. By assumption, there is an embedding $\nu : S \to X$. We form the pushout of μ and ν:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & S \arrow[r, "\nu"] \arrow[d, "\mu"'] & X \arrow[r] \arrow[d] & X/S \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & X/S \arrow[r] & 0, \\
 & S \arrow[r, equal] \arrow[d] & S \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

W.l.o.g., T is a submodule of P. Since $\mathrm{Ext}^1_A(S, X) = 0$, the second column splits, hence $P = X \oplus Y$ with $Y \cong S$. Let $\pi : P \to X$ be the projection. By the assumption on N, the restriction $\pi \restriction T : T \to X$ is not a monomorphism. Since $\mathrm{Ker}(\pi) = Y$ is simple, we have $Y \subseteq T$, so Y is a direct summand in T. This implies that $T \cong S^2$, a contradiction. $\square$

Theorem 21. *Let A be a ring. The following are equivalent for a non-zero module X of finite length:*

a) $X \cong Y$ *whenever Y has finite length,* $^\circ X = {}^\circ Y$, *and* $\lg(X) = \lg(Y)$;

b) (I) $X \cong \bigoplus_{i=1}^m X_i$ *where* X_i *are bricks such that* $\mathrm{Hom}(X_i, X_j) = 0$ *for all* $i \neq j$, *or*

(II) $X \cong S^r \oplus \bigoplus_{i=1}^m X_i$ *and*

i) $r > 0$ *and* S *is a simple module,*

ii) $X_1, \ldots, X_m$ *are bricks such that* $\mathrm{Hom}(X_i, X_j) = 0$ *for all* $i \neq j$,

iii) *If* W *is a submodule of* $\bigoplus_{i=1}^m X_i$ *such that* $s = \lg(W) \leq r$ *then* $W \cong S^s$,

iv) $\mathrm{Ext}_A^1(S, X_i) = 0$ *for each* i.

Proof. a)$\Rightarrow$b) By Lemma 18, there exists a submodule $X' = \bigoplus_{i=1}^m X_i$ of X such that $^\circ X = {}^\circ X'$, all X_is are bricks and $\mathrm{Hom}(X_i, X_j) = 0$ whenever $i \neq j$. If $\lg(X) = \lg(X')$ then $X = X'$ and we are in case (I).

Assume $r = \lg(X) - \lg(X') > 0$. If S is any simple submodule of X' then $^\circ X = {}^\circ Y$ where $Y = S^r \oplus X'$. So $X \cong S^r \oplus \bigoplus_{i=1}^m X_i$ by a). Thus we are in case (II) and the conditions i) and ii) hold.

Moreover, it follows that S is the only simple module, up to isomorphism, that embeds into X', and the socle of every X_i is a finite direct sum of copies of S.

Let W be a submodule of $\bigoplus_{i=1}^m X_i$ such that $s = \lg(W) \leq r$. Then $^\circ Y = {}^\circ(\bigoplus_{i=1}^m X_i) = {}^\circ X$ for $Y = W \oplus S^{r-s} \oplus \bigoplus_{i=1}^m X_i$, so $W \cong S^s$ by a). This proves condition iii).

If $\mathrm{Ext}^1(S, X_j) \neq 0$, then there is a non-split short exact sequence $0 \to X_j \to M \xrightarrow{\beta} S \to 0$. Since S is simple, $M \not\cong S \oplus X_j$. Note that X_j, and hence M, has a submodule isomorphic to S. So we can view β as an endomorphism of M such that $\beta^2 = 0$. Then $^\circ M = {}^\circ(X_j \oplus S)$ by Lemma 17.c). So $^\circ X = {}^\circ(S^{r-1} \oplus M \oplus \bigoplus_{i \neq j} X_i)$, and $M \cong S \oplus X_j$ by a), a contradiction. This proves that $\mathrm{Ext}^1(S, X_j) = 0$ for all j.

b)$\Rightarrow$a) The case (I) has been proved in Theorem 13. We will prove the case (II).

Let X be as in b)(II), and Y be a module such that $^\circ X = {}^\circ Y$ and $\lg(X) = \lg(Y)$. Note that the socle of X is isomorphic to a direct sum of copies of S, by iii). Then Y has the same property: otherwise there exists a simple module $T \not\cong S$ and a non-zero homomorphism $T \to Y$, but $\mathrm{Hom}(T, X) = 0$.

Moreover, we claim that each submodule of Y of length $\leq r$ is isomorphic to a direct sum of copies of S. We have just proved this for $r = 1$. So suppose that U is a submodule of Y such that $\lg(U) \leq r$ and $r \geq 2$. Then there exists a non-zero homomorphism $f_1 : U \to X$. Since $\lg(\mathrm{Im}(f_1)) \leq r$, condition iii) gives that $\mathrm{Im}(f_1)$ is isomorphic to a finite direct sum of copies of S. Suppose $U_1 = \mathrm{Ker}(f_1) \neq 0$. Hence $\mathrm{Hom}(\mathrm{Ker}(f_1), Y) \neq 0$, and there exists a non-zero homomorphism $f_2 : U_1 \to X$. Repeating the previous arguments we find a chain of submodules $U = U_0 \supsetneq U_1 \supsetneq \cdots \supsetneq U_k \supsetneq \cdots$ and a family of homomorphisms $f_i : U_{i-1} \to X$ such that $U_i = \mathrm{Ker}(f_i)$. Since U has finite length, there exists k such that $U_k = 0$, i.e. f_k is a monomorphism. Then U_{k-1} is a finite direct sum of copies of S. Then the exact sequence $0 \to U_{k-1} \to U_{k-2} \xrightarrow{f_{k-1}} \mathrm{Im}(f_{k-1}) \to 0$ splits since $\mathrm{Ext}(\mathrm{Im}(f_{k-1}), U_{k-1}) = 0$ by Lemma 20. It follows that U_{k-2} is a direct sum of copies of S. Repeating this argument we obtain that all U_is are finite direct sums of copies of S, and the claim is proved.

Finally, consider a sequence of submodules $Y = Y_0 > Y_1 > \cdots > Y_k$ as in Lemma 18. Then $^\circ(\bigoplus_{i=1}^m X_i) = {}^\circ Y_k$ and using condition ii) and Theorem 13, we obtain that $\bigoplus_{i=1}^m X_i \cong Y_k$. Notice that for each $i \in \{0, \ldots, k-1\}$, the module Y_i / Y_{i+1} is of length $\leq r$ and it embeds into Y by Lemma 18. By the claim above, we infer that every module Y_i/Y_{i+1} is isomorphic to a finite direct sum of copies of S. If $r = 1$ then $k = 1$ and $Y/Y_1 \cong S$, hence $Y \cong Y_k \oplus S$ by condition iv). If $r > 1$ then Y/Y_k is a direct sum of copies of S by Lemma 20, so again $Y \cong S^r \oplus Y_k \cong X$ by condition iv). □

If A is any ring and $\mathcal{S}$ any non-empty finite set of non-isomorphic simple modules, then the module $X = \bigoplus_{S\in\mathcal{S}} S$ satisfies condition b)(I) of Theorem 21.

Moreover, if S is a simple module and $n > 1$ then $X = S^n$ satisfies condition a) if and only if $\mathrm{Ext}^1_A(S,S) = 0$, by condition b)(II) (here we consider $r = n - 1$).

Given $r > 0$ and $m > 0$, we will now present an example of a ring A and a module X satisfying condition b)(II) for these r and m:

Example 22. Let K be a field and let $(r_i \mid i = 1,\dots,m)$ be a sequence of integers such that $r_i \geq r$ for all i. Let A be the finite dimensional hereditary K-algebra consisting of all $(m+1)\times(m+1)$ upper triangular matrices of the form

$$A = \begin{pmatrix} K & \Theta & K^{r_1} \\ \cdots & \cdots & \cdots \\ \Theta & K & K^{r_m} \\ \Theta & 0 & K \end{pmatrix}$$

where Θ denotes the $1\times(m-1)$ matrix all of whose entries are zero. In other words, if we denote by Q_i the quiver of the generalized r_i-Kronecker algebra (such Q_i has only two vertices: a source and a sink, and r_i arrows), then A is the path algebra of the quiver obtained by identifying the sinks of the quivers Q_i for all $i = 1,\dots,m$.

For $i = 1,\dots,m+1$ let e_i be the i-th diagonal matrix unit (so e_i has exactly one non-zero entry, namely 1 in i-th row and i-th column). Then $\{e_i \mid i = 1,\dots,m+1\}$ is a complete set of primitive idempotents of A. Moreover, $S = e_{m+1}A$ is the unique projective simple module.

For $i = 1,\dots,m$, the projective module $X_i = e_iA$ has length $r_i + 1$, and its socle is isomorphic to S^{r_i}. If W is a submodule of $\bigoplus_{i=1}^m X_i$ of length $s \leq r$, then W is projective, hence a direct sum of some copies of S and X_i ($i = 1,\dots,m$). But each X_i has length $> r$, so no X_i occurs in the decomposition of W, and $W \cong S^s$. Since $\mathrm{End}_A(X_i) \cong e_iAe_i \cong K$, it is easy to see that $X = S^r \oplus \bigoplus_{i=1}^m X_i$ satisfies conditions i)-v) of Theorem 21.b)(II) for the given r and m.

In the following example we will show that the conditions ii)-iv) in Theorem 21.b)(II) are independent.

Example 23. (a) Let A be a commutative local QF-ring which is not a field. If S the simple A-module, then $X = S \oplus A$ satisfies iii) and iv), but not ii).

(b) An example where only iii) fails comes from a modification of Example 22: iii) will fail for the X defined there if $r_i < r$ for some i. Alternatively, we could take any non-completely reducible ring A which has a simple projective module S, and let $\{X_1,\dots,X_m\}$ be a finite set of non-projective pairwise non-isomorphic simple modules.

(c) To show that iv) is independent it is enough to consider a ring A possessing a simple module S such that $\mathrm{Ext}^1_A(S,S) \neq 0$, and take $X = S^r \oplus S$.

Returning to the annihilator classes of the form $X^\perp$, we first observe that if the ring A is as in Theorem 8 then Theorem 21 also characterizes the finite length modules X with the property that $X \cong Y$ for each finite length module Y such that $X^\perp = Y^\perp$ and $\mathrm{lg}(X) = \mathrm{lg}(Y)$. For the remaining case, the solution is given by the following theorem:

Theorem 24. *Let A be a right hereditary ring with at least two non-isomorphic simple modules, and X be a module of finite length.*

(I) *If A has at least two non-isomorphic projective simple modules then the following are equivalent:*
   a) *$X \cong Y$ whenever Y has finite length, $X^\perp = Y^\perp$, and $\lg(X) = \lg(Y)$;*
   b) *$X \cong \bigoplus_{i=1}^m X_i$ is a direct sum of finite length non-projective bricks such that $\operatorname{Hom}(X_i, X_j) = 0$ for all $i \neq j$.*

(II) *If up to isomorphism A has exactly one projective simple module S, then the following are equivalent:*
   a) *$X \cong Y$ whenever Y has finite length, $X^\perp = Y^\perp$, and $\lg(X) = \lg(Y)$;*
   b) *$X \cong S^r \oplus \bigoplus_{i=1}^m X_i$ for some $r \geq 0$, and*
      i) *all projective modules of length $s \leq r$ are isomorphic to S^s,*
      ii) *each X_i is a finite length non-projective brick and $\operatorname{Hom}(X_i, X_j) = 0$ for all $i \neq j$,*
      iii) *$U \cong S^s$ whenever U is a submodule of $\bigoplus_{i=1}^m X_i$ of length $s \leq r$.*

Proof. a)⇒b) Let X be a finite length module as in conditions (I) a) or (II) a). Without loss of generality, we can suppose that X is not projective. By Corollary 19, there exists a submodule $X' \leq X$ such that $X' = \bigoplus_{i=1}^m X_i$ is a direct sum of finite length non-projective bricks such that $\operatorname{Hom}(X_i, X_j) = 0$ if $i \neq j$, and $X'^\perp = X^\perp$. Let $r = \lg(X) - \lg(X')$.

If S and T are simple projective modules then $X^\perp = (S^r \oplus X')^\perp = (T^r \oplus X')^\perp$, hence $X \cong S^r \oplus X' \cong T^r \oplus X'$ by a). So in the case (I), $X = X'$, and b) holds.

In the case (II), we obtain $X = S^r \oplus \bigoplus_{i=1}^m X_i$ where all X_is are finite length non-projective bricks with $\operatorname{Hom}(X_i, X_j) = 0$ for all $i \neq j$. Let U be a submodule of $\bigoplus_{i=1}^m X_i$ of length $s \leq r$. Since A is hereditary, $(\bigoplus_{i=1}^m X_i)^\perp \subseteq U^\perp$, hence $X^\perp = (S^{r-s} \oplus U \oplus \bigoplus_{i=1}^m X_i)^\perp$, and $U \cong S^s$. Similarly, each projective module of finite length $q \leq r$ is isomorphic to S^q. This proves b).

b)⇒a) In the case (I), the implication is a consequence of Theorem 15.

In the case (II), let Y be a module such that $X^\perp = Y^\perp$. Again by Corollary 19, there exists a submodule $Y' \leq Y$ such that $Y' = \bigoplus_{i=1}^n Y_i$ is a direct sum of finite length non-projective bricks, $\operatorname{Hom}(Y_i, Y_j) = 0$ for $i \neq j$, and $Y'^\perp = Y^\perp = X^\perp = X'^\perp$, where $X' = \bigoplus_{i=1}^m X_i$. By Theorem 15, we have $m = n$ and without loss of generality we can assume $Y_i \cong X_i$ for all $i \in \{1, \dots, m\}$. By Lemma 5 there exists a descending chain $Y = Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_k$ such that Z_k is projective, and for each $i \in \{0, \dots, k-1\}$ there exist an index $j_i \in \{1, \dots, m\}$ and a homomorphism $\varphi_i : Z_i \to X_{j_i}$ such that $\operatorname{Ker}(\varphi_i) = Z_{i+1}$.

Let $j \in \{1, \dots, m\}$. Since Y_j is not projective, there exists $i \in \{0, \dots, k-1\}$ such that $Y_j \subseteq Z_i$, but $Y_j \nsubseteq Z_{i+1}$. Then the restriction $\varphi_i \restriction Y_j : Y_j \to X_{j_i}$ is a non-zero homomorphism, hence $j_i = j$ and $\varphi_i \restriction Y_j$ is an isomorphism. Then φ_i is a split epimorphism, and $Z_i = Z_{i+1} \oplus U_i$ where $U_i \cong Y_j \cong X_j$.

So for every $j \in \{1, \dots, m\}$ we can fix an index $i_j \in \{0, \dots, k-1\}$ such that $Z_{i_j} = Z_{i_j+1} \oplus U_j$ with $U_j \cong X_j$. Since $X_j \ncong X_{j'}$ whenever $j \neq j'$, the correspondence $j \mapsto i_j$ is 1-1.

Let $i \in \{0, \dots, k-1\} \setminus \{i_1, \dots, i_m\}$. Since $\sum_{i=0}^{k-1} \lg(Z_i/Z_{i+1}) + \lg(Z_k) = \lg(Y) = \lg(X) = r + \sum_{j=1}^m \lg(X_j)$, we have $r_i = \lg(Z_i/Z_{i+1}) \leq r$, hence $\operatorname{Im}(\varphi_i) \cong S^{r_i}$ is projective. Then $Z_i = Z_{i+1} \oplus U_i$, where $U_i \cong S^{r_i}$.

Moreover, we observe that iii) implies that $\lg(X_j) > r$ for all j. So the index i_j is uniquely determined by j since otherwise we would obtain, using the formula $\sum_{i=0}^{k-1} \lg(Z_i/Z_{i+1}) + \lg(Z_k) = \lg(Y)$, that $\lg(Y) > \lg(X)$.

It follows that for all $i \in \{0, \ldots, k-1\}$ there exists a direct sum decomposition $Z_i = Z_{i+1} \oplus U_i$ such that for every $j \in \{1, \ldots, m\}$ we can find exactly one index $i_j \in \{0, \ldots, k-1\}$ with $U_{i_j} \cong X_j$, and $U_i \cong S^{r_i}$ for all $i \notin \{i_1, \ldots, i_m\}$. Then $Y = U_0 \oplus \cdots \oplus U_{k-1} \oplus Z_k \cong X' \oplus S^t \oplus Z_k$. Since $\lg(Y) = \lg(X)$, Z_k is a projective module of finite length equal to $r - t$, so i) gives that $Z_k \cong S^{r-t}$. This proves that $Y \cong X' \oplus S^r \cong X$. □

Let A be any right hereditary ring with at least two non-isomorphic projective simple modules. If $\mathcal{S}$ is any finite non-empty set of non-isomorphic non-projective simple modules then the module $X = \bigoplus_{S \in \mathcal{S}} S$ clearly satisfies condition (I) b) of Theorem 24.

Finally, given $r > 0$ and $m > 0$, we will present an example of a hereditary ring A and a module X satisfying condition (II) b) for these r and m:

We consider the algebra A from Example 22, but we require $r_i > r$ for all $i = 1, \ldots, m$. Again, we take $S = e_{m+1}R$, the unique simple projective module, but for $i = 1, \ldots, m$, we replace X_i by $\overline{X_i}$, its factor modulo a simple submodule. Let $X = S^r \oplus \bigoplus_{i=1}^m \overline{X_i}$. Then condition i) holds because each projective module is a direct sum of some copies of S and of the X_is ($i = 1, \ldots, m$), and $\lg(X_i) > r$ for all i. Since $\overline{X_i}$ is generated by the coset of the idempotent e_i, we infer that $\mathrm{End}_A(\overline{X_i}) = K$ for all $i = 1, \ldots, m$, and it is easy to see that condition ii) holds. Finally, each submodule of $\overline{X} = \bigoplus_{i=1}^m \overline{X_i}$ of length $s \leq r$ is contained in the socle of $\overline{X}$, so condition iii) also holds.

In the end we mention that the dual statements for Lemma 17, Lemma 18 and Theorem 21 are true. However, a dual of Theorem 24 is not available, as a consequence of Example 28.

Lemma 25. *Let A be a ring, and X, Y, and Z be modules.*

a) $(X \oplus Y)^\circ = X^\circ \cap Y^\circ$;
b) *If* $0 \to X \to Y \to Z \to 0$ *is a short exact sequence then* $(X \oplus Z)^\circ \subseteq Y^\circ \subseteq Z^\circ$;
c) *If* $f \in \mathrm{End}(X)$ *and* $H = \mathrm{Im}(f)$ *then* $X^\circ = (H \oplus X/H)^\circ$. *If moreover* $f^2 = 0$ *then also* $X^\circ = (X/H)^\circ$.

Lemma 26. *Let A be a ring and Y be a non-zero module of finite length. Then there exists a chain of submodules $0 = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_k$ such that*

i) $Y^\circ = (Y/Y_1)^\circ = \cdots = (Y/Y_k)^\circ$,
ii) $Y/Y_k = \bigoplus_{i=1}^m Z_i$ *where each Z_i is a brick such that* $\mathrm{Hom}(Z_i, Z_j) = 0$ *for all $i \neq j$.*
iii) *Every Y_{i+1}/Y_i is an epimorphic image of Y.*

Proof. We proceed in the same way as in the proof of Lemma 18: Suppose that a submodule Y_i has been constructed. If Y/Y_i has no non-zero nilpotent endomorphisms we take $k = i$. Otherwise there exists an endomorphism $0 \neq f : Y/Y_i \to Y/Y_i$ such that $f^2 = 0$, and we put $Y_{i+1} \leq Y$ the only submodule such that $Y_{i+1}/Y_i = \mathrm{Im}(f)$. □

Theorem 27. *Let A be a ring. The following are equivalent for a non-zero module X of finite length:*

a) *$X \cong Y$ whenever Y has finite length, $X^\circ = Y^\circ$, and $\lg(X) = \lg(Y)$;*

b) (I) *$X \cong \bigoplus_{i=1}^m X_i$ where X_i are bricks such that $\operatorname{Hom}(X_i, X_j) = 0$ for all $i \neq j$, or*

(II) *$X \cong S^r \oplus \bigoplus_{i=1}^m X_i$ and*

i) *$r > 0$ and S is a simple module,*

ii) *$X_1, \dots, X_m$ are non-simple bricks such that $\operatorname{Hom}(X_i, X_j) = 0$ for all $i \neq j$,*

iii) *If W is an epimorphic image of $\bigoplus_{i=1}^m X_i$ such that $s = \lg(W) \leq r$ then $W \cong S^s$,*

iv) *$\operatorname{Ext}^1_A(X_i, S) = 0$ for each i.*

Proof. The proof is dual to the proof of Theorem 21. We present some details about the last part of it.

Consider a sequence of submodules $0 = Y_0 < Y_1 < \cdots < Y_k$ as in Lemma 26. Then $(\bigoplus_{i=1}^m X_i)^\circ = (Y/Y_k)^\circ$ and using condition ii) and Theorem 14, we obtain that $\bigoplus_{i=1}^m X_i \cong Y/Y_k$. Then $\lg(Y_k) = r$. Therefore, for each $i \in \{0, \dots, k-1\}$, the module Y_{i+1}/Y_i is of length $\leq r$ and it is an epimorphic image of Y by Lemma 26. Then every module Y_{i+1}/Y_i is isomorphic to a finite direct sum of copies of S.

If $r = 1$ then $k = 1$ and $Y_1 \cong S$, hence $Y \cong (Y/Y_1) \oplus S$ by condition iv).

If $r > 1$ then $Y_1 \cong Y_1/Y_0$ is isomorphic to a direct sum of copies of S. Suppose that Y_i is isomorphic to a direct sum of copies of S. Since the exact sequence $0 \to Y_i \to Y_{i+1} \to Y_{i+1}/Y_i \to 0$ splits by iv), Y_{i+1} is isomorphic to a direct sum of copies of S. Then $Y_k \cong S^r$, hence $Y \cong S^r \oplus Y/Y_k \cong X$ by condition iv). $\square$

If A and B are rings, we will denote by $A \boxplus B$ the direct product (in the category of all rings) of A and B.

Example 28. Let A be the ring from Example 12 and k be a field.

The ring $A \boxplus k \boxplus k$ has two simple injective modules, but by Remark 16, under V = L, it has two non-injective simple modules S_1 and S_2 such that ${}^\perp S_1 = {}^\perp S_2$. So the dual of Theorem 24(I) is not provable for $X = S_1$.

Similarly, we take the ring $A \boxplus k$ to see that the dual of Theorem 24(II) in not provable for $r = 0$ and $X = S_1$.

References

[1] Assem, I., Simson, D., Skowronski, A., Elements of the Representation Theory of Associative Algebras I, Cambridge Univ. Press, Cambridge 2006.

[2] Colpi, R., Tonolo, A., Trlifaj, J., *Perpendicular categories of infinite dimensional partial tilting modules and transfers of tilting torsion classes*, J. Pure Appl. Algebra 211(2007), 223-234.

[3] Eklof, P.C., Mekler, A.H., Almost Free Modules, Revised ed., North-Holland, New York 2002.

[4] Eklof, P.C., Trlifaj, J., *How to make* Ext *vanish*, Bull. London Math. Soc. 33(2001), 41-51.

[5] Göbel, R., Trlifaj, J., *Cotilting and a hierarchy of almost cotorsion groups*, J. Algebra 224(2000), 110-122.

[6] Göbel, R., Trlifaj, J., Approximations and Endomorphism Algebras of Modules, W. de Gruyter, Berlin 2006.

[7] Goodearl, K., Von Neumann Regular Rings, Krieger, Malabar (FL) 1991.

[8] Kerner, O., Trlifaj, J., *Tilting classes over wild hereditary algebras*, J. Algebra 290(2005), 538-556.

[9] Stenström, B., Rings of Quotients, Springer-Verlag, New York 1975.

[10] Trlifaj, J., *Similarities and differences between abelian groups and modules over non-perfect rings*, Contemp. Math. 171(1994), 397-406.

[11] Trlifaj, J., *Whitehead test modules*, Trans. Amer. Math. Soc. 348(1996), 1521-1554.

[12] Wickless, W. J., *An equivalence relation for torsion-free abelian groups of finite rank*, J. Algebra 153(1992), 1-12.
[13] Wisbauer, R., Foundations of Module and Ring Theory, Gordon & Breach, Philadelphia 1991.

"Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, Str. Mihail Kogălniceanu 1, 400084 Cluj-Napoca, Romania
E-mail address: `bodo@math.ubbcluj.ro`

Department of Algebra, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Prague, Czech Republic
E-mail address: `trlifaj@karlin.mff.cuni.cz`