S^*-GROUPS

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ABSTRACT. The main result of this paper (Theorem 4) gives several characterizations for Abelian groups A to have the property that the class of A-adstatic modules is closed with respect submodules of bounded index. Especially, a group has this property if and only if it is flat and faithful with respect to torsion when viewed as a left module over its endomorphism ring.

Associated with every Abelian group A is an adjoint pair (H_A, T_A) of functors between the category of Abelian groups and the category of right modules over the endomorphism ring $E = E(A)$ of A defined by $H_A(G) = \mathrm{Hom}(A, G)$ for all Abelian groups G and $T_A(M) = M \otimes_E A$ for all right E-modules M. The associated canonical maps are $\theta_G : T_A H_A(G) \to G$ and $\phi_M : M \to H_A T_A(M)$. The image of θ_G is called the *A-socle of G*. We say that G is *A-generated* if $S_A(G) = G$. Clearly, this occurs exactly if there exists an epimorphism $A^{(I)} \to G$ for some index-set I. A group G such that θ_G is an isomorphism is called *A-solvable*. Similarly, a right E-module M is *A-adstatic* if $\phi_M : M \to \mathrm{H}_A\mathrm{T}_A(M)$ is an isomorphism. Finally, an Abelian group is *(finitely) A-projective* if it is isomorphic to a direct summand of a (finite) direct sum of copies of A. Finitely A-projective are A-solvable; and finitely generated projective right E-modules are A-adstatic.

Closure properties of the classes of A-generated groups, A-solvable groups or A-adstatic modules play an important role in the theory of Abelian groups since they are closely related to homological properties of A (see [1] and [2]). For instance, the class of A-adstatic modules is closed with respect submodules if and only if A is *flat* (i.e. it is flat as an E-module) and *faithful* (i.e. $T_A(M) \neq 0$ whenever $M \neq 0$) [1].

The group A is a *S-(S^+-)group* if every subgroup U of A, for which A/U is finite, is A-generated (A-solvable). Arnold initiated the study of these groups in [4] in case that A is torsion-free. It was extended to mixed groups in [3]. Dually, we call A an *S^*-group* if every right ideal I of E such that E/I is finite, is A-adstatic. If A is an S^*-group, then A_p is a bounded homogeneous group for every prime p; and $\overline{A} = A/tA$ is p-divisible whenever $A_p \neq 0$. It is the goal of this paper to investigate S^*-groups further. Two notions will be of central importance in this context: A left E-module A

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is *faithful (finitely faithful) with respect to torsion* if $T_A(M) \neq 0$ whenever M is a non-zero E-module with M^+ torsion (bounded). Furthermore, A is *flat with respect to torsion* if $\mathrm{Tor}_E^1(M, A) = 0$ for every right E-module M whose additive group M^+ is torsion.

We begin our discussion with two technical lemmas.

Lemma 1. *Let A be an Abelian group which is flat with respect to torsion.*

a) *The class of A-solvable groups is closed with respect A-generated subgroups of bounded index.*
b) *If A is finitely faithful with respect to torsion, then A is faithful with respect to torsion.*

Proof. a) Let $0 \to H \stackrel{\alpha}{\to} G \stackrel{\beta}{\to} K \to 0$ be an exact sequence such that G is A-solvable, H is A-generated and K is bounded. Observe that θ_H is onto since $S_A(H) = H$. To see that it is one-to-one, consider the induced sequence $0 \to H_A(H) \stackrel{H_A(\alpha)}{\longrightarrow} H_A(G) \stackrel{H_A(\beta)}{\longrightarrow} H_A(K)$. Since the additive group of the submodule $M = \mathrm{Im}(H_A(\beta))$ of $H_A(K)$ is bounded, $\mathrm{Tor}_1^R(M, A) = 0$, and the top-row of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_AH_A(H) & \xrightarrow{T_AH_A(\alpha)} & T_AH_A(G) & \longrightarrow & T_A(M) & \longrightarrow & 0 \\ & & \downarrow \theta_H & & \downarrow \theta_G & & \downarrow \theta & & \\ 0 & \longrightarrow & H & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & K & \longrightarrow & 0. \end{array}$$

is exact. Hence, θ_H is one-to-one.

b) Let M be a right E-module such that M^+ is torsion and $T_A(M) = 0$. If $x \in M$, then the sequence $0 \to T_A(xE) \to T_A(M)$ is exact since $\mathrm{Tor}_1^R(M/xE, A) = 0$ because M/xE is torsion as an Abelian group. Therefore, $T_A(xE) = 0$. However, xE is bounded as an Abelian group because x is of finite order. Since A is finitely faithful with respect to torsion, $xE = 0$. $\square$

The next result discusses finitely faithful groups:

Lemma 2. *Let A be finitely faithful with respect to torsion. If G and K are A-solvable such that K is bounded, and if $\alpha : G \to K$ is an epimorphism, then $H_A(\alpha)$ is an epimorphism. Moreover, if K is finitely A-projective, then α splits.*

Proof. Let $N = \mathrm{Coker}(H_A(\alpha))$, and consider the commutative diagram

$$\begin{array}{ccccccc} T_AH_A(G) & \xrightarrow{T_AH_A(\alpha)} & T_AH_A(K) & \longrightarrow & T_A(N) & \longrightarrow & 0 \\ \theta_G \downarrow \cong & & \theta_K \downarrow \cong & & & & \\ G & \xrightarrow{\alpha} & K & \longrightarrow & 0 & & \end{array} .$$

A simple diagram chase shows $T_A(N) = 0$. Moreover, N^+ is bounded and A is finitely faithful with respect to torsion. Thus, $N = 0$; and $H_A(\alpha)$ is epic.

If K is finitely A-projective, then $H_A(K)$ is projective. Hence, $T_AH_A(\alpha)$ splits. The last commutative diagram yields that α splits. □

We want to remind the reader that an Abelian group A is *torsion reduced* if tA is reduced. An Abelian p-group is *homogeneous* if it is is isomorphic to $\oplus_I \mathbb{Z}/p^n\mathbb{Z}$ for some $n < \omega$ and some index-set I. Finally, let A_p denote the p-component of A.

Proposition 3. *If A is a torsion-reduced Abelian group which is finitely faithful with respect to torsion, then A_p is homogeneous for all primes p; and A/tA is p-divisible whenever $A_p \neq 0$.*

Proof. Suppose that A_p is not homogeneous for a prime p. Then, there exist two direct summands G and K of A such that $G \cong \mathbb{Z}(p^k)$ and $H \cong \mathbb{Z}(p^l)$ with $l < k < \infty$ since A_p is reduced. However, we can find an epimorphism $G \to K$ which does not split. But, this is impossible by Lemma 2 since G and K are finitely A-projective. The proof for the last statement is similar. □

Theorem 4. *The following are equivalent for a torsion reduced abelian group A:*

a) *A is flat and faithful with respect to torsion.*
b) *If M is an A-adstatic right E-module, and U is a submodule of M such that $(M/U)^+$ is bounded, then U is A-adstatic.*
c) *A is an S^*-group.*
d) i) *A is flat with respect to torsion.*
   ii) *ϕ_M is a monomorphism for every right E-module M such that M^+ is torsion.*
e) i) *A is flat with respect to torsion.*
   ii) *A finitely generated right E-module M such that M^+ is bounded is A-adstatic if and only if $T_A(M)$ is A-solvable.*

Proof. a) $\Rightarrow$ b): Suppose that $0 \to L \to M \to N \to 0$ is an exact sequence of right E-modules such that M is A-adstatic and N^+ is bounded. Since A is flat with respect to torsion, we obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow \phi_L & & \cong \downarrow \phi_M & & \downarrow \phi_N & & \\ 0 & \longrightarrow & H_AT_A(L) & \longrightarrow & H_AT_A(M) & \longrightarrow & H_AT_A(N) & & \end{array}$$

whose rows are exact. Therefore, ϕ_L is monic; and $\mathrm{Coker}(\phi_L) \cong \mathrm{Ker}(\phi_N)$ is torsion as an Abelian group. Since A is flat with respect to torsion, we obtain an exact sequence

$$0 \to T_A(L) \overset{T_A(\phi_L)}{\longrightarrow} T_AH_AT_A(L) \longrightarrow T_A(\mathrm{Coker}(\phi_L)) \to 0.$$

On the other hand, the sequence $0 \to T_A(L) \to T_A(M) \to T_A(N) \to 0$ is exact since A is flat with respect to torsion. By Part a) of Lemma 1, $T_A(L)$ is A-solvable. Hence, $T_A(\phi_L)$ is an isomorphism too since $\theta_{T_A(L)}T_A(\phi_L) =$

$1_{T_A(L)}$. Consequently, $T_A(\mathrm{Coker}(\phi_L)) = 0$. An application of Part b) of Lemma 1 yields $\mathrm{Coker}(\phi_L) = 0$; and L is A-adstatic.

b) $\Rightarrow$ *c)* is obvious.

c) $\Rightarrow$ *a)*: To show that A is flat with respect to torsion, we first show $\mathrm{Tor}_1^E(M, A) = 0$ whenever M is a finitely generated right E-module with M^+ bounded. Assume that M is generated by k elements, and proceed by on induction on k.

For $k = 1$, choose an exact sequence $0 \to I \to E \to M \to 0$ where I is a right ideal of E. By c), I is A-adstatic. Applying T_A yields the exact sequence $0 \to \mathrm{Tor}_1^E(M, A) \to T_A(I) \to T_A(E)$. Since H_A is left exact, the commutative diagram

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & I & \longrightarrow & E \\ & & & & \wr\big\downarrow \phi_I & & \wr\big\downarrow \phi_E \\ 0 & \longrightarrow & H_A(\mathrm{Tor}_1^E(M, A)) & \longrightarrow & H_A T_A(I) & \longrightarrow & H_A T_A(E) \end{array}$$

has exact rows. Therefore, $H_A(\mathrm{Tor}_1^E(M, A)) = 0$. Observe that $\mathrm{Tor}_1^E(M, A)$ is bounded since M^+ is bounded. Let p be a prime p with $\mathrm{Tor}_1^E(M, A)_p \neq 0$. Then, A is p-divisible since otherwise we can find a non-zero homomorphism $A \to A/pA \to \mathrm{Tor}_1^E(M, A)$. Because A is torsion reduced, $A_p = 0$; and multiplication by p is an automorphism of A. However, multiplication by p is also an automorphism of $\mathrm{Tor}_1^E(M, A)$ in this case, a contradiction.

Let U be a submodule of M which is generated by $k - 1$ elements. By induction hypothesis, we have $\mathrm{Tor}_1^E(U, A) = 0$. Because M/U is cyclic, $\mathrm{Tor}_1^E(M/U, A) = 0$. Thus, $\mathrm{Tor}_1^E(M, A) = 0$.

Let M be a right E-module with M^+ torsion. Since M is the direct limit of the system $\mathcal{F}$ of its finitely generated submodules, we obtain

$$\mathrm{Tor}_1^E(M, A) \cong \lim_{\rightarrow}(\mathrm{Tor}_1^E(U, A) \mid U \in \mathcal{F}) = 0.$$

Thus, A is flat with respect to torsion.

To prove that A is faithful with respect to torsion, let M be a right E-module such that M^+ is torsion and $T_A(M) = 0$. If $M \neq 0$, then there is a non-zero $x \in M$. Since we have already shown that A is flat with respect to torsion, $\mathrm{Tor}_1^E(M/xE, A) = 0$. We obtain the exact sequence $0 = \mathrm{Tor}_1^E(M/xE, A) \to T_A(xE) \to T_A(M) = 0$. Thus, we may assume that M is cyclic. As before, we choose a right ideal I and an exact sequence $0 \to I \to E \to M \to 0$. Observe that I is A-adstatic by c) and $\mathrm{Tor}_1^E(M, A) = 0$. We obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \wr\big\downarrow \phi_I & & \wr\big\downarrow \phi_E & & & & \\ 0 & \longrightarrow & H_A T_A(I) & \longrightarrow & H_A T_A(E) & \longrightarrow & 0 & & \end{array}$$

which shows $M = 0$ completing the proof.

c) $\Rightarrow$ *d)*: By what has been shown so far, A is flat with respect to torsion. Let M be a right E-module such that M^+ is torsion. Suppose that there is a non-zero $x \in M$ such that $\phi_M(x) = 0$. Since $0 = \mathrm{Tor}^1_E(M/xE, A)$, we obtain the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & H_AT_A(xE) & \longrightarrow & H_AT_A(M) \\ & & \uparrow{\scriptstyle \phi_{xE}} & & \uparrow{\scriptstyle \phi_M} \\ 0 & \longrightarrow & xE & \longrightarrow & M \end{array}$$

from which we obtain $\phi_{xE}(x) = 0$. Select a right ideal I of E such that $E/I \cong xE$, and consider the induced commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & E & \longrightarrow & xE & \longrightarrow & 0 \\ & & \cong\downarrow{\scriptstyle \phi_I} & & \cong\downarrow{\scriptstyle \phi_E} & & \downarrow{\scriptstyle \phi_{xE}} & & \\ 0 & \longrightarrow & H_AT_A(I) & \longrightarrow & H_AT_A(E) & \longrightarrow & H_AT_A(xE), & & \end{array}$$

in which ϕ_I is an isomorphism by c). The Snake-Lemma yields that ϕ_{xE} is a monomorphism. Then, $x = 0$; and ϕ_M is a monomorphism.

d) $\Rightarrow$ *c)* : Let I be a right ideal of E such that E/I is bounded. Then, the bottom sequence in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & E & \longrightarrow & E/I & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_I} & & \cong\downarrow{\scriptstyle \phi_E} & & \downarrow{\scriptstyle \phi_{E/I}} & & \\ 0 & \longrightarrow & H_AT_A(I) & \longrightarrow & H_AT_A(E) & \longrightarrow & H_AT_A(E/I) & & \end{array}$$

is exact since A is flat with respect to torsion. By the Snake-Lemma, ϕ_I is an isomorphism.

c) $\Rightarrow$ *e)*: It is obvious that $T_A(M)$ is an A-solvable if M is A-adstatic.

Conversely, let M be a finitely generated right E-module such that M^+ is bounded and $T_A(M)$ is A-solvable. Choose a projective resolution $0 \to U \to P \to M \to 0$ with P finitely generated. By Lemma 2, we obtain that the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_U} & & \downarrow{\scriptstyle \theta_P} & & \downarrow{\scriptstyle \theta_M} & & \\ 0 & \longrightarrow & \mathrm{H}_A\mathrm{T}_A(U) & \longrightarrow & \mathrm{H}_A\mathrm{T}_A(P) & \longrightarrow & \mathrm{H}_A\mathrm{T}_A(M) & \longrightarrow & 0 \end{array}$$

has exact rows since A is flat with respect to torsion by what has been shown so far. Since U is A-adstatic by the equivalence of b) and c), the Snake Lemma yields that θ_M is an isomorphism.

e) $\Rightarrow$ *c)* is a direct consequence of Lemma 1. $\square$

The condition that A is torsion reduced is essential in the previous theorem. If $A = \mathbb{Z}(p^\infty)$, then $E = J_p$, the ring of p-adic integers. Using the exact sequence $0 \to J_p \to \mathbb{Q}_p \to \mathbb{Q}_p/J_p \to 0$ (of right J_p-modules), where

$\mathbb{Q}_p$ is the field of all p-adic rationals, we obtain that $\mathbb{Z}(p^\infty)$ is not flat with respect to torsion. Moreover, since $J_p/pJ_p \otimes \mathbb{Z}(p^\infty) = 0$, the group $\mathbb{Z}(p^\infty)$ is not finitely-faithful with respect to torsion either. On the other hand, $\mathbb{Z}(p^\infty)$ is an S^*-group since every J_p-submodule of J_p^n is free.

Corollary 5. *Let A be a torsion-free Abelian group such that $\mathbb{Q}E$ is semi-simple Artinian. Then, A is an S^*-group if and only if A is faithfully flat as an E-module.*

Proof. By the last Theorem, it remains to show that an S^*-group of the given form is faithfully flat. For this consider a right E-module M. If M^+ is torsion-free, then $\mathbb{Q}M$ is a projective $\mathbb{Q}E$-module, and $\mathrm{Tor}_1^E(\mathbb{Q}M, A) = 0$. Moreover, there is an exact sequence $0 \to U \to \oplus_I E/n_iE \to \mathbb{Q}M/M \to 0$ for an index-set I and non-zero integers $\{n_i | i \in I\}$. Since $\mathrm{Tor}_2^E(E/n_iE, A) = 0$ in view of the fact that A is torsion-free, we obtain the exact sequence $0 = \mathrm{Tor}_2^E(\oplus_I E/n_iE, A) \to \mathrm{Tor}_2^E(\mathbb{Q}M/M, A) \to \mathrm{Tor}_1^E(U, A) = 0$ where the last Tor vanishes by the last theorem. Thus, $\mathrm{Tor}_2^E(\mathbb{Q}M/M, A) = 0$, and the same holds for $\mathrm{Tor}_1^E(M, A)$.

If M^+ is not torsion-free, then consider $0 \to tM \to M \to M/tM \to 0$ which induces $0 = \mathrm{Tor}_1^E(tM, A) \to \mathrm{Tor}_1^E(M, A) \to \mathrm{Tor}_1^E(M/tM, A) = 0$ by the last theorem. Thus, A is flat as an E-module.

Let I be a right ideal of E. Choose a right ideal J such that $I \oplus J$ is essential in E. Since $\mathbb{Q}E$ is semi-simple Artinian, $E/(I \oplus J)$ is bounded. Since A is an S^*-group, $\phi_{I \oplus J}$ is an isomorphism, and the same holds for ϕ_I. If $T_A(I) = 0$, then consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_AT_A(I) & \longrightarrow & H_AT_A(E) & \longrightarrow & 0 \\ & & \cong\uparrow\phi_I & & \cong\uparrow\phi_E & & \\ 0 & \longrightarrow & I & \longrightarrow & E & & \end{array}$$

from which we obtain $I = E$. $\square$

The last corollary may fail if $\mathbb{Q}E$ is not semi-simple Artinian:

Example 6. *There exists an flat S^*-group A which is not faithful.*

Proof. Let $A = \mathbb{Z} \oplus \mathbb{Q}$, and consider the idempotents e_1 and e_2 associated with the projections of A onto $\mathbb{Z}$ and $\mathbb{Q}$ respectively. Let I be a right ideal of E such that $nE \subseteq I$ for some non-zero integer n. Since $(e_2E)^+$ is divisible, we have $nE = ne_1E \oplus e_2E$. Hence, $I = (e_1E \cap I) \oplus e_2E$. Since $e_1E \cap I$ is a right ideal of $e_1Ee_1 \cong \mathbb{Z}$, we have that $I \cong E$. Hence, ϕ_I is an isomorphism. $\square$

Example 7. *There exists a flat with respect to torsion group which is not finitely faithful.*

Proof. The group $\mathbb{Z}(p) \oplus \mathbb{Z}(p^2)$ is projective by [5] over its endomorphism ring, but it is not finitely faithful. $\square$

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