GROUPS WHICH ARE DETERMINED BY SUBGROUP LATTICES

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Abstract. Although the lattice of all (normal) subgroups of an Abelian group does not determine the group up to an isomorphism, an Abelian group A can be determined by the lattice of all (normal) subgroups of other groups, e.g. if A is an Abelian group and G is a group such that $\mathbb{Z}\times A$ and $\mathbb{Z}\times G$ have isomorphic (normal) subgroup lattices then A and G are isomorphic groups. We present some results with this flavor. In the end of the paper, we discuss the cancellation property for subgroup lattices.

2000 *Mathematics Subject Classification*: 20E15, 20K27, 06C99

1 Introduction

It is known that the lattice $L(G)$ of all subgroups of a group G does not determine the group, that is, there exist non-isomorphic groups with isomorphic subgroup lattices. The simplest example is given by the groups $\mathbb{Z}(2)=\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}(3)=\mathbb{Z}/3\mathbb{Z}$.

However, if we restrict the class of all groups to some specific subclasses, subgroup lattices can determine some groups. For example, R. Baer proved in [2] that if A and A' are Abelian p-groups such that $L(A)\cong L(A')$, then $A\cong A'$ (see Theorem 3).

But, in general, an Abelian p-group is not determined by its subgroup lattice in the class of all p-groups (with modular subgroup lattice):

Theorem 1 [3] *Let G be a non-Hamiltonian locally finite p-group with modular subgroup lattice. There exists an Abelian p-group A such that $L(A)\cong L(G)$.*

[1]The authors are supported by the grant PN2-ID489 (CNCSIS)

Similar results are also true for other classes of Abelian groups. Sato proved in [21] the following result.

Theorem 2 *Let G be a non-Abelian group with elements of infinite order and modular subgroup lattice. Then there exists an Abelian group A such that $L(A) \cong L(G)$.*

Moreover, there exist torsion-free Abelian groups of rank 1 (i.e. groups which are isomorphic to subgroups of $\mathbb{Q}$, the group of all rational numbers) which are not determined by their subgroup lattices, even in the class of rank 1 torsion-free Abelian groups. If p is a prime and $R_p = \{\frac{m}{p^k} \mid m \in \mathbb{Z},\ k \in \mathbb{N}\} \leq \mathbb{Q}$ then

$$L(R_p) = \{\frac{n}{p^k}\mathbb{Z} \mid n \in \mathbb{N},\ k \in \mathbb{N} \cup \{\infty\}, \gcd(n, p^k) = 1\},$$

where $\frac{n}{p^\infty}\mathbb{Z} = nR_p$. If $p \neq q$ are primes and $n = q^s n_1$ is an integer such that $\gcd(n_1, pq) = 1$ then we denote $n' = p^s n_1$. The function

$$\varphi : L(R_p) \to L(R_q),\ \varphi(\frac{n}{p^k}\mathbb{Z}) = \frac{n'}{q^k}\mathbb{Z}$$

is a lattice isomorphism, but since they have different types $R_p \ncong R_q$. The general result concerning subgroup lattices of rank 1 Abelian torsion-free groups was proved by L. Fuchs in [14, pp. 305] and states that two such groups are lattice-isomorphic if and only if their types can be obtained from each other by a suitable permutation of the primes. In conclusion, $\mathbb{Q}$ is determined by its subgroup lattice, whereas, not all of its subgroups enjoy this property. We mention here that recently Călugăreanu and Rangaswamy gave in [10] a complete solution to the problem of deciding when two Abelian groups of will have the lattices of their subgroups isomorphic.

As seen earlier, the cyclic groups are not determined by their subgroup lattice (not even in the class of cyclic groups). The only cyclic group determined by its subgroup lattice is the infinite one ([2, Theorem 3.2]).

Among the well-known non-Abelian groups: the quaternion groups and the dihedral 2-groups are determined by their subgroup lattice. As a consequence, every hamiltonian 2-group is determined by its subgroup lattice ([2]). Moreover, Yakovlev proved in [25] that the free groups are determined by their subgroup lattice.

The present paper has semi-expository character. Its main aim is to exhibit a point of view on the problem which asks to determine a group (up to an isomorphism) using some subgroup lattices.

If G is a group, $L(G)$ denotes the subgroup lattice of G, while $\mathcal{N}(G)$ denotes the lattice of all normal subgroups of G. If n is an integer, $\mathbb{Z}(n)$ denotes the cyclic group $\mathbb{Z}/n\mathbb{Z}$. We denote by Grp the class of all groups, and by Lat the class of all lattices.

2 Groups determined by projectivities

We consider a map $\mathcal{S} : Grp \to Lat$ such that $\mathcal{S}(G)$ is a sublattice of $L(G)$, for all $G \in Grp$. If $V : Grp \to Grp$ is a map and $\mathcal{C}$ is a class of groups, we say that a group $G \in \mathcal{C}$ is *determined by* V *and* $\mathcal{S}$*-projectivities in* $\mathcal{C}$ if $G \cong H$ whenever $H \in \mathcal{C}$ and $\mathcal{S}(V(G)) \cong \mathcal{S}(V(H))$. If $\mathcal{C}$ is the class of all groups we say that G is *determined by* V *and* $\mathcal{S}$*-projectivities*. A group G is *determined by* $\mathcal{S}$*-projectivities* if it is determined by 1_{Grp} and $\mathcal{S}$-projectivities, i.e. if $G \cong H$ whenever $\mathcal{S}(G) \cong \mathcal{S}(H)$.

We discuss two kinds of projectivities: if $\mathcal{S}(G) = L(G)$ (respectively if $\mathcal{S}(G) = \mathcal{N}(G)$, the lattice of all normal subgroups of G), we say that G is *determined by* V *and (*$\mathcal{N}$*-)projectivities.*

General results on Abelian groups determined by projectivities were found by Baer in [2]. Moreover, Brandl in [4] and Curzio in [11] have found similar theorems concerning $\mathcal{N}$-projectivities. We summarize these results in the following theorem.

Theorem 3 *The following statements are true:*

i) Every Abelian group of torsion-free rank ≥ 2 *(i.e. it has a subgroup isomorphic to* $\mathbb{Z} \times \mathbb{Z}$*) is determined by (*$\mathcal{N}$*-)projectivities.*

ii) Every Abelian p-group is determined by projectivities in the class Ab_p *of all Abelian p-groups.*

iii) Every Abelian p-group which is not locally cyclic (i.e. which has a subgroup isomorphic to $\mathbb{Z}(p) \times \mathbb{Z}(p)$*) is determined by projectivities in Ab, the class of all Abelian groups. Consequently, every torsion group* T *whose primary components are not locally cyclic is determined by projectivities in the class Ab.*

iv) Every Abelian p-group which is not locally cyclic is determined by $\mathcal{N}$*-projectivities.*

Proof. The reader can find nice proofs for i), ii), and iv) in [22, Section 2.6 and Section 9.1].

The statement iii) is proved in [14, 80.2]. Since every projectivity preserves socles, we also can use [22, 2.2.6] in order to prove that if A is an Abelian p-group which is not locally cyclic and G is an Abelian group such that $L(A) \cong L(G)$ then G is a p-group. ■

We have already seen that in general an Abelian torsion group is not determined by projectivities in Ab, the class of all Abelian groups. However, such groups are determined by other subgroup lattices. The following result is an easy consequence of some known results. We state it here since it contains the germ idea for the main subject of the paper.

Proposition 4 *i) If $B \neq 0$ is a torsion-free Abelian group and $B \times \mathcal{T}$ is the class of all groups isomorphic to $B \times T$, where T is a torsion Abelian group, then every group $G \in B \times \mathcal{T}$ is determined by projectivities in the class $B \times \mathcal{T}$. Consequently, every torsion Abelian group is determined by $B \times -$ and projectivities in the class $\mathcal{T}$ of all torsion Abelian groups.*

ii) If A and G are finitely generated Abelian groups and $n > 1$ is an integer, then $A \cong G$ whenever $L(A^n) \cong L(G^n)$, i.e. every finitely generated Abelian group is determined by $(-)^n$ and projectivities in the class of all finitely generated Abelian groups.

Proof. i) is a consequence of [22, 2.6.15]. The statement ii) is a consequence of Theorem 3 and the structure theorem of finitely generated Abelian groups. ■

Remark 5 If the groups B and T are as in the statement i) from the previous Proposition, we cannot generalize and affirm that $B \times T$ is determined by projectivities. For example, if $B = \mathbb{Z}$ and T contains an element of order 8 or an element of order p^2 , for some prime $p > 2$, there exists a non-Abelian group H such that $L(\mathbb{Z} \times T) \cong L(H)$, as seen in [22, Exercise 5, pag100].

The point of view presented in the beginning of the section has appeared in [19] for $V(G) = G^2$, in [9] for $V(G) = G^n$ and in [7] for $V(G) = B \times G$, where B is a fixed group. In all these papers the following metatheorem is used:

Theorem 6 *Let $V : Grp \to Grp$ be a map and $\mathcal{S} : Grp \to Lat$ such that $\mathcal{S}(G)$ is a sublattice of $L(G)$ for all $G \in Grp$. Suppose that G is a group such that there exists a class $\mathcal{C}$ of groups with the following properties:*

- *$V(G) \in \mathcal{C}$;*

- *$V(G)$ is determined by projectivities in $\mathcal{C}$;*
- *if $\mathcal{S}(V(G)) \cong \mathcal{S}(V(H))$ then $V(H) \in \mathcal{C}$.*

Then G is determined by V and $\mathcal{S}$-projectivities if and only if G is determined by V (i.e. $V(G) \cong V(H) \Rightarrow G \cong H$).

This metatheorem was used in the mentioned papers in the case when $\mathcal{C}$ is the class of all Abelian groups. Therefore, in order to apply this metatheorem in our case, we should establish sufficient conditions such that $V(H)$ is Abelian, whenever $V(G)$ is.

3 The case $\mathcal{S} = L$

We will present here some results obtained in [7] and [19] concerning the problem: "Find sufficient conditions for the commutativity of a group using its subgroup lattice." We sketch some of the proofs for the reader's convenience.

Lemma 7 [7] *Let K and G be groups such that for every $g \in G$ there exists $k \in K$ with $ord(g) \mid ord(k)$ (here every non-zero positive integer is considered to be a divisor of ∞).*

a) If $L(K \times G)$ is modular then every subgroup of G is normal.

b) If $L(K \times G)$ is isomorphic to $L(A)$ for an Abelian group A, then G is an Abelian group.

Proof. For a), we consider $H \leq G$, $g \in G$, $H^g = g^{-1}Hg$, $k \in K$ such that $ord(g) \mid ord(k)$. Using the modularity law for the following subgroups of $K \times G$: $X = 1 \times \langle H, H^g \rangle$, $Y = \langle (k, g) \rangle$ and $Z = 1 \times H$, we deduce $X = Z$ and this implies that every subgroup of G is normal.

Applying the structure theorem of Hamiltonian groups, [22, Exercise **1**, p. 68], b) follows from the fact that no Abelian group is lattice isomorphic to the quaternion group H_8 (according to [13, Theorem 3.1]). ∎

Theorem 8 *Let B be an Abelian group. The following statements are true for a group G:*

a) If B is not a torsion group and $L(B \times G)$ is modular then G is Abelian.

b) If B is an unbounded p-group, G is a p-group and $L(B \times G)$ is modular then G is Abelian.

c) [19] *If G is a group such that $L(G^n)$ is modular for some integer $n > 1$ then G is Abelian.*

Proof. Statement a) follows directly from Lemma 7 and Theorem 2. Statement b) is a consequence of Lemma 7 and of the fact that a Hamiltonian group may be p-group only when $p = 2$.

For statement c), using again the previous lemma, we deduce that every subgroup of G is normal. Supposing G is not Abelian, we have a direct decomposition $G^n \cong H_8 \times H_8 \times K$. But $L(H_8 \times H_8)$ is not modular. To prove this, it is enough to consider the following subgroups of $H_8 \times H_8$: $X = \langle (i, i) \rangle$, $Y = \langle (i, j) \rangle$ and $Z = \langle (i, i), (1, -1) \rangle$. For these subgroups we observe that $X \leq Z$, but they do not satisfy the modularity law. ■

Recall that a group B has the *cancellation property (with respect a class $\mathcal{C}$)* if $B \times A \cong B \times G$ (and $A, G \in \mathcal{C}$) implies $A \cong G$. If $n > 0$ is an integer, the group A has the *n-root property* if $A \cong G$ whenever $A^n \cong G^n$.

Corollary 9 *Let B be an Abelian group. The following statements are true:*

a) If B is not a torsion group, $A \cong G$ whenever A is an Abelian group, G is a group and $L(B \times A) \cong L(B \times G)$ if and only if B has the cancellation property with respect to Ab.

b) If B is an unbounded p-group, $A \cong G$ whenever A is an Abelian p-group, G is a p-group and $L(B \times A) \cong L(B \times G)$ if and only if B has the cancellation property with respect to Ab.

c) If $n > 1$ is an integer, $B \cong G$ whenever G is a group and $L(B^n) \cong L(G^n)$ if and only if B has the n-root property.

Proof. We consider $V : Grp \to Grp$, $V(X) = B \times X$ for a) and b), respectively $V(X) = X^n$ for c), and we apply the previous metatheorem together with Theorem 3 and Theorem 8. Moreover, for a) we apply Proposition 4 in the case A is torsion. For b) we remark that Theorem 3 cannot be applied in the case $B \cong \mathbb{Z}(p^\infty)$ and $A = 0$. For this situation, we use again [13, Theorem 3.1] to deduce that $B \times G$ is a cocyclic Abelian group, and this is possible if and only if $G = 0$. ■

Remark 10 It is known that countable torsion Abelian groups and countable mixed Abelian groups of torsion-free rank 1 share the square-root property. Hence, these groups are determined by $(-)^2$ and projectivities. Moreover, the Abelian groups with semilocal endomorphism rings have the n-th root property (see [12, Proposition 4.8]), for any positive integer $n \geq 2$ and hence they are determined by $(-)^n$ and projectivities, for any integer $n \geq 2$. These groups were studied in [8]. Other mixed groups with n-root property were studied in [6].

We also recall that divisible Abelian groups of finite rank share the cancellation property with respect to Ab. In particular, $\mathbb{Z}(p^\infty)$ and $\mathbb{Q}$ possess this property. However, $\mathbb{Z}$ also has the cancellation property with respect to Ab (see [1, Corollary 8]).

Corollary 9 provides a partial answer for the problems: "When B is an Abelian group, under which conditions Abelian groups are determined by $B\times -$ and projectivities" and "For some positive integer $n \geq 1$, which groups are determined by $(-)^n$ and projectivities?"

In what regards the second problem, the case of Abelian groups is solved by the Corollary. For the case of non-Abelian groups, when $n = 2$, we just mention that the Rottlaender groups (see [20]) don't make part of the solution.

When a group is not determined by projectivities, it might be possible that one of its powers it is so. Baer's results from 1939 imply that the cube of an Abelian group is determined by projectivities in Ab. In what concerns non-Abelian groups, Michio Suzuki proved in 1951 that the square of finite simple group is determined by projectivities. This result was generalized by Schmidt who proved that the square of a finite, perfect, centerless group G is determined by projectivities. However, there are groups with the property that none of their powers is determined by projectivities. When p is a prime $\mathbb{Z}(p)$ is such a group (see [20, Section 6]).

4 The case $V = \mathcal{N}$

For the lattice of all normal subgroups we have similar (not *verbatim*!) results. In what follows we shall use the following result.

Theorem 11 [5] *Let $B \neq 0$ be a torsion-free Abelian group. If G is a group such that $\mathcal{N}(B \times G) \cong \mathcal{N}(A)$ for some Abelian group A then G is Abelian.*

Proof. Let $\varphi : \mathcal{N}(B \times G) \to \mathcal{N}(A)$ be an isomorphism. Every subgroup of B is normal in $B \times G$, hence the restriction of φ to B induces a projectivity from B onto $\varphi(B)$. Consequently $\varphi(B)$ is not a torsion group. It follows that

A is not a torsion group too. If the torsion-free rank of A is > 1 the conclusion follows from [22, Theorem 9.1.12]. If A is of torsion-free rank 1, using the direct decomposition $A = \varphi(B) \times \varphi(G)$, we deduce that G is a subgroup of the centralizer of $\varphi^{-1}(\langle x \rangle)$, for all $x \in A \setminus T(A)$. Therefore, G is contained in the center of $B \times G$ and hence is Abelian. ∎

Remark 12 The hypothesis "B is not a torsion group" is essential in the previous theorem. For example, if we take $B = \mathbb{Z}(5)$ and $G = S_3$, then $\mathcal{N}(B \times G) \cong \mathcal{N}(\mathbb{Z}(5) \times \mathbb{Z}(4))$.

Remark 13 Theorem 11 still holds if we assume that all involved groups are p-groups, as a consequence of [22, Theorem 9.1.11].

Remark 14 The condition from the hypothesis of the previous theorem can be replaced with weaker one: if B and G are as above and $\mathcal{N}(B \times G) \cong L(H)$, for some (not necessarily Abelian) group H, then G is Abelian. Indeed, if H is Abelian we are in the conditions of the previous theorem. But if H is not Abelian, we deduce that H is not a torsion group (in the same way as in the proof of the Theorem 11). Since $L(H)$ is modular we can apply Theorem 2 and we find ourselves again in the conditions of the previous theorem.

Corollary 15 *Let $B \neq 0$ be an Abelian group. The following statements are true:*

a) If B is torsion-free, $A \cong G$ whenever A is an Abelian group, G is a group and $\mathcal{N}(B \times A) \cong \mathcal{N}(B \times G)$ if and only if B has the cancellation property with respect to Ab.

b) If B is a p-group, $A \cong G$ whenever $A \neq 0$ is an Abelian p-group, G is a group and $\mathcal{N}(B \times A) \cong \mathcal{N}(B \times G)$ if and only if B has the cancellation property with respect to Ab.

c) If $n > 1$ is an integer, $B \cong G$ whenever G is a group and $\mathcal{N}(B^n) \cong \mathcal{N}(G^n)$ if and only if B has the n-root property.

Proof. a) Suppose B has the cancellation property with respect to *Ab* and $\mathcal{N}(B \times A) \cong \mathcal{N}(B \times G)$. The fact that G is an Abelian group follows from Theorem 11. By [22, Theorem 2.6.10 and Theorem 2.6.15], we get $B \times A \cong B \times G$ and so $A \cong G$.

Statements b) and c) are consequences of Theorems 3 and 11. ∎

Remark 16 In the proof of a) we cannot deduce directly that $\varphi(B)$ is Abelian, even if $\mathcal{N}(B) \cong \mathcal{N}(\varphi(B))$. In contrast with the case of subgroup lattices, it is not known when a rank 1 torsion-free Abelian group B (even for $B = \mathbb{Z}$) is determined by $\mathcal{N}$-projectivities. However, under additional assumptions Curzio proved the following result.

Theorem 17 [11] *If G is a rank 1 torsion-free Abelian group, K a soluble group such that $\mathcal{N}(G) \cong \mathcal{N}(K)$ then K is Abelian and of torsion-free rank 1.*

Using the normal subgroup lattice, a commutativity criterion for the nilpotent groups is provided in [5].

Theorem 18 *A nilpotent group G is Abelian if and only if there exists an abelian group A such that $\mathcal{N}(G) \cong \mathcal{N}(A)$.*

Corollary 19 *If G is a finite p-group such that $\mathcal{N}(G) \cong \mathcal{N}(A)$ for some Abelian group A, then G is Abelian.*

Corolarry 19 does not work for the subgroup lattices of finite p-groups as a consequence of 1.

Remark 20 In Corollary 9 b) the hypothesis "B is unbounded" is essential: if G is a non-Hamiltonian locally finite p-group with modular subgroup lattice then $\mathbb{Z}(p) \times G$ is also a non-Hamiltonian locally finite p-group with modular subgroup lattice, as a consequence of Iwasawa's characterization [22, 2.4.14]. The existence of an Abelian p-group C such that $L(C) \cong L(\mathbb{Z}(p) \times G)$ is assured by Theorem 1. Using this projectivity, we deduce the existence of a direct decomposition $C \cong \mathbb{Z}(p) \times A$. In conclusion, there exist an Abelian group A and a non-Abelian group G such that $L(\mathbb{Z}(p) \times A) \cong L(\mathbb{Z}(p) \times G)$.

In Corollary 15 b) we have no such restrictions for the group B.

Corollary 21 *Let A be an Abelian group and G a group. The following statements are true.*

(i) If $\mathbb{Z} \times A$ and $\mathbb{Z} \times G$ have isomorphic (normal) subgroup lattices then $A \cong G$.

(ii) If $\mathbb{Q} \times A$ and $\mathbb{Q} \times G$ have isomorphic (normal) subgroup lattices then $A \cong G$.

(iii) If A is an Abelian p-group and G is a p-group such that $\mathbb{Z}(p^\infty) \times A$ and $\mathbb{Z}(p^\infty) \times G$ have isomorphic subgroup lattices then $A \cong G$.

(iii') If A is an Abelian p-group and G is a group such that the groups $\mathbb{Z}(p^k) \times A$ and $\mathbb{Z}(p^k) \times G$, have isomorphic normal subgroup lattices, for some $k \in \mathbb{N}^\star \cup \{\infty\}$, then $A \cong G$.

(iv) If A is a finitely generated Abelian group, G is a group, and there exists $n > 1$ such that A^n and G^n have isomorphic (normal) subgroup lattices then $A \cong G$.

Proof. We only note that the proof is based on the fact that $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ have the cancellation property, and finitely generated Abelian groups have the n-root property. Since a finite group has the cancellation property (see [16]), $\mathbb{Z}(p^k)$ enjoys this property too. Full proofs for the case of subgroup lattices can be found in [7], [9] and [19]. ■

Remark 22 Since there exist non-isomorphic groups G and H such that $\mathbb{Z} \times G \cong \mathbb{Z} \times H$ (see [24, Theorem 13]), the group $\mathbb{Z}$ does not have the cancellation property with respect to non-Abelian groups. However, when $\mathbb{Z} \times G \cong \mathbb{Z} \times H$, for some non-isomorphic groups G and H, these groups must be infinite. Hence, our next question is whether the Corollary 21(i) can be extended to non-Abelian finite groups. The answer is still negative and it is presented in Section 5.

Corollary 23 *Let A be an Abelian group. If G is a group and B is a finite rank torsion-free Abelian group such that $L(B\times A) \cong L(B\times G)$ (or $\mathcal{N}(B\times A) \cong \mathcal{N}(B \times G)$) then there exists a positive integer n such that $A^n \cong G^n$.*

5 An open question

We have seen that in the case of non-Abelian groups, the group $\mathbb{Z}$ does not have the cancellation property, hence it is not cancellable from the subgroup lattices. A more general cancellation property is investigated.

A group B has the *power cancellation property (with respect a class $\mathcal{C}$)* if $B \times A \cong B \times G$ (and $A, G \in \mathcal{C}$) implies $A^n \cong G^n$, for some positive integer n.

Hirshon proved in [17, Theorem 1] that $\mathbb{Z}$ has the power cancellation property and later Goodearl established in [15, Theorem 5.1] that actually the torsion-free Abelian groups of finite rank possess this property . Corollary 23 follows from Goodearl's result.

The question "Does the property $L(\mathbb{Z}\times G_1)\cong L(\mathbb{Z}\times G_2)$ (or $\mathcal{N}(\mathbb{Z}\times G_1)\cong\mathcal{N}(\mathbb{Z}\times G_2)$) imply $G_1^n\cong G_2^n$, for some integer $n>0$" seems to be natural. The answer is negative one, see [5], and it uses some classes of groups constructed in [23] and [18]. We present here, for the sake of completeness, a particular example.

Example 24 Let us recall here the groups constructed in [23, V (i)]. We fix two primes p and q such that $q\equiv 1(\mathrm{mod}\ 3)$, $p\equiv 1(\mathrm{mod}\ 3q)$, an integer m of order 3 in $\mathbb{Z}(p)^\star$ and an integer n of order 3 in $\mathbb{Z}(q)^\star$. Let G_1 be the group

$$G(m,n)=\langle s,t,u\mid s^p=t^q=u^3=1, st=ts, us=s^m u, ut=t^n u\rangle.$$

We denote by G be the underlying set of $G(m,n)$, and by $\circ$ the operation on $G(m,n)$, hence $G_1=(G,\circ)$.

We consider the binary term $T=xy[x,y]^{p-1}$, and denote by G_2 the group defined on the set G by the operation $x\star y=T(x,y)$, where $T(x,y)$ is the image of the term function induced by T on G_1. In [23] it is proved that $G_2=(G,\star)$ is a group isomorphic to $G(m^2,n)$, and $G_1\ncong G_2$. If we repeat the construction, using this time T and G_2 we obtain a group $G_3\cong G(m^4,n)=G(m,n)$. This means that G_1 and G_2 are term equivalent groups (see [18]). Moreover, for every integer $n>0$, the groups G_1^n and G_2^n are term equivalent and it follows $L(G_1^n)\cong L(G_2^n)$.

Using the same term function T, it is not hard to see that the groups $\mathbb{Z}\times G_1$ and $\mathbb{Z}\times G_2$ are term equivalent (see [5, Corollary 13]). Therefore, $L(\mathbb{Z}\times G_1)\cong L(\mathbb{Z}\times G_2)$ and $\mathcal{N}(\mathbb{Z}\times G_1)\cong\mathcal{N}(\mathbb{Z}\times G_2)$ as a consequence of [18, Lemma 2.6]. But $G_1^n\ncong G_2^n$ for all $1\le n<\omega$ since $G_1\ncong G_2$ and these groups are finite groups. Hence, we have presented the promised example from Remark 22.

So it seems to be reasonable to formulate the following

Conjecture 25 *If B is a finite rank torsion-free Abelian group and G_1, G_2 are groups (non-necessarily Abelian) such that $L(B\times G_1)\cong L(B\times G_2)$ (or $\mathcal{N}(B\times G_1)\cong\mathcal{N}(B\times G_2)$) then there exists a positive integer n such that $L(G_1^n)\cong L(G_2^n)$ (respectively $\mathcal{N}(G_1^n)\cong\mathcal{N}(G_2^n)$).*

This conjecture can be viewed as a part of the more general problem: Find (characterize) the groups which have the *power cancellation property for (normal) subgroup lattices*: if $L(A\times G_1)\cong L(A\times G_2)$ (respectively $\mathcal{N}(A\times G_1)\cong\mathcal{N}(A\times G_2)$) then there exists a positive integer n such that $L(G_1^n)\cong L(G_2^n)$ (respectively, $\mathcal{N}(G_1^n)\cong\mathcal{N}(G_2^n)$).

References

[1] Arnold D., *Finite Rank Torsion-Free Abelian Groups and Rings*, Lecture Notes in Math. 931, Springer - Verlag, New-York, 1982.

[2] Baer, R., *The significance of the system of subgroups for the structure of the group*, Amer. J. Math., **61** (1939), 1–44.

[3] Baer, R., *Crossed isomorphisms*, Amer. J. Math., **66** (1944), 341–404.

[4] Brandl, R., *On groups with certain lattices of normal subgroups*, Arch. Math. (Basel), **47** (1986), 6–11.

[5] Breaz, S., *Commutativity conditions using normal subgroup lattices*, Rend. Sem. Mat. Univ. Padova , to appear.

[6] Breaz, S., *On a class of mixed groups with semi-local $WALK$-endomorpism ring*, Comm. Algebra **30** (2002), 4473–4485.

[7] Breaz, S., Călugăreanu, *Every Abelian group is determined by a subgroup lattice*, Stud. Sci. Math. Hung., **45** (2008), 135–137.

[8] Călugăreanu, G., *Abelian groups with semilocal endomorphism rings*, Comm. Algebra, **30** (2002), 4105 - 4111.

[9] Călugăreanu, G., *Abelian groups determined by subgroup lattices of direct powers*, Arch. Math. (Basel) **86** (2006), 97–100.

[10] Călugăreanu, G., Rangaswamy, K. M., *A solution to a problem on lattice isomorphic Abelian groups*, in Models, Modules and Abelian Groups, In Memory of A.L.S. Corner, de Gruyter 2008, 249–256 .

[11] Curzio, M., *Una caratterizzazione reticolare dei gruppi abeliani*, Rend. Math. e Appl., **24** (1965), 1–10.

[12] Facchini, A., *Module theory. Endomorphism rings and direct sum decompositions in some classes of modules*, Progress in Mathematics **167**, Birkhäuser, Basel, 1998.

[13] Fuchs L., *Infinite Abelian Groups*, vol.I, Academic Press, New-York and London, 1970.

[14] Fuchs, L., *Abelian Groups*, Pergamon Press, New York, 1960.

[15] Goodearl, K., *Power cancellation of groups and modules*, Pacific J. Math., **64** (1976), 387–411.

[16] Hirshon, R., *On cancellation in groups*, Amer. Math. Monthly, **76** (1969), 1037–1039.

[17] Hirshon, R., *The cancellation of an infinite cyclic group in direct products*, Archiv der Mathematik **26**, (1975), 134–138.

[18] Kearnes, K.A., Sendrei, Á., *Groups with identical subgroup lattices in all powers*, J. Group Theory **7** (2004), 385–402.

[19] Lukács, E., Pálfy, P.P., *Modularity of the subgroup lattice of a direct square*. Arch. Math. (Basel), **46** (1986), 18–19.

[20] Pálfy, P.P., *Groups and Lattices*, Mathematical Institute of the Hungarian Academy of Sciences, Preprint **3**, 2001.

[21] Sato, S., *Note on the lattice isomorphisms between Abelian groups and non-Abelian groups*, Osaka Math. J., **3** (1951), 215–220.

[22] Schmidt R., *Subgroup Lattices of Groups*, de Gruyter Expositions in Mathematics 14, de Gruyter, Berlin, 1994.

[23] Street, A.P., *Subgroup-determining functions on groups*, Ill. J. Math., **12** (1968), 99–120.

[24] Walker, E.A., *Cancellation in direct sums of groups*, Proc. A.M.S., **7** (1956), 898902

[25] Yakovlev, B. V., *Conditions under which a lattice is isomorphic to the lattice of subgroups of a group* (Russian) Algebra i Logika, **13** (1974), 694-712.

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