ON SOME DUALITIES INDUCED BY RIGHT ADJOINT CONTRAVARIANT FUNCTORS

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Abstract. We characterize some dualities which are induced by pairs of contravariant functors which are adjoint on the right.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $\mathrm{F} : \mathcal{A} \leftrightarrows \mathcal{B} : \mathrm{G}$ a pair of contravariant functors which are adjoint on the right. Then the natural transformation

$$\eta_{X,Y} : \mathrm{Hom}_{\mathcal{A}}(X, \mathrm{G}(Y)) \to \mathrm{Hom}_{\mathcal{B}}(Y, \mathrm{F}(X)),$$

which corresponds to this duality, induces two natural transformations

$$\delta : 1_{\mathcal{A}} \to \mathrm{GF},\ \delta_X = \eta^{-1}_{X,\mathrm{F}(X)}(1_{\mathrm{F}(X)}) \text{ and } \zeta : 1_{\mathcal{B}} \to \mathrm{FG},\ \zeta_Y = \eta^{-1}_{\mathrm{G}(Y),Y}(1_{\mathrm{G}(Y)}).$$

An object X is called δ(respectively ζ)-*reflexive* if δ_X (respectively ζ_X) is an isomorphism. We will denote by Refl_δ (respectively Refl_ζ) the classes of F-reflexive (respectively G-reflexive) objects. A main topic is the study of dualities induced by F and G between some full subcategories of $\mathcal{A}$ and $\mathcal{B}$. The restrictions of F and G to the classes of reflexive objects induce a duality $\mathrm{F} : \mathrm{Refl}_\delta \leftrightarrows \mathrm{Refl}_\zeta : \mathrm{G}$. Moreover, if $\mathrm{F} : \mathcal{C} \leftrightarrows \mathcal{D} : \mathrm{G}$ is a duality then $\mathcal{C} \subseteq \mathrm{Refl}_\delta$ and $\mathcal{D} \subseteq \mathrm{Refl}_\zeta$ (see [11]).

We also fix an δ-reflexive object Q, and, following ideas from [4], we will call the triple $\mathfrak{D} = (Q, \mathrm{F}, \mathrm{G})$ a *pointed pair of right adjoint contravariant functors*. Let $V = \mathrm{F}(Q)$. Then $\mathrm{add}(Q) \subseteq \mathrm{Refl}_\delta$ and $\mathrm{add}(V) \subseteq \mathrm{Refl}_\zeta$ (recall that $\mathrm{add}(X)$ denotes the class of all summands of finite direct sums of copies of X). We will denote by Faith_δ (Faith_ζ) the classes of all objects $X \in \mathcal{A}$ ($X \in \mathcal{B}$) such that δ_X (ζ_X) is a monomorphism, and we will call they δ-*faithful* (respectively ζ-*faithful*) objects. We recall that the natural transformations δ and ζ satisfy the identities

$$\mathrm{F}(\delta_X) \circ \zeta_{\mathrm{F}(X)} = 1_{\mathrm{F}(X)} \text{ for all } X \in \mathcal{A}$$

and

$$\mathrm{G}(\zeta_Y) \circ \delta_{\mathrm{G}(Y)} = 1_{\mathrm{G}(Y)} \text{ for all } Y \in \mathcal{B},$$

hence $\mathrm{F}(\mathcal{A}) \subseteq \mathrm{Faith}_\zeta$ and $\mathrm{G}(\mathcal{B}) \subseteq \mathrm{Faith}_\delta$.

Example 1.1. The typical example of such functors is the following: Let R and S be unital rings and Q an S-R-bimodule. Then the contravariant functors $\Delta = \mathrm{Hom}_R(-, Q) : \text{Mod-}R \to S\text{-Mod}$ and $\Delta' = \mathrm{Hom}_S(-, Q) : S\text{-Mod} \to \text{Mod-}R$ are right adjoint. If S is the endomorphism ring of Q then (Q, Δ, Δ') is a pointed pair of right adjoint contravariant functors.

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The study of dualities induced by this pair of functors is an important topic in Module Theory. The starting point was the papers [8] and [1]. During the time this topic developed important concepts as (f)-cotilting and costar module (see [6] and [10] for complete surveys on the subjects).

Another important example was exhibited by Castaño-Iglesias in [3].

Example 1.2. Let G be a group. If $R = \bigoplus_{x\in G} R_x$ and $S = \bigoplus_{x\in G} {}_xS$ are two G-graded unital rings, we will denote by $\mathrm{Mod}_{\mathrm{gr}}$-$R$ (respectively, by S-$\mathrm{Mod}_{\mathrm{gr}}$) the category of all G-graded unital right R- (respectively, left S-) modules (see [9]).

If $Q, M \in \mathrm{Mod}_{\mathrm{gr}}$-$R$ we consider the G-graded abelian group $\mathrm{HOM}_R(M,Q)$ whose homogeneous component in x is

$$_x\mathrm{HOM}_R(M,Q) = \{f \in \mathrm{Hom}_R(M,Q) \mid f(M_y) \subseteq Q_{xy},\ \text{for all } y \in G\}.$$

We note that $\mathrm{HOM}_R(Q,Q) = \mathrm{END}_R(Q)$ has a canonical structure of G-graded unital ring. If $M, N \in S$-$\mathrm{Mod}_{\mathrm{gr}}$ we consider the G-graded abelian group $\mathrm{HOM}_S(M,Q)$ whose homogeneous component in x is

$$\mathrm{HOM}_S(M,Q)_x = \{f \in \mathrm{Hom}_R(M,Q) \mid f({}_yM) \subseteq_{yx} Q,\ \text{for all } y \in G\}.$$

Then we have a pair of contravariant functors

$$\mathrm{H}^{\mathrm{gr}}_R = \mathrm{HOM}_R(-,Q_R) : \mathrm{Mod}_{\mathrm{gr}}\text{-}R \leftrightarrows S\text{-}\mathrm{Mod}_{\mathrm{gr}} : \mathrm{HOM}_S(-,{}_SQ) = {}_S\mathrm{H}^{\mathrm{gr}}.$$

If $Q \in \mathrm{Mod}_{\mathrm{gr}}$-$R$ and $S = \mathrm{END}_R(Q)$, then $(Q, \mathrm{H}^{\mathrm{gr}}_R, {}_S\mathrm{H}^{\mathrm{gr}})$ is a pointed pair of right adjoint contravariant functors.

In this note we continue a approach initiated by Castagño in [3]. In this paper the author generalizes the notion of costar module introduced by Colby and Fuller in [5] to Grothendieck categories. We continue this kind of study, generalizing a duality exhibited in [2, Theorem 2.8] to abelian categories.

2. Right pointed pairs of contravariant functors

In the following $\mathfrak{D}$ will denote a pointed pair of right adjoint contravariant functors $(Q, \mathrm{F}, \mathrm{G})$ between the abelian categories $\mathcal{A}$ and $\mathcal{B}$.

Lemma 2.1. *Let $\mathfrak{D}$ be a pointed pair of right adjoint contravariant functors. If*

$$(\sharp)\ 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

is an exact sequence in $\mathcal{A}$ then the unique homomorphism α, for which the diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\delta_Y} & & \downarrow{\scriptstyle\delta_Z} & & \\
0 & \longrightarrow & \mathrm{G}(\mathrm{Im}(\mathrm{F}(f))) & \longrightarrow & \mathrm{GF}(Y) & \xrightarrow{\mathrm{GF}(g)} & \mathrm{GF}(Z) & &
\end{array}$$

is commutative, is given by the formula $\alpha = \mathrm{G}(j) \circ \delta_X$, where $j : \mathrm{Im}(\mathrm{F}(f)) \to \mathrm{F}(X)$ is the inclusion map.

Proof. The existence of α comes from the universal property of the kernel. Moreover, α is unique.

Let $\mathrm{F}(f) = j \circ p$ be the canonical decomposition of $\mathrm{F}(f)$. Since $(\sharp)$ is an exact sequence, the sequence

$$0 \to \mathrm{F}(Z) \xrightarrow{\mathrm{F}(g)} \mathrm{F}(Y) \xrightarrow{p} \mathrm{Im}(\mathrm{F}(f)) \to 0$$

is exact, hence the sequence

$$0 \to \mathrm{G}(\mathrm{Im}(\mathrm{F}(f))) \overset{\mathrm{G}(p)}{\to} \mathrm{GF}(Y) \overset{\mathrm{GF}(g)}{\to} \mathrm{GF}(Z)$$

is also exact.

If we denote $\mathrm{G}(j)\circ\delta_X$ by α we have: $\mathrm{G}(p)\circ\alpha = \mathrm{G}(p)\circ\mathrm{G}(j)\circ\delta_X = \mathrm{G}(j\circ p)\circ\delta_X = \mathrm{GF}(f)\circ\delta_X = \delta_Y\circ f$ hence the following diagram is commutative with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\delta_Y} & & \downarrow{\scriptstyle\delta_Z} & & \\
0 & \longrightarrow & \mathrm{G}(\mathrm{Im}(\mathrm{F}(f))) & \xrightarrow{\mathrm{G}(p)} & \mathrm{GF}(Y) & \xrightarrow{\mathrm{GF}(g)} & \mathrm{GF}(Z) & &
\end{array}$$

□

The following result is a version for [3, Lemma 2.2] and [3, Proposition 2.3].

Lemma 2.2. *Let $\mathfrak{D}$ be a pointed pair of right adjoint contravariant functors.*

a) *An object $X \in \mathcal{A}$ is δ-faithful and $\mathrm{F}(X) \in \mathrm{gen}(V)$ if and only if there exists a monomorphism $f : X \to Q^n$ such that $\mathrm{F}(f)$ is an epimorphism.*

b) *F is exact with respect an exact sequence $0 \to Y \overset{f}{\to} X \overset{g}{\to} Z \to 0$ with $X \in \mathrm{Refl}_\delta$ and $Z \in \mathrm{Faith}_\delta$ if and only if $\mathrm{Im}(\mathrm{F}(f)) \in \mathrm{Refl}_\zeta$.*

Proof. a) Suppose that $X \in \mathrm{Faith}_\delta$ and there exists an epimorphism $V^n \overset{p}{\to} \mathrm{F}(X) \to 0$. Applying the functor G we obtain an monomorphism $\mathrm{G}(p) : \mathrm{GF}(X) \to \mathrm{G}(V^n) \cong Q^n$. Let $f = \mathrm{G}(p)\circ\delta_X$. Then $\mathrm{F}(f) = \mathrm{F}(\delta_X)\circ\mathrm{FG}(p)$. Then

$$\mathrm{F}(f)\circ\zeta_{V^n} = \mathrm{F}(\delta_X)\circ\mathrm{FG}(p)\circ\zeta_{V^n} = p,$$

hence $\mathrm{F}(f)$ is an epimorphism.

The converse implication is obvious.

b) Let $0 \to Y \overset{f}{\to} X \overset{g}{\to} Z \to 0$ be an exact sequence in $\mathcal{A}$ such that $X \in \mathrm{Refl}_\delta$ and $Z \in \mathrm{Faith}_\delta$. Let $\mathrm{F}(f) = j\circ p$ be the canonical factorization of $\mathrm{F}(f)$. Since ζ is a natural transformation we have the identity $\zeta_{\mathrm{F}(Y)}\circ j = \mathrm{FG}(j)\circ\zeta_{\mathrm{Im}(\mathrm{F}(f))}$.

By Lemma 2.1, the following diagram is commutative with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{f} & X & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\delta_X} & & \downarrow{\scriptstyle\delta_Z} & & \\
0 & \longrightarrow & \mathrm{G}(\mathrm{Im}(\mathrm{F}(f))) & \xrightarrow{\mathrm{G}(p)} & \mathrm{GF}(X) & \xrightarrow{\mathrm{GF}(g)} & \mathrm{GF}(Z) & &
\end{array}$$

where $\alpha = \mathrm{G}(j)\circ\delta_Y$.

Since X is F-reflexive and Z is δ-faithful it follows, from Snake Lemma, that α is an isomorphism hence $\mathrm{F}(\alpha)$ is an isomorphism.

We have $j = 1_{\mathrm{F}(Y)}\circ j = \mathrm{F}(\delta_Y)\circ\zeta_{\mathrm{F}(Y)}\circ j = \mathrm{F}(\delta_Y)\circ\mathrm{FG}(j)\circ\zeta_{\mathrm{Im}(\mathrm{F}(f))} = \mathrm{F}(\alpha)\circ\zeta_{\mathrm{Im}(\mathrm{F}(f))}$.

Since j is a monomorphism, $j = \mathrm{F}(\alpha)\circ\zeta_{\mathrm{Im}(\mathrm{F}(f))}$ and $\mathrm{F}(\alpha)$ is an isomorphism we conclude that $\mathrm{F}(f)$ is an epimorphism if and only if $\mathrm{Im}(\mathrm{F}(f)) \in \mathrm{Refl}_\zeta$. □

Theorem 2.3. *The following are equivalent for a pair $\mathfrak{D}$:*

a) $\mathrm{F} : \mathrm{cog}(Q) \rightleftarrows \mathrm{pres}(V)\cap\mathrm{Faith}_\zeta : \mathrm{G}$ *is a duality;*

b) i) $\mathrm{cog}(Q) = \mathrm{cop}(Q)$*;*

ii) F *is exact with respect exact sequences* $0 \to X \to Q^n \to Y \to 0$ *with* $Y \in \mathrm{cog}(Q)$.

Proof. a)⇒b) Let $X \in \mathrm{cog}(Q)$. From a) we have $\mathrm{F}(X) \in \mathrm{pres}(V)$ and $X \in \mathrm{Refl}_\delta$. Then there exists an exact sequence $V^m \to V^n \to \mathrm{F}(X) \to 0$ and hence the sequence $0 \to \mathrm{GF}(X) \to \mathrm{G}(V^n) \to \mathrm{G}(V^m)$ is exact. It follows that $0 \to X \to Q^n \to Q^m$ is exact which shows that $X \in \mathrm{cop}(Q)$, hence $\mathrm{cog}(Q) = \mathrm{cop}(Q)$.

Let $0 \to X \xrightarrow{f} Q^n \xrightarrow{g} Y \to 0$ be an exact sequence with $Y \in \mathrm{cog}(Q)$. Since $\mathrm{F}(Y) \in \mathrm{pres}(V)$ and $0 \to \mathrm{F}(Y) \xrightarrow{\mathrm{F}(g)} \mathrm{F}(Q^n) \xrightarrow{\mathrm{F}(f)} \mathrm{Im}(\mathrm{F}(f)) \to 0$ is exact we obtain that $\mathrm{Im}(\mathrm{F}(f)) \in \mathrm{pres}(V)$. But $\mathrm{Im}(\mathrm{F}(f)) \in \mathrm{Faith}_\zeta$ because $\mathrm{F}(X) \in \mathrm{Faith}_\zeta$. So $\mathrm{Im}(\mathrm{F}(f)) \in \mathrm{pres}(V) \cap \mathrm{Faith}_\zeta \subseteq \mathrm{Refl}_\zeta$. By Lemma 2.2, the sequence

$$0 \to \mathrm{F}(Y) \xrightarrow{\mathrm{F}(g)} \mathrm{F}(Q^n) \xrightarrow{\mathrm{F}(f)} \mathrm{F}(X) \to 0$$

is exact.

b)⇒a) Let $X \in \mathrm{cog}(Q)$. There exists an exact sequence $0 \to X \xrightarrow{f_1} Q^{m_1} \xrightarrow{f_2} Q^{m_2}$ hence the sequence $0 \to X \xrightarrow{f_1} Q^{m_1} \xrightarrow{f_2} Y \to 0$ is exact with $Y \in \mathrm{cog}(Q)$, where Y is $\mathrm{Im} f_2$. By ii) we obtain that the diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f_1"] \arrow[d, "\delta_X"'] & Q^{m_1} \arrow[r, "f_2"] \arrow[d, "\delta_{Q^{m_1}}"'] & Y \arrow[r] \arrow[d, "\delta_Y"'] & 0 \\
0 \arrow[r] & \mathrm{GF}(X) \arrow[r, "\mathrm{GF}(f_1)"] & \mathrm{GF}(Q^{m_1}) \arrow[r, "\mathrm{GF}(f_2)"] & \mathrm{GF}(Y) &
\end{tikzcd}$$

is commutative with exact sequences. Moreover, all vertical arrows are monomorphisms and $\delta_{Q^{m_1}}$ is an isomorphism. From the Snake Lemma we obtain δ_X is an isomorphism, hence $X \in \mathrm{Refl}_\delta$. Therefore $\mathrm{cog}(Q) \subseteq \mathrm{Refl}_\delta$.

Since $Y \in \mathrm{cog}(Q) = \mathrm{cop}(Q)$ there exists an exact sequence

$$0 \to Y \xrightarrow{g_1} Q^{n_1} \xrightarrow{g_2} Q^{n_2},$$

hence the sequence $0 \to Y \xrightarrow{g_1} Q^{n_1} \xrightarrow{g_2} Z \to 0$ is exact with $Z \in \mathrm{cog}(Q)$, where Z is $\mathrm{Im} g_2$. The sequences

$$0 \longrightarrow \mathrm{F}(Y) \xrightarrow{\mathrm{F}(f_2)} \mathrm{F}(Q^{m_1}) \xrightarrow{\mathrm{F}(f_1)} \mathrm{F}(X) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{F}(Z) \xrightarrow{\mathrm{F}(g_2)} \mathrm{F}(Q^{n_1}) \xrightarrow{\mathrm{F}(g_1)} \mathrm{F}(Y) \longrightarrow 0$$

are exact. Then the sequence

$$V^{n_1} \xrightarrow{\mathrm{F}(g_1 f_2)} V^{m_1} \xrightarrow{\mathrm{F}(f_1)} \mathrm{F}(X) \longrightarrow 0$$

is exact, hence $\mathrm{F}(X) \in \mathrm{pres}(V)$. But $\mathrm{F}(X) \in \mathrm{Faith}_\zeta$, so $\mathrm{F}(X) \in \mathrm{pres}(V) \cap \mathrm{Faith}_\zeta$. Therefore $\mathrm{F} : \mathrm{cog}(Q) \to \mathrm{pres}(V) \cap \mathrm{Faith}_\zeta$ is well-defined.

Let $A \in \mathrm{pres}(V) \cap \mathrm{Faith}_\zeta$. There is an exact sequence $V^m \xrightarrow{f} V^n \xrightarrow{g} A \to 0$, and applying G we obtain that the sequence $0 \to \mathrm{G}(A) \to Q^n \to Q^m$ is exact. Then $\mathrm{G}(A) \in \mathrm{cog}(Q)$. Therefore G is well defined.

Since the sequence $V^m \xrightarrow{f} V^n \xrightarrow{g} A \to 0$ is exact we have that the sequence

$$0 \to \mathrm{G}(A) \xrightarrow{\mathrm{G}(g)} \mathrm{G}(V^n) \xrightarrow{\mathrm{G}(f)} \mathrm{Im}(\mathrm{G}(f)) \to 0$$

is exact with $\mathrm{Im}(\mathrm{G}(f)) \in \mathrm{cog}(Q)$. From b)ii) we have that the sequence

$$0 \to \mathrm{F}(\mathrm{Im}(\mathrm{G}(f))) \overset{\mathrm{FG}(f)}{\to} \mathrm{FG}(V^n) \overset{\mathrm{FG}(g)}{\to} \mathrm{FG}(A) \to 0$$

is exact.

In the commutative diagram

$$\begin{tikzcd} V^n \arrow[r, "\zeta_{V^n}"] \arrow[d, "g"'] & \mathrm{FG}(V^n) \arrow[d, "\mathrm{FG}(g)"] \\ A \arrow[r, "\zeta_A"] & \mathrm{FG}(A) \end{tikzcd}$$

$\mathrm{FG}(g)$ and ζ_{V^n} are epimorphisms, so ζ_A is an epimorphisms. Since ζ_A is a monomorphism ($A \in \mathrm{Faith}_\zeta$) we obtain that $A \in \mathrm{Refl}_\zeta$. Therefore $\mathrm{pres}(V) \cap \mathrm{Faith}_\zeta \subseteq \mathrm{Refl}_\zeta$. □

Suppose that $\mathfrak{D} = (Q, \Delta, \Delta')$ is the (classical) pointed pair of right adjoint contravariant functors from Example 1.1. By [2, Theorem 3.4] it satisfies the equivalent conditions from Theorem 2.3 if and only if the conditions:

i) Δ is exact with respect exact sequences $0 \to X \to Q^n \to Y \to 0$ if and only if $Y \in \mathrm{cog}(Q)$,
ii) $\mathrm{F}(\mathrm{cog}(Q)) \subseteq \mathrm{gen}(V)$.

are satisfied. In the proof of this result it is used the fact that, in this particular setting, the class $\mathrm{Pres}(V) \cap \mathrm{Faith}_\zeta$ is closed with respect kernels of epimorphisms.

The property i) and the closure with respect kernels of epimorphisms are not valid for the general case, as it is showed in the next example.

Example 2.4. Let p be a prime integer, $\mathbb{J}_p$ the ring of p-adic integers and $\mathbb{Z}(p^\infty) \cong \mathbb{Z}_p/\mathbb{Z}$, where $\mathbb{Z}_p = \{\frac{m}{p^k} \mid m, k \in \mathbb{Z},\ k \geq 0\} \leq \mathbb{Q}$. Observe that $\mathbb{J}_p$ is the endomorphism ring of the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$. Moreover $\mathbb{Z}(p^\infty)$ is injective as a $\mathbb{Z}$-module, and also as a $\mathbb{J}_p$-module (see [7]).

We consider the functors

$$\mathrm{F} = \mathrm{Hom}_{\mathbb{J}_p}(-, \mathbb{Z}(p^\infty)) : \mathbb{J}_p\text{-Mod} \leftrightarrows \text{Mod-}\mathbb{Z} : \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}(p^\infty)) = \mathrm{G}.$$

If $Q = \mathbb{J}_p$ then $V = \mathrm{F}(Q) = \mathbb{Z}(p^\infty)$, and it is not hard to see that Q is F-reflexive.

If $K \in \mathrm{cog}(Q)$ then it is a finitely generated torsion-free $\mathbb{J}_p$-module, hence it is free. Then $\mathrm{cog}(Q) = \{\mathbb{J}_p^n \mid n \in \mathbb{N}\} = \mathrm{cop}(Q)$. Moreover, the $\mathbb{J}_p$-module $\mathbb{Z}(p^\infty)$ is injective, hence $(Q, \mathrm{F}, \mathrm{G})$ satisfies the condition b) in Theorem 2.3.

However,

(1) there exists an exact sequence

$$(\star)\ 0 \to p\mathbb{J}_p \to \mathbb{J}_p \to \mathbb{J}_p/p\mathbb{J}_p \to 0$$

such that F is exact with respect $(\star)$, but $\mathbb{J}_p/p\mathbb{J}_p \notin \mathrm{cog}(Q)$.

(2) since we have an exact

$$0 \to \mathbb{Z}(p) \to \mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty) \to 0,$$

the class $\mathrm{Pres}(V) \cap \mathrm{Faith}_\zeta$ is not closed with respect kernels of epimorphisms.

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