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Report

on the implementation of the project
PN-III-P4-ID-PCE-2020-0454 (contract number 75/2021)

Contributions to Silting Theory

2021– 2023

(A) The objectives

The main objectives of the project were concentrated on the following directions:

(I) Contributions to silting theory,

(II) Special classes and objects (in various contexts/categories).

According the the project plan, the concrete research topics, associated to these general objectives, were the following:

(I)(A) Silting and cosilting complexes in the derived category associated to a Grothendieck category:
- Classes (co)generated by (co)silting objects;
- Structures associated to Grothendieck categories which are connected to (co)silting complexes;
- Applications of the results. The case of group representations.

(I)(B) The transfer of the (co)silting properties via functors:
- The ascent-descent property for cotilting and cosilting complexes;
- Transferring the (co)silting property via functors.

(II)(A) Approximations and definable classes:
- Definable classes and pure-injective objects in triangulated categories;
- Rings and approximations; splitting properties.

(II)(B) Pure-injective objects in Grothendieck and triangulated categories:
- Pure-injective objects and direct products;
- Approximations and identities various categories.

The activities carried out to achieve these objectives are in accordance with those foreseen in the project proposal. Mainly, they were of the following types: information and/or research seminars, independent research activities, research visits, participation in scientific conferences, etc.

(B) The results

The main results are included in scientific papers that are in various stages of development or evaluation. We give below a description of these papers.

Scientific papers that are submitted for publications.

1. Simion Breaz: *A characterization of (co)silting objects*, https://arxiv.org/abs/2303.06843

The subject of this paper is connected with the directions (IA), (IB) şi (IIB). We prove that a characterization for silting objects that is already known in well generated triangulated categories is valid for the case of triangulated categories with coproducts. The proof can be dualized, and we obtain in this way a characterization for cosilting objects in triangulated categories with products: an object U from a triangulated category with coproducts is cosilting if and only if

(C1) $U \in {}^{\perp_{>0}}U$,
(C2) ${}^{\perp_{>0}}U$ is closed under direct products,
(C3) ${}^{\perp_{\mathbb{Z}}}U = 0$,

where ${}^{\perp_{>0}}U = \{X \mid \mathrm{Hom}(X, U[i] = 0)$ for all integers $i > 0\}$, iar ${}^{\perp_{\mathbb{Z}}}U = \{X \mid \mathrm{Hom}(X, U[i] = 0)$ for all integers $i\}$. Such a characterization was known only in the hypothesis that the object is pure-injective. In the last part of the paper, we apply these results to the case of (co)intermediate (co)silting objects, i.e. in the cases when we can replicate, by using some t-structures, the case of (co)silting complexes that are provided by bounded complexes of projectives (injectives).

2. Simion Breaz, George Ciprian Modoi: *Derived equivalences induced by good silting complexes*, to appear in Functor Categories, Model Theory, Algebraic Analysis and Constructive Methods, editor Alexander Martsinkovsky, book series Springer Proceedings in Mathematics and Statistics.

The subject of this paper is in connection with the themes (IA) şi (IB). It refers to the equivalences induced by silting complexes in derived categories associated to the categories of dg-modules over dg-algebras. In the first part of the paper we consider the case when the involved dg-modules are compact. It is proved that if U is a dg-B-A-bimodule that is compact over B then the derived functor associated to the covariant Hom induced by U as an A-module is fully faithful. If U is also compact as an A-module, then it induces an equivalence of derived categories. Some of these results can be extended in the case when our object is not compact: the good silting objects over coconective dg-algebras. In this situation we obtain equivalences at the level of the module categories that are similar with those that are valid for the classical case of the equivalences induced by tilting modules for module categories.

3. Andrei Marcus, Virgilius-Aurelian Minuță: *An exact sequence for the graded Picent*, to appear in Journal of Group Theory.

The subject of this paper is in connection with the themes (IA) and (IIB). Let G be a finite group, let k be a commutative ring, and let A be a crossed product of the k-algebra B with G. We denote by $\mathrm{hU}(A)$ the group of homogeneous units of A. For all $g \in G$, we choose $u_g \in \mathrm{hU}(A) \cap A_g$.

Consider the centralizer $\mathrm{C}_A(B)$ in A of the 1-component B. We know that $\mathrm{C}_A(B)$ is a G-graded G-algebra, and let

$$G[B] := \{g \in G \mid A_g \simeq B \text{ as } (B, B)\text{-bimodules}\}$$

be the inertia group of B as a (B, B)-bimodule. Then it is well known that $G[B]$ is a normal subgroup of G, and

$$\mathcal{C} := \mathrm{C}_A(B)_{G[B]} = \bigoplus_{h \in G[B]} \mathrm{C}_A(B)_h$$

is a strongly $G[B]$-graded G-acted subalgebra of $\mathrm{C}_A(B)$. We denote $H := G[B]$, $\mathcal{Z} := \mathcal{C}_1 = \mathrm{Z}(B)$ and $Z = \mathrm{U}(\mathcal{Z})$.

We denote by $\mathrm{Pic}_k(B)$ the Picard group of B relative to k. Recall that $\mathrm{Picent}(B)$ is the set of isomorphism classes $[P]$ of invertible (B, B)-bimodules P satisfying $zp = pz$ for all $p \in P$ and $z \in \mathcal{Z}$ (we will say that P is a bimodule over $\mathcal{Z}$). Then $\mathrm{Picent}(B)$ is a subgroup of $\mathrm{Pic}_k(B)$.

We define the groups $\mathrm{Pic}_k^{\mathrm{gr}}(A)$ and $\mathrm{Picent}^{\mathrm{gr}}(A)$ in a similar manner. Let $\mathrm{Pic}_k^{\mathrm{gr}}(A)$ be the group of isomorphism classes $[\tilde{P}]$ of G-graded invertible (A, A)-bimodules $\tilde{P}$ over k; and let $\mathrm{Picent}^{\mathrm{gr}}(A)$ be the subgroup of $\mathrm{Pic}_k^{\mathrm{gr}}(A)$, consisting of isomorphism classes $[\tilde{P}]$ of G-graded (A, A)-bimodules $\tilde{P}$ satisfying:

(1) $\tilde{P} \otimes_A \tilde{P}^* \simeq A$ and $\tilde{P}^* \otimes_A \tilde{P} \simeq A$ as G-graded (A, A)-bimodules, where $\tilde{P}^*$ is the A-dual of $\tilde{P}$.
(2) $\tilde{p}_g c = {}^g c \tilde{p}_g$ for all $g \in G$, $c \in \mathcal{C}$, $\tilde{p}_g \in \tilde{P}_g$.

Such bimodules are called G-graded (A, A)-bimodules over $\mathcal{C}$ and they form a category denoted by $A\text{-Gr}/\mathcal{C}\text{-}A$.

We denote

$$\mathrm{Aut}_{\mathcal{C}}^{\mathrm{gr}}(A) = \{\alpha \in \mathrm{Aut}_k(A) \mid \alpha(A_g) = A_g \text{ for all } g \in G, \text{ and } \alpha(c) = c \text{ for all } c \in \mathcal{C}\}.$$

Let $\tilde{P}$ be a G-graded (A,A)-bimodule over $\mathcal{C}$. For $\alpha,\beta \in \mathrm{Aut}^{\mathrm{gr}}_{\mathcal{C}}(A)$, we define a new G-graded (A,A)-bimodule ${}_{\alpha}\tilde{P}_{\beta}$ over $\mathcal{C}$, where ${}_{\alpha}\tilde{P}_{\beta} = \tilde{P}$ as k-modules, and with multiplication given by

$$a \cdot \tilde{p} \cdot a' = \alpha(a)\tilde{p}\beta(a').$$

For $u \in \mathrm{U}(A)$, denote $\iota_u : A \to A$, $\iota_u(a) = uau^{-1}$. We consider $\mathrm{Inn}_1(A) = \{\iota_u \mid u \in \mathrm{U}(B)\}$. Using this notations, we prove that:

1) There is an exact sequence of groups

$$1 \longrightarrow \mathrm{Inn}_1(A) \longrightarrow \mathrm{Aut}^{\mathrm{gr}}_{\mathcal{C}}(A) \xrightarrow{\varphi} \mathrm{Picent}^{\mathrm{gr}}(A),$$

where $\varphi(\alpha) = [{}_1A_{\alpha}]$.

2) If $[\tilde{P}]$, $[\tilde{Q}] \in \mathrm{Picent}^{\mathrm{gr}}(A)$, then $\tilde{P} \simeq \tilde{Q}$ as a G-graded $(A,\mathcal{C})$-bimodule over $\mathcal{C}$ if and only if there is $\alpha \in \mathrm{Aut}^{\mathrm{gr}}_{\mathcal{C}}(A)$ such that $\tilde{Q} \simeq {}_1\tilde{P}_{\alpha}$ as G-graded (A,A)-bimodules over $\mathcal{C}$. In particular,

$$\mathrm{Im}\,\varphi = \{[\tilde{P}] \in \mathrm{Picent}^{\mathrm{gr}}(A) \mid \tilde{P} \simeq A \text{ as } G\text{-graded } (A,\mathcal{C})\text{-bimodules over } \mathcal{C}\}.$$

We introduce the group $\mathrm{Pic}_{\mathcal{C}}(B)$ of isomorphism classes of invertible (B,B)-bimodules P satisfying $cp = pc$, for all $c \in \mathcal{C}$, $p \in P$. We denote by $\mathrm{Aut}^{\mathrm{gr}}_{kG}(\mathcal{C})$ the set of all k-linear automorphisms of $\mathcal{C}$ that preserve gradings and are also G-algebra maps.

In one of our main results, we present the Picent version of the Beattie-del Río exact sequence:

$$1 \longrightarrow \mathrm{H}^1(G/H,Z) \xrightarrow{\Phi} \mathrm{Picent}^{\mathrm{gr}}(A) \xrightarrow{\Psi} \mathrm{Picent}(B)^{G/H} \xrightarrow{\Theta} \mathrm{H}^2(G/H,Z).$$

By using this exact sequence, we built the following commutative diagram, with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{H}^1(G/H,Z) & \xrightarrow{\Phi} & \mathrm{Picent}^{\mathrm{gr}}(A) & \xrightarrow{\Psi} & \mathrm{Picent}(B)^{G/H} & \xrightarrow{\Theta} & \mathrm{H}^2(G/H,Z) \\
 & & \downarrow{\scriptstyle \mathrm{Inf}} & & \downarrow{\scriptstyle \subseteq} & & \downarrow{\scriptstyle \subseteq} & & \downarrow{\scriptstyle \mathrm{Inf}} \\
1 & \longrightarrow & \mathrm{H}^1(G,Z) & \xrightarrow{\Phi_k} & \mathrm{Pic}^{\mathrm{gr}}_k(A) & \xrightarrow{\Psi_k} & \mathrm{Pic}_k(B)^G & \xrightarrow{\Theta_k} & \mathrm{H}^2(G,Z) \\
 & & \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \tilde{\sigma}} & & \downarrow{\scriptstyle \sigma} & & \\
1 & \longrightarrow & \mathrm{H}^1(H,Z)^{G/H} & \xrightarrow{\phi} & \mathrm{Aut}^{\mathrm{gr}}_{kG}(\mathcal{C}) & \xrightarrow{\psi} & \mathrm{Aut}_k(\mathcal{Z})^G & &
\end{array}$$

$$\mathrm{Tg} : \mathrm{H}^1(H,Z)^{G/H} \longrightarrow \mathrm{H}^2(G/H,Z)$$

Moreover, there is a group homomorphism $\theta : \mathrm{H}^1(H, Z)^{G/H} \to \mathrm{Pic}^{\mathrm{gr}}_{\mathcal{Z}}(A_H)^{G/H}$ such that the following diagram is commutative.

$$\begin{array}{ccc} & & \mathrm{H}^1(H,Z)^{G/H} \\ & \swarrow{\scriptstyle\theta} & \downarrow{\scriptstyle\mathrm{Tg}} \\ \mathrm{Pic}^{\mathrm{gr}}_{\mathcal{Z}}(A_H)^{G/H} & \xrightarrow{\quad\Theta\quad} & \mathrm{H}^2(G/H,Z). \end{array}$$

Finally, we proved that there exists good isomorphisms in butterfly-like situations: Let $\hat{G}$ be another finite group and let $\hat{A}$ be a $\hat{G}$-graded crossed product with 1-component B. Assume that the maps $\varepsilon : \mathrm{hU}(A) \to \mathrm{Aut}_k(B)$ and $\hat{\varepsilon} : \mathrm{hU}(\hat{A}) \to \mathrm{Aut}_k(B)$ induced by conjugation satisfy $\mathrm{Im}\,\varepsilon = \mathrm{Im}\,\hat{\varepsilon}$. Denote $\hat{H} = \hat{G}[B]$. Then $G/H \simeq \hat{G}/\hat{H}$, and we have an isomorphism $\alpha : \mathrm{Picent}^{\mathrm{gr}}(A) \to \mathrm{Picent}^{\mathrm{gr}}(\hat{A})$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{H}^1(G/H, Z) & \xrightarrow{\Phi} & \mathrm{Picent}^{\mathrm{gr}}(A) & \xrightarrow{\Psi} & \mathrm{Picent}(B)^{G/H} & \xrightarrow{\Theta} & \mathrm{H}^2(G/H, Z) \\ & & \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\alpha} & & \| & & \downarrow{\scriptstyle\simeq} \\ 1 & \longrightarrow & \mathrm{H}^1(\hat{G}/\hat{H}, Z) & \xrightarrow{\hat{\Phi}} & \mathrm{Picent}^{\mathrm{gr}}(\hat{A}) & \xrightarrow{\hat{\Psi}} & \mathrm{Picent}(B)^{\hat{G}/\hat{H}} & \xrightarrow{\hat{\Theta}} & \mathrm{H}^2(\hat{G}/\hat{H}, Z). \end{array}$$

4. I. Simion, D. Testerman: *On minimal epimorphic subgroups in simple algebraic groups of rank 2*, accepted by Journal of Pure and Applied Algebra, https://arxiv.org/abs/2303.17440

Within the activities associated with objectives (IA) and (IIA), we also considered epimorphisms in several algebraic categories. For example, the category of linear algebraic groups admits non-surjective epimorphisms. Let $\phi : H \to G$ be a homomorphism of linear algebraic groups defined over an algebraically closed field. The map ϕ is an epimorphism if it admits right cancellation, i.e. whenever $\psi_1 \circ \phi = \psi_2 \circ \phi$ for homomorphisms ψ_1, ψ_2 we have $\psi_1 = \psi_2$. An *epimorphic subgroup* $H \subseteq G$ is a subgroup for which the inclusion map is an epimorphism. If ϕ is a non-surjective epimorphism then $\phi(H)$ is a proper epimorphic subgroup of G.

A study of epimorphic subgroups in linear algebraic groups was initiated by Bien and Borel in [C. R. Acad. Sci., Paris, Sér. I 315, No. 6, 649-653 (1992), Zbl 0767.20017, MR1183796; C. R. Acad. Sci., Paris, Sér. I 315, No. 13, 1341-1346 (1992), Zbl 0780.20028, MR1198999] where criteria for recognizing epimorphic subgroups are established. Their work complements results of Bergman from the unpublished manuscript 'Epimorphisms of Lie algebras', where non-surjectivity of epimorphisms is studied for Lie algebras, and extends the list of categories for which the difference between surjectivity and epimorphicity was studied by Reid in [Invent. Math. 9, 295-307 (1970), Zbl 0191.13503, MR0260829]. It follows from the results of Bien and Borel that minimal epimorphic subgroups are solvable and that maximal proper epimorphic subgroups are parabolic subgroups. Presently, a classification of epimorphic subgroups appears out of reach.

In this article, the authors consider simple groups of rank 2 defined over fields of positive characteristic. The main result of the paper states that, for a simple algebraic group of rank 2 defined over an algebraically closed field of characteristic $p > 0$, the minimal dimension of a closed epimorphic subgroup is 3.

5. Simion Breaz, Tomasz Brzezinski, Bernard Rybolowicz, Paolo Saracco: *Heaps of modules: Categorical aspects*,
https://arxiv.org/abs/2311.01979

The approached research topic falls within the directions (IA) and (IB) and continues to research the areas initiated in the work [8] from this list.

The main subject consists in the study of the category of heaps of modules over a truss, with a special attention to categorical constructions (limits and colimits) realized in this category. We often use various interpretations of the category of heaps of modules by using categories of pointed modules or affine modules over some rings associated to the initial trusses.

It is proved that there exists an isomorphism of categories between the category of heaps over a fixed truss$T$ and the category of affine modules over the ring $R(T)$, associated to T by using a universal property. This isomorphism can be used to describe the above mentioned categorical constructions. The limits can be computed in the same way as in the category of pointed modules over T, but for colimits a careful analysis is required. For instance, in order to construct the coproducts, we need to add copies of the Doroh extension of the ring $R(T)$.

We also consider two ways to introduce exact sequences in the category of heaps of modules. One of them use the exact sequences that came from the abelian category of pointed T-modules, and the other one use the Barr exact sequences. We provide, in the last part of the paper, a characterization for the Barr exact sequences by using the exact sequences that came from the exact structure of the category of pointed T-modules.

6. Simion Breaz, Michal Hrbek, George Ciprian Modoi: *Silting, cosilting, and extensions of commutative rings*,
https://arxiv.org/abs/2204.01374

This paper concerns research activities related to all the research topics previously mentioned: (I)(A), (I)(B) and (II)(B). Its main purpose is to study the behaviour of the bounded silting complexes when applying ring morphisms induced functors (scalar extension/induction covariants functors or scalar extension/coinduction contravariants functors), especially for the commutative rings case.

More precisely, we consider a morphism $\lambda : R \to S$ of (commutative) rings and the transfers determined by the induced derived functors between the derived categories $\mathbf{D}(R)$ and $\mathbf{D}(S)$: $- \otimes^{\mathbf{L}}_R S : \mathbf{D}(R) \to \mathbf{D}(S)$ and $\mathbf{R}\mathrm{Hom}_R(S, -) : \mathbf{D}(R) \to \mathbf{D}(S)$. We also consider the topological transfer of the cofinite type cosilting complexes which can be done via the natural map $\lambda^\star$: $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ induced by λ between the Zarisky spectra of the rings R and S.

We prove that the silting property is preserved by the functor $-\otimes^{\mathbf{L}}_R S$ and that $\mathbf{R}\mathrm{Hom}_R(S, -)$ preserves the objects properties of being pure-injective and cosilting. Moreover, these functors preserve the (co)finite types of the considered (co)silting objects. For the (co)finite type (co)silting objects we can also apply the previously mentioned topological transfer $\lambda^\star$ because it is continuous with respect to Hochster topologies. We prove that these transfers are (up to an equivalence) the transfers provided by the derived functors. If λ is faithfully flat then $\lambda^\star$ is also closed and this allows us to identify the (co)finite type (co)silting complexes from $\mathbf{D}(S)$ which result, up to an equivalence, by using the derived functors.

On the otherside, as for the n-tilting modules, it is not clear if the derived of scalar extension functors induced by faithfully flat morphisms reflect the silting and cosilting properties for bounded complexes of length at least 3. In the last part of the paper we study this problem. We prove that the previous described phenomenon is present for the most of the interesting cases: it happens for cosilting complexes which are duals of some projective complexes, and, for the silting case, it happens for morphisms which give Zariski localizations and for all the morphisms which have as domains noetherian rings, rings with finite pure global dimension (in particular, for rings of cardinality $\aleph_n$ with $n > 0$). As a corollary, we prove that the n-tilting property of modules behaves in the same way. In this respect, we use a lemma stated under the hypothesis that "the pure-injective objects generate the derived category". It is still an open problem to decide if this hypothesis is valid for all the commutative rings.

We also use and prove a result which may be an independent interest: a characterization of the bounded silting complexes in which the standard condition of the class $T^{\perp_{>0}}$ to be closed under direct sums is replaced by the condition $\mathrm{Add}(T) \subseteq T^{\perp_{>0}}$. This result also generalizes a similar characterization proved for n-tilting modules by Positselski and Šťovíček.

7. Simion Breaz, Andrei Marcus, George Ciprian Modoi: *Support τ-tilting modules and semibricks over group graded algebras*, J. Algebra 637 (2024), 90–111.
https://arxiv.org/abs/2209.02992

This paper concerns research activities related to the research topics (I)(A) and (I)(B). We consider a k-algebra A of finite dimension which is strongly G-graded (over the finite group G) and we denote by B its 1-component (k is a field). Under these circumstances, the induction functor $\mathrm{Ind}_B^A = A \otimes_B -$ and the scalar restriction functor Res_B^A form a Frobenius pair (are biadjoint functors). Moreover, Ind_B^A is separable. If $\mathrm{char}(k)$ does not divide the order of G

then Res_B^A is also separable. We study how are the τ-tilting pairs transferred by these functors. Our study is based on the following important information: if B is self-injective then Ind_B^A commutes with the projective covers: it takes minimal projective presentations into minimal projective. This result extends a result proved by E. C. Dade for Res_B^A.

Using these results, we prove that if M is an s-τ-tilting B-module then $\mathrm{Ind}_B^A M$ is also s-τ-tilting if and only if M is G-invariant and that a similar result holds for Res_B^A too. We also present connections with some other objects/structures related to s-τ-tilting modules (like semibricks and finite functorial torsion classes). These results generalize and explain the phenomena presented by Koshio and Kozakai for finite groups representations.

8. Simion Breaz, Tomasz Brzeziński, Bernard Rybolowicz, Paolo Saracco: *Heaps of modules and affine spaces*, Annali di Matematica Pura ed Applicata (1923–). https://doi.org/10.1007/s10231-023-01369-0 (2023)

This paper concerns research activities related to (I)(A). Its purpose is to realize a detailed study of some structures related to module categories which enable the extension of some Baer-Kaplansky type theorems (which identify objects by means of their endomorphisms) from particular classes of abelian groups (torsion groups) to some results valid for all modules. These structures, called *heaps of modules*, can be characterized, from a geometrical perspective, as affine spaces associated to some module categories.

For this, we consider an extension of the ring notion, truss, in which the additive operation is replaced by a ternary operation which satisfies the Mal'cev associativity and idempotence axioms. Various structures over these ring generalizations (T-groups, T-modules, heaps of T-modules) are described and connected. The main result describes an equivalence between the category of the heaps of T-modules and the category of the affine spaces over T.

In the final part of the paper we present some examples of various structures from different fields which can be organized as heaps of modules: algebraic structures associated to Yang-Baxter type equations solutions, a kind of geometric structures (connections) and contractions in homotopy categories associated to module categories (in particular, splittings).

9. Simion Breaz, Cristian Rafiliu: *Decompositions of matrices by using commutators*, Linear Algebra and its Applications, 662 (2023), 39–48; https://arxiv.org/abs/2209.03195

This paper concerns research activities related to (I)(A). For the study of endomorphism rings associated to free modules of infinite rank we use two of their properties: any element of such a ring is a commutator and such a ring is isomorphic to any of the square matrix rings with entries in this ring. In this paper we study decompositions of 3×3 matrices with entries in an arbitrary ring for which the trace is a commutator. We prove that given three splitting polynomials of degree 3 with roots in the center of the considered ring, these matrices can be written as sums of 3 matrices, each of them annihilated by one of the given polynomials.

These decompositions are later used for getting some informations on the endomorphism rings of infinite rank free modules, and also for studying bounded operators associated to the complex Hilbert spaces. It is also proved that the simple rings obtained by factorizing endomorphism rings of infinite dimensional vector spaces through the corresponding maximal ideals can be decomposed, using the ternary operations associated to the addition (heaps), using three trusses which induce the "brace" structures defined by Rump for constructing solutions of the set theoretic Yang-Baxter equation.

10. Cs. Szántó, I. Szöllősi: *On some Ringel-Hall polynomials associated to tame indecomposable modules*, Journal of Pure and Applied Algebra, vol. 228 (2024), article number 107555, available online 7 November 2023, 40 pages, https://doi.org/10.1016/j.jpaa.2023.107555.

This paper concerns research activities related to the first two research topics previously mentioned: (I)(A) and (I)(B). Let k be an arbitrary field and Q an acyclic quiver of tame type (i.e. of type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$). Consider the path algebra kQ and the category of finite dimensional right modules Mod-kQ. The rational Ringel-Hall algebra $\mathcal{H}(kQ)$ of the algebra kQ has as $\mathbb{Q}$-basis the isomorphism classes $[M]$ from Mod-kQ and the multiplication is defined by $[N_1][N_2] = \sum_{[M]} F^M_{N_1 N_2}[M]$. The structure constants $F^M_{N_1 N_2} = |\{U \subseteq M |\ U \cong N_2,\ M/U \cong N_1\}|$ are called Ringel-Hall numbers.

Far reaching analogues of the classical Hall algebras (associated with discrete valuation rings), these Ringel-Hall algebras were introduced by Ringel for a large class of rings, namely finitary rings, including in particular path algebras of quivers over finite fields. Ringel-Hall algebras provided a new approach to the study of quantum groups using the representation theory of finite dimensional algebras and they can also be used successfully in the theory of cluster algebras. Moreover they play an essential role in the investigation of the structure of the module category. So in this way we can relate them to tilting and silting theory.

Due to a result of Hubery we know that in tame cases the Ringel-Hall numbers are rational polynomials in q with respect to so-called decomposition classes of modules, so we can call them Ringel-Hall polynomials. If we are looking to Ringel-Hall polynomials associated to indecomposable modules in various tame cases, we do not have too much information about them. In a previously reported paper the authors determined all the Ringel-Hall polynomials associated to indecomposable modules in the special tame case of type $\tilde{D}_4$.

In this paper we determine all the tame Ringel-Hall polynomials associated to indecomposable modules of defect belonging to the set $\{-2, -1, 0, 1, 2\}$. Compared to the case $\tilde{D}_4$ this is a much more harder task, needing the introduction of new, generic tools. It is surprising that, as in the $\tilde{D}_4$ case, also in the general tame case we will essentially have only three families of

Ringel-Hall polynomials (with a single member for each degree). The first family corresponds to non-regular indecomposables and the second describes those involving non-homogeneous regulars. It turns out that the second family is very closely related to the first, so these two families are essentially the same, involving indecomposables of discrete type (preinjectives, preprojectives and non-homogeneous regulars). The third family corresponds to the continuous cases (when one of the indecomposables is regular homogeneous).

11. Simion Breaz: *On a theorem of Stelzer for some classes of mixed groups.* Mediterr. J. Math. 19, No. 4, Paper No. 159, 14 p. (2022).

The subject of this paper is connected with the research direction (IIB). We provide classes of mixed (abelian) groups that have the property that if a group from such a class has the cancellation property then its Walk-endomorphism ring has the unit lifting property. We prove a version of a theorem of Stelzer to self-small mixed groups of finite torsion-free rank.

Moreover, in some particular cases we can use this property to obtain characterizations for the cancellation property by using the stable range. For instance, in Teorema 4.6 we prove that a strongly indecomposable self-small mixed group of torsion-free rank at most 4 has the cancellation property if and only if it is of the form $B \oplus H$, where B is a finite group and H is a group whose Walk-endomorphism rink has 1 in its stable range.

12. Cs. Szántó, I. Szöllősi: *Ringel-Hall polynomials associated to a quiver of type $\tilde{D}_4$*, Periodica Mathematica Hungarica (2023), published online 19 September 2023, 25 pages, https://doi.org/10.1007/s10998-023-00549-y

The subject of the paper is connected with the directions (IA) and (IIB). Let k be an arbitrary finite field with q elements and Q a quiver of tame type $\tilde{D}_4$. Consider the path algebra kQ and the category of finite dimensional right modules Mod-kQ. The rational Ringel-Hall algebra $\mathcal{H}(kQ)$ of the algebra kQ has as $\mathbb{Q}$-basis the isomorphism classes $[M]$ from Mod-kQ and the multiplication is defined by $[N_1][N_2] = \sum_{[M]} F^M_{N_1N_2}[M]$. The structure constants $F^M_{N_1N_2} = |\{U \subseteq M |\ U \cong N_2,\ M/U \cong N_1\}|$ are called Ringel-Hall numbers.

Far reaching analogues of the classical Hall algebras (associated with discrete valuation rings), these Ringel-Hall algebras were introduced by Ringel for a large class of rings, namely finitary rings, including in particular path algebras of quivers over finite fields. Ringel-Hall algebras provided a new approach to the study of quantum groups using the representation theory of finite dimensional algebras and they can also be used successfully in the theory of cluster algebras. Moreover they play an essential role in the investigation of the structure of the module category. So in this way we can relate them to tilting and silting theory.

Due to a result of Hubery we know that in tame cases the Ringel-Hall numbers are rational polynomials in q with respect to so-called decomposition classes of modules, so we can call them Ringel-Hall polynomials. If we are looking to Ringel-Hall polynomials associated to indecomposable modules in various tame cases, we do not have too much information about them. In the present paper the authors determine all the Ringel-Hall polynomials associated to indecomposable modules in the special tame case of type $\tilde{D}_4$.

13. S. Breaz, Y. Zhou: *When is every non central-unit a sum of two nilpotents?* apărută în Algebra and coding theory, editor A. Leroy, Contemp. Math. 785 (2023), 47–55.

In connection with (IIB), in this paper we study the rings that have the property that all non-central units are sums of two nilpotents. In the main result we prove that if such a ring is not commutative then it is simple, and if it is commutative then it is local with nil Jacobson radical. An example of a simple non-commutative ring which has the property is provided. Moreover, we prove that we can consider this construction minimal since, if we add the hypothesis that the simple ring is right Goldie then it is a field.

14. Cs. Szántó, I. Szöllősi: *Combinatorial methods in the representation theory of finite dimensional tame algebras*, monograph 2023.

Consider the category of representations of tame (affine, Euclidean) quivers over a finite field, i.e the category of finite dimensional modules over the path algebra kQ, where Q is of type $\mathbb{A}_m$ $(m \geq 1)$, $\mathbb{D}_m$ $(m \geq 4)$, $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ and k is finite. Note that kQ with k finite is a finitary ring, i.e the group of extensions of modules is finite. Our aim is to count certain monomorphisms, epimorphisms, automorphisms and extensions mainly for indecomposables. For this we will need many tools, but of special importance is the so called Schofield induction via orthogonal exceptional pairs.

Knowing the number of extensions leads us to Ringel-Hall algebras with a large spectrum of applications. More precisely the structure coefficients of the Ringel-Hall algebra associated to kQ (called Ringel-Hall numbers) are (up to automorphisms) the numbers of extensions with given middle middle terms.

In case of Ringel-Hall algebras corresponding to Dynkin quivers and tame quivers we know due to Ringel, Hubery, respectively Deng and Ruan that the structure coefficients of the multiplication (the Ringel-Hall numbers) are polynomials in the number of elements of the base field. We will call these polynomials Ringel-Hall polynomials. If we are looking at Ringel-Hall polynomials associated to indecomposable modules, the ones in the Dynkin case are known (due to Ringel) and have degree up to 5, however we do not have too much information about the Ringel-Hall ones in the tame case.

The present monograph records the progresses made by the authors in the last 15 years regarding tame Ringel-Hall polynomials and their various applications, using a large spectrum of combinatorial, computational and representation theoretical tools.

We will apply our knowledge on Ringel-Hall polynomials in the theory of Gabriel-Roiter measures. The Gabriel-Roiter measure (GR measure for short) was introduced by Gabriel in order to give a combinatorial interpretation of the induction scheme used by Roiter in his proof of the first Brauer-Thrall conjecture. Ringel used it as a foundation tool for the representation theory of Artin algebras.

We also determine cardinalities of Kronecker quiver Grassmannians via Ringel-Hall numbers, a result of great importance in cluster theory.

(C) Disemination of the results

The disemination of the results was done through the following international conference talks or research workshops abroad:

(1) S. Breaz: The 10-th Congress of Romanian Mathematicians, 30.06-05.07 2023, Pitesti, Romania. Talk: Change of scalars and bounded silting complexes.
(2) S. Breaz: Trends in Representation Theory and Related Topics, 12-16 Septembrie 2023, Babeş-Bolyai University Cluj-Napoca, Romania, `https://math.ubbcluj.ro/trtrt2023`. Talk: On a characterization of (co)silting objects.
(3) Cs. Szanto: Summer school, Quiver Representations, Quiver Varieties and Combinatorics (BIP Blended Intensive Program), 22-26 Mai 2023, Bologna University, Italia, `https://eventi.unibo.it/bip-quiver/summer-school`. Talk: Ringel-Hall polynomials associated to tame quivers.
(4) Cs. Szanto: Trends in Representation Theory and Related Topics, 12-16 Septembrie 2023, Babeş-Bolyai University Cluj-Napoca, Romania, `https://math.ubbcluj.ro/trtrt2023`. Talk: Ringel-Hall polynomials associated to tame quivers.
(5) A. Marcus: 17th International Conference on Applied Mathematics and Computer Science, 11 – 13 Iulie 2023, Technical University Cluj-Napoca. Talk: Support tau-tilting modules and semibricks over group graded algebras.
(6) G. C. Modoi: The 10-th Congress of Romanian Mathematicians, 30.06-05.07 2023, Pitesti, Romania. Talk: Migration of silting-like properties via adjoint pairs.
(7) I.-I. Simion: The Tenth Congress of Romanian Mathematicians, June 30-July 5, 2023, Pitesti, Romania. Talk: On the product expansion of normal subsets in simple groups.
(8) V.-A. Minuță: The Tenth Congress of Romanian Mathematicians, 30 June- 5 July 2023, Pitești, Romania. Talk: An exact sequence for the graded Picent group
(9) V.-A. Minuță: 17th International Conference on Applied Mathematics and Computer Science, 11-13 July 2023, Cluj-Napoca, Romania. Talk: An exact sequence for the graded Picent group

(10) I.-I. Simion: Trends in Representation Theory and Related Topics, September 12-16, 2023, Cluj-Napoca, Romania, `https://math.ubbcluj.ro/trtrt2023`. Talk: Epimorphic subgroups in simple algebraic groups.

(11) V.-A. Minuță: Trends in Representation Theory and Related Topics, 12-16 September 2023, Cluj-Napoca, Romania, `https://math.ubbcluj.ro/trtrt2023`. Talk: Relations between module triples, and derived equivalences for wreath products

(12) Simion Breaz: Malga Seminar Padova-Verona, Universita di Verona 24 mai 2022. Talk: Silting complexes and extensions of commutative rings.

(13) Simion Breaz: Functor Categories, Model Theory, and Constructive Category Theory, Universidad de Almeria, 11-15.07.2022. Talk: Change of scalars functors and silting complexes.

(14) Simion Breaz: Algebra Seminar, Charles University, Prague 10-12 octombrie 2022. Talk: The Baer-Kaplansky Theorem and Heaps of Modules.

(15) Simion Breaz: Hopf algebras, monoidal categories and related topics IMAR, Bucuresti, 27-29.07.2022. Talk: Heaps of Modules

(16) George Ciprian Modoi: Functor Categories, Model Theory, and Constructive Category Theory, Universidad de Almeria, 11-15.07.2022. Talk: Not necessarily compact approximability via silting theory.

(17) Tudor Micu: Young Researchers' Conference on Non-Archimedean and Tropical Geometry, Universität Regensburg, 1-5 August 2022. Talk: The Structure of Special Fibers through Valuations.

(18) Csaba Szanto: Algebra Seminar, Renyi Institute, Budapest, 7 martie 2022. Talk: Ringel-Hall polynomials associated to a quiver of type $\tilde{D}_4$.

(19) Andrei Marcus: Hopf algebras, monoidal categories and related topics IMAR, Bucuresti, 27-29.07.2022. Talk: Tilting complexes over a G-graded G-algebra.

(20) S. Breaz: Homological Methods in Representation Theory. A conference in honour of Lidia Angeleri Hugel; 03 - 08 October 2021; Fraueninsel (Chiemsee), Abtei Frauenworth, Germany; Talk: Transfer of homological properties along some canonical functors.

(D) Documentation/research visits

Our team members participated to several conferences and they made some research visits. Among these, we would like to mention:

(1) S. Breaz: Homological Algebra and Representation Theory, 10 –14 July 2023, Karlovasi, Samos, Greece.

(2) G. C. Modoi: Homological Algebra and Representation Theory, 10 –14 July 2023, Karlovasi, Samos, Greece.

(3) G. C. Modoi: Trends in Representation Theory and Related Topics, 12-16 Septembrie 2023, Babeş-Bolyai University Cluj-Napoca.

(4) A. Marcus: The 10-th Congress of Romanian Mathematicians, 30.06-05.07 2023, Pitesti, Romania.
(5) A. Marcus: Trends in Representation Theory and Related Topics, 12-16 Septembrie 2023, Babeş-Bolyai University Cluj-Napoca.
(6) T. Micu: Trends in Representation Theory and Related Topics, 12-16 Septembrie 2023, Babeş-Bolyai University Cluj-Napoca.
(7) C. Pelea: Trends in Representation Theory and Related Topics, 12-16 Septembrie 2023, Babeş-Bolyai University Cluj-Napoca.
(8) T. Micu: - Summer School on Higher Structures in Algebra and Geometry; 13-18 August 2023; Nordfjordeid, Norway.
(9) T. Micu: Tau-Tilting Research School; 5-8 September 2023; Köln, Germany.
(10) T. Micu: Representation Theory and Triangulated Categories; 26-30 September 2022; Paderborn, Germany.
(11) V.-A. Minuță: Representation Theory and Triangulated Categories; 26-30 September 2022; Paderborn, Germany.
(12) C.-G. Modoi: Representation Theory and Triangulated Categories; 26-30 September 2022; Paderborn, Germany.
(13) C.-G. Modoi: Homological Methods in Representation Theory. A conference in honour of Lidia Angeleri Hugel; 03 - 08 October 2021; Fraueninsel (Chiemsee), Abtei Frauenworth, Germany.

(E) Conferences, talks given by invited speakers as part of our grant activities:

We organized the conference *Trends in Representation Theory and Related Topics*, 12-16 September 2023, Babeş-Bolyai University Cluj-Napoca, Romania: https://math.ubbcluj.ro/trtrt2023. The following researchers were invited speakers at this conference: Ana Agore (Institute of Mathematics of the Romanian Academy), Ana Bălibanu (Louisiana State University), Tomasz Brzeziński (Swansea University), Stefaan Caenepeel (Vrije Universiteit Brussel), Călin Chindriș (University of Missouri), Dan Ciubotaru (University of Oxford), Sorin Dăscălescu (Bucharest University), Meinolf Geck (Universität Stuttgart), Dolors Herbera (Universitat Autònoma de Barcelona), Michal Hrbek (Institute of Mathematics of the Czech Academy of Sciences), Alexander Martsinkovsky (Northeastern University), Gigel Militaru (Bucharest University), Bernard Rybołowicz (Heriot-Watt University), Manuel Saorín (Universidad de Murcia), Radu Stancu (Université de Picardie), Alberto Tonolo (Università degli Studi di Padova), Jan Trlifaj (Univerzita Karlova v Praze), Jan Žemlička (Univerzita Karlova v Praze).

Also, the following researchers gave lectures during our grant seminars: Tomasz Brzeziński, Kenny De Commer, Attila Maróti, Yuta Kozakai, Emil Horobet, Jonathan Gruber.

(F) Indicators achieved

Here is an outline of our grant accomplishments. Our initial plan envisaged the submission of eight papers as estimated verifiable results.

The obtained **results** are the following:

- 13 submitted papers; among these,
  - 8 papers are accepted by Web of Science indexed journals (and they are in different stages of the editorial process),
  - 2 papers are accepted in Contemporary Mathemathics series collective volumes and Springer Proceedings in Mathematics and Statistics;
  - 3 papers are submitted to Web of Science indexed journals;
- 20 talks at international conferences and research workshops organized by teams in other universities;
- 10 participation at international conferences;
- one monograph, in press;
- 1 international conference organized as part of this project.

(G) Estimated impact

We estimate that the new information concerning silting theory that we obtained in our research grant activities will be appreciated by this field scientific community. Our results concern several settings: the general theory of silting objects in triangulated categories as well as some particular situations (derived categories over commutative rings or dg-algebras, or τ-tilting theory for some generalizations of group algebras) which require specific approach and special techniques. Our team members investigations also provided important results on module categories over finite dimensional algebras (like group algebras or path algebras over some special quivers) or on some methods to associate affine structures to module categories. These results were well received by specialists and the papers which contain them were accepted for publication in highly visible journals.

Cluj-Napoca,
28.11.2023

Prof. dr. Simion-Sorin Breaz