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Report

on the implementation of the project
PN-III-P4-ID-PCE-2020-0454 (contract number 75/2021)
Contributions to Silting Theory
for the year 2022

(A) The scientific description

We continued the research activities with respect to the algebraic structures related to silting theory. We also investigated connections with other types of mathematical objects.

The research topics we approached respected the project plan. These are:

(I)(A) Grothendieck categories associated structures related to (co)silting complexes.

(I)(B) Cosilting properties transfer using functors.

(II)(A) Definable classes and pure-injective objects in triangulated categories.

Our results are included in some papers which are in different work stages. We describe these results and we mention the papers which include them.

Submitted papers:

1. Simion Breaz, Michal Hrbek, George Ciprian Modoi: *Silting, cosilting, and extensions of commutative ring*, https://arxiv.org/abs/2204.01374

This paper concerns research activities related to all the research topics previously mentioned: (I)(A), (I)(B) and (II)(B). Its main purpose is to study the behaviour of the bounded silting complexes when applying ring morphisms induced functors (scalar extension/induction covariants functors or scalar extension/coinduction contravariants functors), especially for the commutative rings case.

More precisely, we consider a morphism $\lambda : R \to S$ of (commutative) rings and the transfers determined by the induced derived functors between the derived categories $\mathbf{D}(R)$ and $\mathbf{D}(S)$: $- \otimes^{\mathbf{L}}_R S : \mathbf{D}(R) \to \mathbf{D}(S)$ and $\mathbf{R}\mathrm{Hom}_R(S, -) : \mathbf{D}(R) \to \mathbf{D}(S)$. We also consider the topological transfer of the cofinite type cosilting complexes which can be done via the natural

map $\lambda^\star : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ induced by λ between the Zarisky spectra of the rings R and S.

We prove that the silting property is preserved by the functor $-\otimes^{\mathbf{L}}_R S$ and that $\mathbf{R}\mathrm{Hom}_R(S,-)$ preserves the objects properties of being pure-injective and cosilting. Moreover, these functors preserve the (co)finite types of the considered (co)silting objects. For the (co)finite type (co)silting objects we can also apply the previously mentioned topological transfer $\lambda^\star$ because it is continuous with respect to Hochster topologies. We prove that these transfers are (up to an equivalence) the transfers provided by the derived functors. If λ is faithfully flat then $\lambda^\star$ is also closed and this allows us to identify the (co)finite type (co)silting complexes from $\mathbf{D}(S)$ which result, up to an equivalence, by using the derived functors.

On the otherside, as for the n-tilting modules, it is not clear if the derived of scalar extension functors induced by faithfully flat morphisms reflect the silting and cosilting properties for bounded complexes of length at least 3. In the last part of the paper we study this problem. We prove that the previous described phenomenon is present for the most of the interesting cases: it happens for cosilting complexes which are duals of some projective complexes, and, for the silting case, it happens for morphisms which give Zariski localizations and for all the morphisms which have as domains noetherian rings, rings with finite pure global dimension (in particular, for rings of cardinality $\aleph_n$ with $n > 0$). As a corollary, we prove that the n-tilting property of modules behaves in the same way. In this respect, we use a lemma stated under the hypothesis that "the pure-injective objects generate the derived category". It is still an open problem to decide if this hypothesis is valid for all the commutative rings.

We also use and prove a result which may be an independent interest: a characterization of the bounded silting complexes in which the standard condition of the class $T^{\perp_{>0}}$ to be closed under direct sums is replaced by the condition $\mathrm{Add}(T) \subseteq T^{\perp_{>0}}$. This result also generalizes a similar characterization proved for n-tilting modules by Positselski and Šťovíček.

2. Simion Breaz, Andrei Marcus, George Ciprian Modoi: *Support τ-tilting modules and semibricks over group graded algebras*,
https://arxiv.org/abs/2209.02992

This paper concerns research activities related to the research topics (I)(A) and (I)(B). We consider a k-algebra A of finite dimension which

is strongly G-graded (over the finite group G) and we denote by B its 1-component (k is a field). Under these circumstances, the induction functor $\mathrm{Ind}_B^A = A \otimes_B -$ and the scalar restriction functor Res_B^A form a Frobenius pair (are biadjoint functors). Moreover, Ind_B^A is separable. If $\mathrm{char}(k)$ does not divide the order of G then Res_B^A is also separable. We study how are the τ-tilting pairs transferred by these functors. Our study is based on the following important information: if B is self-injective then Ind_B^A commutes with the projective covers: it takes minimal projective presentations into minimal projective. This result extends a result proved by E. C. Dade for Res_B^A.

Using these results, we prove that if M is an s-τ-tilting B-module then $\mathrm{Ind}_B^A M$ is also s-τ-tilting if and only if M is G-invariant and that a similar result holds for Res_B^A too. We also present connections with some other objects/structures related to s-τ-tilting modules (like semibricks and finite functorial torsion classes). These results generalize and explain the phenomena presented by Koshio and Kozakai for finite groups representations.

3. Simion Breaz, Tomasz Brzeziński, Bernard Rybolowicz, Paolo Saracco: *Heaps of modules and affine spaces*, https://arxiv.org/abs/2203.07268

This paper concerns research activities related to (I)(A). Its purpose is to realize a detailed study of some structures related to module categories which enable the extension of some Baer-Kaplansky type theorems (which identify objects by means of their endomorphisms) from particular classes of abelian groups (torsion groups) to some results valid for all modules. These structures, called *heaps of modules*, can be characterized, from a geometrical perspective, as affine spaces associated to some module categories.

For this, we consider an extension of the ring notion, truss, in which the additive operation is replaced by a ternary operation which satisfies the Mal'cev associativity and idempotence axioms. Various structures over these ring generalizations (T-groups, T-modules, heaps of T-modules) are described and connected. The main result describes an equivalence between the category of the heaps of T-modules and the category of the affine spaces over T.

In the final part of the paper we present some examples of various structures from different fields which can be organized as heaps of modules: algebraic structures associated to Yang-Baxter type equations solutions, a kind

of geometric structures (connections) and contractions in homotopy categories associated to module categories (in particular, splittings).

4. Simion Breaz, Cristian Rafiliu: *Decompositions of matrices by using commutators*, https://arxiv.org/abs/2209.03195

This paper concerns research activities related to (I)(A). For the study of endomorphism rings associated to free modules of infinite rank we use two of their properties: any element of such a ring is a commutator and such a ring is isomorphic to any of the square matrix rings with entries in this ring. In this paper we study decompositions of 3×3 matrices with entries in an arbitrary ring for which the trace is a commutator. We prove that given three splitting polynomials of degree 3 with roots in the center of the considered ring, these matrices can be written as sums of 3 matrices, each of them annihilated by one of the given polynomials.

These decompositions are later used for getting some informations on the endomorphism rings of infinite rank free modules, and also for studying bounded operators associated to the complex Hilbert spaces. It is also proved that the simple rings obtained by factorizing endomorphism rings of infinite dimensional vector spaces through the corresponding maximal ideals can be decomposed, using the ternary operations associated to the addition (heaps), using three trusses which induce the "brace" structures defined by Rump for constructing solutions of the set theoretic Yang-Baxter equation.

5. Cs. Szántó, I. Szöllősi, *On some Ringel-Hall polynomials associated to tame indecomposable modules.*

This paper concerns research activities related to the first two research topics previously mentioned: (I)(A) and (I)(B). Let k be an arbitrary field and Q an acyclic quiver of tame type (i.e. of type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$). Consider the path algebra kQ and the category of finite dimensional right modules Mod-kQ. The rational Ringel-Hall algebra $\mathcal{H}(kQ)$ of the algebra kQ has as $\mathbb{Q}$-basis the isomorphism classes $[M]$ from Mod-kQ and the multiplication is defined by $[N_1][N_2] = \sum_{[M]} F^M_{N_1N_2}[M]$. The structure constants $F^M_{N_1N_2} = |\{U \subseteq M |\ U \cong N_2,\ M/U \cong N_1\}|$ are called Ringel-Hall numbers.

Far reaching analogues of the classical Hall algebras (associated with discrete valuation rings), these Ringel-Hall algebras were introduced by Ringel for a large class of rings, namely finitary rings, including in particular path algebras of quivers over finite fields. Ringel-Hall algebras provided a new

approach to the study of quantum groups using the representation theory of finite dimensional algebras and they can also be used successfully in the theory of cluster algebras. Moreover they play an essential role in the investigation of the structure of the module category. So in this way we can relate them to tilting and silting theory.

Due to a result of Hubery we know that in tame cases the Ringel-Hall numbers are rational polynomials in q with respect to so-called decomposition classes of modules, so we can call them Ringel-Hall polynomials. If we are looking to Ringel-Hall polynomials associated to indecomposable modules in various tame cases, we do not have too much information about them. In a previously reported paper the authors determined all the Ringel-Hall polynomials associated to indecomposable modules in the special tame case of type $\tilde{D}_4$.

In this paper we determine all the tame Ringel-Hall polynomials associated to indecomposable modules of defect belonging to the set $\{-2, -1, 0, 1, 2\}$. Compared to the case $\tilde{D}_4$ this is a much more harder task, needing the introduction of new, generic tools. It is surprising that, as in the $\tilde{D}_4$ case, also in the general tame case we will essentially have only three families of Ringel-Hall polynomials (with a single member for each degree). The first family corresponds to non-regular indecomposables and the second describes those involving non-homogeneous regulars. It turns out that the second family is very closely related to the first, so these two families are essentially the same, involving indecomposables of discrete type (preinjectives, preprojectives and non-homogeneous regulars). The third family corresponds to the continuous cases (when one of the indecomposables is regular homogeneous).

(B) Overview of the results

There are 5 papers finalized in 2022 and submitted. These are the papers presented in the previous section of this report. Hence we reached the result indicators target value (3 submitted papers).

The following papers, written in 2021, were accepted for publication. They were revised and prepared for publication according to the referees' suggestions.

1. Simion Breaz: On a theorem of Stelzer for some classes of mixed groups. Mediterr. J. Math. 19, No. 4, Paper No. 159, 14 p. (2022).

2. S. Breaz, Y. Zhou: When is every non central-unit a sum of two nilpotents? va apărea în Contemporary Math.

Disemination of the results. It was also done through the following international conference talks or research workshops abroad:

1. Simion Breaz: Malga Seminar Padova-Verona, Universita di Verona 24 mai 2022. Talk: Silting complexes and extensions of commutative rings.

2. Simion Breaz: Functor Categories, Model Theory, and Constructive Category Theory, Universidad de Almeria, 11-15.07.2022. Talk: Change of scalars functors and silting complexes.

3. Simion Breaz: Algebra Seminar, Charles University, Prague 10-12 octombrie 2022. Talk: The Baer-Kaplansky Theorem and Heaps of Modules.

4. Simion Breaz: Hopf algebras, monoidal categories and related topics IMAR, Bucuresti, 27-29.07.2022. Talk: Heaps of Modules

5. George Ciprian Modoi: Functor Categories, Model Theory, and Constructive Category Theory, Universidad de Almeria, 11-15.07.2022. Talk: Not necessarily compact approximability via silting theory.

6. Tudor Micu: Young Researchers' Conference on Non-Archimedean and Tropical Geometry, Universität Regensburg, 1-5 August 2022. Talk: The Structure of Special Fibers through Valuations.

7. Csaba Szanto: Algebra Seminar, Renyi Institute, Budapest, 7 martie 2022. Talk: Ringel-Hall polynomials associated to a quiver of type $\tilde{D}_4$.

8. Andrei Marcus: Hopf algebras, monoidal categories and related topics IMAR, Bucuresti, 27-29.07.2022. Talk: Tilting complexes over a G-graded G-algebra.

(B) Summary

The grant activities went according to the initial schedule, corresponding to the project main objectives and to the planned interim steps. They were performed by the team, in dissemination and research workshops, or individually. These workshops were held online on MsTeams platform with access provided by Babes-Bolyai University and on-site. Participants were team members and colleagues and students interested in the mathematical objects we study or in related phenomena. The team members also attended scientific conferences on topics strongly related to our project research interest.

Approached research topics:

(I)(A) Grothendieck categories associated structures related to (co)silting complexes.

(I)(B) Cosilting properties transfer using functors.

(II)(A) Definable classes and pure-injective objects in triangulated categories.

Dissemination/research workshops:

During our research workshop sessions there were presented and studied results from the literature which concern the following subjects:

(1) Silting, cosilting, tau-tilting modules.

(2) Silting objects in triangulated categories.

(3) Identities in rings and approximations. Splitting properties.

The grant members also attended the dissemination activities of some international conferences (online or on-site).

The following **results** were obtained:

- 5 papers are submitted,
- 2 papers written in 2021 have been accepted for publication,
- 6 international conferences talks,
- 2 talks at research seminars organized by other universities.

Cluj-Napoca,
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