Babeş-Bolyai University, Cluj-Napoca
Faculty of Mathematics and Computer Science

Scientific report

regarding implementation of the project
PN-III-P4-ID-PCE-2020-0454 (contract 75/2021)
Contributions to silting theory

(A) Phase summary

The grant activities have taken place according to the plan determined in the initial project, the main objectives of the project and the established intermediate steps. The topics approached this year have been:

(I)(A) (Co-)generated classes of (co-)silting objects.

(I)(B) Silting/Cosilting properties preserved/reflected by functors.

(II)(A) Pure injective objects and direct products.

The research activities have been done both individually and as a team, through research and information seminars. The research seminars have taken place on the MS Teams platform, made available by Babeş-Bolyai University. For this a dedicated channel was created within the Team of the Algebra group of the Faculty. Access to this channel was given to members of the group, but also to colleagues and students interested in the types of mathematical objects that we study. A separate team for individual discussions was also created on the same platform.

Research/Information seminars

In the research seminar results from the specialty literature were presented and studied, treating the following topics:

(a) **Silting objects associated to group algebras.** For this we began with an analysis of the paper [R. Koshio, Y. Kozakai: On support τ-tilting modules over blocks covering cyclic blocks, J. Algebra 580 (2021), 84–103], where compact 2-silting complexes are studied (i.e. τ-tilting modules) over group algebras. In this context we recalled the basic results of τ-tilting theory, published in [T. Adachi, O. Iyama, and I. Reiten, τ-tilting theory, Compos. Math. 150 (2014), 415–452] and [T. Aihara and O. Iyama, Silting

mutation in triangulated categories, J. Lond. Math. Soc. (2) 85 (2012), no. 3, 633–668]. Additionally, results pertaining to a new direction in the study of equivalences associated to group algebras, butterfly equivalences, were presented, based on the article [A. Marcus, Andrei; V.-A. Minuţă, Character triples and equivalences over a group graded G-algebra. J. Algebra 565 (2021), 98–127].

(b) **The transfer of tilting complexes through the scalar extension functor.** Starting from the results published in [R. Koshio, Y. Kozakai: On support τ-tilting modules over blocks covering cyclic blocks, J. Algebra 580 (2021), 84–103], one can remark that in order to ensure a good transfer of the τ-tilting properties it is necessary to impose an invariance condition regarding some inertia subgroups. This condition appears naturally in the study of tilting complexes, see [A. Marcus: Tilting complexes for group graded algebras, J. Group Theory 6 (2003), 175–193]. Hence, we moved to the study of the transfer of tilting/silting properties in the context of G-categories. This was done in the seminar through the investigation of the paper [C. Amiot, T. Brüstle: Derived equivalences between skew-gentle algebras using orbifolds, arXiv:1912.04367]. Such invariance conditions appear naturally and in a more formal context in [J-i. Miyachi. Extensions of rings and tilting complexes, J. of Pure and Appl. Alg. 105 (1995) 183-194]. These were also discussed during the seminar.

(c) **(Co)silting objects in triangulated categories.** In the seminar we presented results regarding the properties of (co)silting objects in the more general context of triangulated categories. In doing so we employed the papers [L. Angeleri-Hügel, Silting objects. Bull. Lond. Math. Soc. 51 (2019), 658–690], [L. Angeleri Hügel, F. Marks, J. Vitória, Torsion pairs in silting theory. Pac. J. Math. 291 (2017), 257–278] and [C. Psaroudakis and J. Vitória, Realisation functors in tilting theory, Math. Z. 288 (2018), 965–1028]. The goal of these presentations was to identify the structures associated to some (co)silting objects/classes that can be easily transferred using functors. For example, we studied the ways in which the notion of (co)silting is associated to t-structures (TTF-triples) in triangulated categories.

Research activities

(a) **Silting and cosilting objects associated to Grothendieck categories and triangulated categories.** Research on this topic commenced with the identification of fundamental results from the silting theory associated to Grothendieck and triangulated categories, respectively. One objective of this research was the identification of a common space where these results can be stated. In doing so we focused on the concept of extriangulated categories, defined in [H. Nakaoka, Y. Palu, Extriangulated categories, Hovey twin cotorsion pairs and model structures. Cah. Topol. Géom. Différ. Catég. 60 (2019) no. 2, 117–193]. Because in the study of (co)silting objects one very often employs finiteness conditions (for instance, compactly generated TTF-triples or purity conditions), we began to study such conditions in extriangulated categories. In the process we needed to define "compactness" so that both locally finitely presented Grothendieck categories and compactly generated triangulated categories can be modeled.

(b) **The transfer of (co)silting properties using functors. Ascent-descent properties.** We studied situations where, given a pair of adjoint functors $\varphi^* : \mathcal{C} \rightleftarrows \mathcal{D} : \varphi_*$ between well-generated triangulated categories and T a silting object in $\mathcal{C}$, the object $\varphi^*(T)$ is also silting. In particular, we obtained a characterization stating that if T is silting, then $\varphi^*(T)$ is silting if and only if $\mathcal{C}(T, \varphi_*\varphi^*(T)[i]) = 0$ for all $i > 0$. In this situation, we described the manner in which the functors φ_* and φ^* move the classes in the TTF triples associated to T and $\varphi^*(T)$, respectively. We applied this result to pairs of Frobenius functors (i.e. when φ^* and φ_* are two-sided adjoints) to deduce that T and $\varphi^*(T)$ are both silting if and only if $\varphi_*\varphi^*(T) \in \mathrm{Add}(T)$. This result shows that, in studying the transfer of silting/tilting objects associated to group algebras, restricting to the G-invariant hypotheses is natural. Moreover, the results are applied to the scalar extension/restriction functors associated to morphisms of commutative rings. We proved that if $\varphi : R \to S$ is a morphism of commutative rings, then for any silting object T in the derived category $\mathbf{D}(R)$ associated to R, the induced object $\varphi^*(T)$ from the derived category $\mathbf{D}(S)$ of S is silting, where φ^* is the derived functor associated to extension of scalars. Furthermore, if φ is faithfully flat, then φ^* induces an injective correspondence between the equivalence classes of silting objects in $\mathbf{D}(R)$ and the similar classes in $\mathbf{D}(S)$.

(c) **Contributions to the theory of approximation: rings and applications.** This research direction was used in order to obtain preliminary information on the methods that we wish to apply and the contexts that we intend to work in. Additionally, it is employed as an alternative research direction, as stated in the project proposal. Thus, for some categories of mixed abelian groups (that satisfy certain finiteness conditions) we studied properties regarding simplification of direct sums. We proved that, for certain classes (in particular for self-small groups of torsionfree rank at most 4), the property of simplification in direct sums can be characterized using the ring of (quasi-)endomorphisms of the group. Another direction was the study of nilpotent elements in a ring. In this context we proved that if all the invertible elements of a ring that are not central are sums of two nilpotent elements, then the ring is commutative or simple. The commutative case was also characterized: it is the case of local rings with nilpotent Jacobson radical. In this same context, of approximations in module categories, we began studying algebraic structures called "heaps of modules". They are modules over "trusses", structures similar to rings, wherein the addition is replaced by a ternary operation satisfying certain conditions introduced by Mal'cev. It was noticed that a preenvelope/precover induces such structures, hence we considered them worth studying in detail. Let us remark that such structures also occur naturally in other mathematical fields (for instance, in theories dealing with solutions of Yang-Baxter - type equations)

We also had realized a preliminary study of the following context: Let k be an arbitrary finite field with q elements and Q a quiver of tame type $\tilde{D}_4$. Consider the path algebra kQ and the category of finite dimensional right modules -ModkQ. The rational Ringel-Hall algebra $\mathcal{H}(kQ)$ of the algebra kQ has as $\mathbb{Q}$-basis the isomorphism classes $[M]$ from -ModkQ and the multiplication is defined by $[N_1][N_2] = \sum_{[M]} F^M_{N_1N_2}[M]$. The structure constants $F^M_{N_1N_2} = |\{U \subseteq M |\ U \cong N_2,\ M/U \cong N_1\}|$ are called Ringel-Hall numbers.

Far reaching analogues of the classical Hall algebras (associated with discrete valuation rings), these Ringel-Hall algebras were introduced by Ringel for a large class of rings, namely finitary rings, including in particular path algebras of quivers over finite fields. Ringel-Hall algebras provided a new approach to the study of quantum groups using the representation theory of finite dimensional algebras and they can also be used successfully in the

theory of cluster algebras. Moreover they play an essential role in the investigation of the structure of the module category. So in this way we can relate them to tilting and silting theory.

Due to a result of Hubery we know that in tame cases the Ringel-Hall numbers are rational polynomials in q with respect to so-called decomposition classes of modules, so we can call them Ringel-Hall polynomials. If we are looking to Ringel-Hall polynomials associated to indecomposable modules in various tame cases, we do not have too much information about them. The results obtained for this subject were inclded in a paper, submitted for publications, that has as main aim to determine all the Ringel-Hall polynomials associated to indecomposable modules in the special tame case of type $\tilde{D}_4$.

(B) Results

The research hitherto undertaken was synthesized in the following scientific articles.

(1) Completed and submitted papers

1. S. Breaz: *On a theorem of Stelzer for self-small mixed groups.*

2. S. Breaz, Y. Zhou: *When is every non central-unit a sum of two nilpotents?*

3. Cs. Szanto, I. Szollosi: *Ringel-Hall polynomials associated to a quiver of type* $\tilde{D}_4$

(2) Papers at an advanced stage of completion

1. S. Breaz, G.-C. Modoi: *Migration of silting objects via adjoint pairs*

2. S. Breaz, F. Pop: *On extriangulated categories*

3. S. Breaz, T. Brzeziński, B. Rybołowicz, P. Saracco: *Heaps of modules. First properties and applications*

Participations in conferences, workshops, research seminars

1. **S. Breaz**: participation at the conference *Homological Methods in Representation Theory*, A conference in honour of Lidia Angeleri Hugel, October 3 - 8, 2021; Fraueninsel (Chiemsee), Abtei Frauenworth, Germany. Talk: *Transfer of homological properties along some canonical functors.*

2. **G.-C. Modoi**: participation at the conference *Homological Methods in Representation Theory*, A conference in honour of Lidia Angeleri Hugel, October 3 - 8, 2021; Fraueninsel (Chiemsee), Abtei Frauenworth, Germany.

(C) Conclusions

The objectives of this stage have been accomplished.

From an organizational point of view, the members of the team have familiarized themselves with the specialized literature and have identified techniques and contexts (in addition to the ones proposed in the initial projects, other recent approaches have been assimilated) that will be used for accomplishing the research.

From the point of view of the research results, we have obtained new information related to the objectives established in the project proposal. They will be disseminated through the already submitted papers, along with the ones currently at an advanced stage of completion.

We mention that the initial plan had the measurable objective for 2021 of two submitted articles and this objective was fulfilled.

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