SOLUTIONS TO THE TRAINING EXERCISES FOR THE FINAL EXAM

(1) Evaluate the limit $\lim_{(x,y)\to(1,1)} \frac{3xy}{x^2+4y^2}$ or prove it does not exists. Same question with $\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+4y^2}$

Answer: At $(1,1)$ the denomintor x^2+4y^2 has limit (and value) 5, so we can apply the usual rules about the limit of a quotient of two functions, and we get $3/5$ for the limit $\lim_{(x,y)\to(1,1)} f(x,y)$ where $f(x,y) = \frac{3xy}{x^2+4y^2}$.

However at $(0,0)$, x^2+4y^2 vanishes, and we can't apply this method.

If we look at the limit of the function along the x-axis (y=0) we have $f(x,0) = 0$ for $x \neq 0$ so the limit is 0. If we look at the restriction to the function over the diagonal line $y = x$, we have $f(x,y) = \frac{3x^2}{5x^2} = 3/5$ if $x \neq 0$, so the limit along this line is $5/3$. Since that limit is different of the first one, the function $f(x,y)$ has no limit when (x,y) goes to $(0,0)$.

(2) Find the domain and the partial derivatives of $f(x,y) = \sqrt{x^2+y^4}$. Same question for $f(x,y,z) = \frac{x}{y-z}$.

answer : The domain for the given $f(x,y)$ is the whole plane $\mathbb{R}^2$ because $x^2+y^4 \geq 0$ and it makes sense to take the square root of a number ≥ 0. The partial derivatives are $f_x(x,y) = \frac{x}{\sqrt{x^2+y^4}}$ and $f_y(x,y) = 2\frac{y^3}{\sqrt{x^2+y^4}}$.

The domain for $f(x,y,z)$ is $\{(x,y,z), y \neq z\}$, that is all the space $\mathbb{R}^3$ excepted the plane $y = z$. The partial derviatives are $f_x(x,y,z) = \frac{1}{y-z}$, $f_y(x,y,z) = -\frac{x}{y-z^2}$ and $f_z(x,y,z) = \frac{x}{y-z^2}$

(3) Find all second partial derivatives of $f(x,y,z) = x^2y - y^2x + zxy$.

answer : Easy.

(4) Find the equation of the tangent plane of the surface $z = 3x^2 - y^2 + 2z$ at $(1,-2,1)$.

Answer : This equation can be rewritten $z = -3x^2 + y^2$, hence it is the equation of the graph of the function $f(x,y) = -3x^2 + y^2$. We check that $f(1,-2) = -3+4 = 1$, hence $(1,-2,1)$ is really a point of the graph, as the question says. The equation of the tangent plane was given in class :

$$z - f(1,-2) = (x-1)f_x(1,-2) + (y+2)f_y(1,-2).$$

Here $f_x(x,y) = -6x$ so $f_x(1,-2) = -6$ and $f_y(x,y) = 2y$ so $f_y(1,2) = -4$. Thus the equation of the tangent plane is

$$z - 1 = (x-1)\times(-6) + (y+2)\times(-4)$$

or

$$6x + 4y + z = -1.$$

(5) If $z = y + f(x^2 - y^2)$ and f is differentiable show that $y\frac{dz}{dx} + x\frac{dz}{ddy} = x$.

Answer : We compute $\frac{dz}{dx}$. We get $\frac{dy}{dx} + \frac{df(x^2-y^2)}{dx}$. The first term of this sum is 0 because y is seen as a constant when computing the x-partlai derivative. To compute the second, simply note that you have to compute the derivative of the function of one variable (x) $f(x^2 - y^2)$ (here y^2 is nothing but a constant). So the chain rule for one-variable function gives $\frac{df(x^2-y^2}{dx} = 2xf'(x^2 - y^2)$. So

$$\frac{dz}{dx} = 2xf'(x^2 - y^2).$$

With the same method, we compute

$$\frac{dz}{dy} = 1 - 2yf'(x^2 - y^2).$$

Then

$$y\frac{dz}{dx} + x\frac{dz}{dy} = 2xyf'(x^2 - y^2) + x = 2xyf'(x^2 - y^2) = x.$$

(6) Let $f(x, y, z) = x^2y^3z$. Find the gradient vector of f at $(1, 1, 1)$. When is the directional derivatives along a unit vector $\vec{u}$ at that point equal to 0 ?

Answer : The gradient vector is $\nabla f(1, 1, 1) = \langle 2, 3, 1\rangle$. The directional derivative along $\vec{u}$ is $D_{\vec{u}}f(1, 1, 1) = \nabla f(1, 1, 1) \bullet \vec{u} = 2a+3b+c$ if $\vec{u} = \langle a, b, c\rangle$. Hence $D_{\vec{u}}(f)$ is 0 if and only $\vec{u}$ is orthogonal to $\nabla f(1, 1, 1)$, that is if and only if $2a = 3b + c = 0$.

(7) Find the local maxima, minima, and saddle points of the function $f(x, y) = xye^{y/2}$

Answer : We first look for the critical points, that is points were $f_x(x, y) = f_y(x, y) = 0$. We compute $f_x(x, y) = ye^{y/2}$ and $f_y(x, y) = \frac{1}{2}xye^{y/2} + xe^{y/2} = x(1+y/2)e^{y/2}$. Thus $f_x(x, y)$ vanishes if and only if $y = 0$ and $f_y(x, y)$ vanishes if and only if $x = 0$ or $y = -2$. Therefore, the only critical point is $(0, 0)$.

Next we use the second-derivative test, to find out which critical points are local minima, local maxima, saddle points, or neither of those three. We compute $f_{xx}(x, y) = 0$, $f_{xy}(x, y) = (1 + y/2)e^{y/2}$, $f_{yy}(x, y) = xe^{y/2}(1 + y/4)$. So at the critical points $(0, 0)$, we have $f_{xx}(0, 0) = 0$, $f_{xy}(0, 0) = 1$ and $f_{yy}(0, 0) = 0$ and $D = f_{xx}(0, 0)f_{yy}(0, 0) - f_{x,y}(0, 0) = -1 < 0$, hence $(0, 0)$ is a saddle point.

(8) Find the minimum and maximum values of $f(x, y, z) = xyz$ on the closed ball of center the origin and radius 3 (that is, the set of points whose distance to origin is ≤ 3)

Answer We first look for the critical point in the open ball of center the origin and radius 3. They are the points (x, y, z) such that $f_x(x, y, z) = f_y(x, y, z) = f_z(x, y, z) = 0$, that is $yz = xz = xy = 0$. So the critical points are all the points for which at least two out of three coordinates are zero. We remark that at those points $f(x,y,z) = 0$. (We don't have to determine if those points are local minimum, maximum, saddle points or nothing - we will simply have to compare the values at those critical points with the extremal values on the boundary - Nevertheless it is easy to see in this case that critical point are neither local minimum nor local maximum - Can you see why ?)

On the boundary, that is the sphere $x^2 + y^2 + z^2 = 9$, we use Lagrange's multipliers to find extremal values. Since the partial derivatives of $g(x,y,z) = x^2 + y^2 + z^2$ are $2x$, $2y$ and $2z$

$$\begin{aligned} yz &= \lambda\, 2x \\ xz &= \lambda\, 2y \\ xy &= \lambda\, 2z \\ x^2 + y^2 + z^2 &= 9 \end{aligned}$$

We may first remark that x,y,z are all non zero. For suppose for example that $x = 0$ (the same reasoning would work if y or z wass 0, because the equation are symmetric with respect to x, y and z). Then by the first equation $yz = 0$ so either y or z is 0. If y is zero then z is zero by the third equation and if z is zero, then y is zero by the second equation : in any case, we see that of x is 0, so are y and z. But then x, y, z cannot be a solution of the fourth equation, sonce $0 \neq 9$.

Now multiply the three first equation : we get $(xyz)^2 = 8\lambda^3 xyz$. We have just seen that xyz is not zero, so we can deduce $xyz = 8\lambda^3$. Note that λ is non zero too.

Multiplying the first equation by x we get $xyz = 2\lambda x^2$ so $8\lambda^3 = 2\lambda x^2$ and finally we get $x^2 = 4\lambda^2$. Similarly, we get $y^2 = 4\lambda^2$ and $z^2 = 4\lambda^2$ usingthe second and third equation.

In particular $x^2 = y^2 = z^2$ but sicne $x^2 + y^2 + z^2 = 9$, we have $x^2 = y^2 = z^2 = 3$.

So the solutions of our system are the eight points

$$(z, y, z) = (\pm\sqrt{3}, \pm\sqrt{3}, \pm\sqrt{3})$$

and the value of f at those points are either $3\sqrt{3}$ or $-3\sqrt{3}$.

Comparing with the value (0) found at critical point in the open ball we see that the maximum value of f is $3\sqrt{3}$ and the minimum $-3\sqrt{3}$.

(9) Find the minimum amd maximum values of $f(x,y) = \frac{1}{x} + \frac{1}{y}$ subject to the constraint $\frac{1}{x^2} + \frac{1}{y^2} = 1$

Answer : We use Lagrange multipliers. Note that $f_x(x,y) = -1/x^2$ and $f_y(x,y) = -1/x^2$. The constraint is $g(x,y) = 1$, with $g(x,y) = \frac{1}{x^2} + \frac{1}{y^2}$ so

$g_x(x, y) = -2/x^3$ and $g_y(x, y) = -2/y^3$. We thus have to solve

$$\begin{aligned} -\frac{1}{x^2} &= \lambda\frac{-2}{x^3} \\ -\frac{1}{y^2} &= \lambda\frac{-2}{y^3} \\ \frac{1}{x^2} + \frac{1}{y^2} &= 1 \end{aligned}$$

The first two equations may be rewritten as $x = 2\lambda$, $y = 2\lambda$. So $x = y$, and the third equation gives $\frac{2}{x^2} = 1$ so $x = \sqrt{2}$. So the two solutions of the system are $(x, y) = (\sqrt{2}, \sqrt{2})$ and $(x, y) = (-\sqrt{2}, -\sqrt{2})$ giving the values $\sqrt{2}$ and $-\sqrt{2}$ which are respectively the maximal and minimum value of $f(x, y)$ with the given constraint.