Sheet 6: integrals

1. Prove that the volume under the surface
$$f(x,y) = \frac{y}{x}, \quad \text{for} \quad e \le x \le e^3 \quad \text{and} \quad 1 \le y \le 2,$$
is $V = 3$.

2. Prove that the volume under the surface
$$f(x,y) = \frac{y}{1+x^2}, \quad \text{for} \quad 0 \le x \le 1 \quad \text{and} \quad 0 \le y \le 1,$$
is $V = \pi/8$.

3. By sketching first the integration region, prove that
$$I = \int\int_R \frac{dxdy}{\sqrt{x+2y}} = \frac{3}{2},$$
in the region determined by the conditions
$$R = \{(x,y) : x - 2y \le 1 \quad \text{and} \quad x \ge y^2 + 1\}.$$

4. Sketch the region of integration for
$$I = \int_{y=0}^{y=4} dy \int_{x=y/2}^{x=\sqrt{y}} e^{y/x} dx,$$
and reverse the order of integration in order to evaluate $I = e^2 - 1$.

5. Find the Jacobian of the following changes of coordinates:
$$\begin{array}{llll} (a) & x = 2u + 4v, & y = 3u + 5v, & \\ (b) & x = \alpha\beta, & y = \alpha^2 + \beta^2, & \\ (c) & x = v + w, & y = u + w, & z = u + v. \end{array}$$
Hence, write the element of volume (or surface) in each case.

6. Show that in polar coordinates, the equation of the circle $(x-1)^2 + y^2 = 1$ takes the form $r = 2\cos\theta$ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.
Hence, by using the cylindrical coordinates find the volume of the solid enclosed in the vertical cylinder $(x-1)^2 + y^2 = 1$, bounded below by the plane $z = 0$ and bounded above by the cone $z = 2 - \sqrt{x^2 + y^2}$.

7. Show that the equation of the semicircle $x^2 + y^2 - ay = 0$ with $x \ge 0$ in the polar coordinates takes the form
$$r = a\sin\theta \qquad 0 \le \theta \le \frac{\pi}{2}.$$
Hence using the cylindrical coordinates find the volume of the solid that is inside of the ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1,$$
above the xy-plane and inside the vertical cylinder $x^2 + y^2 - ay = 0$, $x \ge 0$.

8. Find the Jacobian of the transformation of coordinates:
$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$
where
$$0 \le \theta \le 2\pi, \qquad 0 \le r \le \infty, \qquad -\infty \le z \le \infty.$$
Using the cylindrical coordinates above, determine the mass of the solid bounded by the cone $z^2 = x^2 + y^2$, $z \le 0$ and the cylinder $x^2 + y^2 = a^2$, given that the density of the solid is defined by the function $(x^2 + y^2)z$.