SEEMOUS 2018, Iaşi, Romania, March 1, 2018

Competition problems

$\boxed{\textbf{1.}}$ Let $f : [0,1] \to (0,1)$ be a Riemann integrable function. Show that

$$\frac{2\ \int_0^1 x f^2(x)\,\mathrm{d}x}{\int_0^1 (f^2(x)+1)\,\mathrm{d}x} < \frac{\int_0^1 f^2(x)\,\mathrm{d}x}{\int_0^1 f(x)\,\mathrm{d}x}.$$

$\boxed{\textbf{2.}}$ Let $m,\ n, p,\ q \geq 1$ and let the matrices $A \in \mathcal{M}_{m,n}(\mathbb{R})$, $B \in \mathcal{M}_{n,p}(\mathbb{R})$, $C \in \mathcal{M}_{p,q}(\mathbb{R})$, $D \in \mathcal{M}_{q,m}(\mathbb{R})$ be such that

$$A^t = BCD,\ B^t = CDA,\ C^t = DAB,\ D^t = ABC.$$

Prove that $(ABCD)^2 = ABCD$.

$\boxed{\textbf{3.}}$ Let $A,\ B \in \mathcal{M}_{2018}(\mathbb{R})$ such that $AB = BA$ and $A^{2018} = B^{2018} = I$, where I is the identity matrix. Prove that if $\mathrm{Tr}(AB) = 2018$, then $\mathrm{Tr}\,A = \mathrm{Tr}\,B$.

$\boxed{\textbf{4.}}$ **(a)** Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial function. Prove that

$$\int_0^\infty e^{-x} f(x)\,\mathrm{d}x = f(0) + f'(0) + f''(0) + \cdots .$$

(b) Let f be a function which has a Taylor series expansion at 0 with radius of convergence $R = \infty$. Prove that if $\displaystyle\sum_{n=0}^{\infty} f^{(n)}(0)$ converges absolutely then $\displaystyle\int_0^\infty e^{-x} f(x)\,\mathrm{d}x$ converges and

$$\sum_{n=0}^{\infty} f^{(n)}(0) = \int_0^\infty e^{-x} f(x)\,\mathrm{d}x.$$

Solutions

$\boxed{1.}$ Use $2f(x) \leq f^2(x) + 1$ and $xf^2(x) < f^2(x)$ and integrate.

$\boxed{2.}$ Denote $M = ABCD$. We have $M = AA^t$ so $M \geq 0$ (i.e. positive semi-definite).
$M^3 = ABCDABCDABCD = D^tC^tB^tA^t = (ABCD)^t = ABCD = M$.
So, the minimal polynomial of the matrix M divides $x^3 - x \implies$ the eigenvalues $\subseteq \{-1, 0, 1\}$. But -1 cannot be an eigenvalue since $M \geq 0$. Hence the minimal polynomial divides $x^2 - x \implies M^2 - M = 0$.

$\boxed{3.}$ $n := 2018$. $AB = BA \implies A, B$ are diagonalizable in a same basis: $A = P^{-1}DP,\ B = P^{-1}EP$, where D, E are diagonal matrices containing the eigenvalues λ_k, μ_k $(k = 1 \ldots n)$.
$A^n = B^n = I \implies D^n = E^n = I \implies$ the eigenvalues λ_k, μ_k $(k = 1 \ldots n)$ have modulus 1.
$n = \mathrm{Tr}(AB) = \sum_{k=1}^n \lambda_k \mu_k$. The triangle inequality implies $\lambda_k \mu_k = 1 \implies \lambda_k = 1/\mu_k = \overline{\mu_k} \implies \sum_{k=1}^n \lambda_k = \sum_{k=1}^n \overline{\mu_k} = \sum_{k=1}^n \mu_k$ (because the matices are real, so the eigenvalues are in conjugate pairs). $\implies \mathrm{Tr}(A) = \mathrm{Tr}(B)$ [the eigenvalues are actually the same].

$\boxed{4.}$ **(a)** Repeated integration by parts. [Actually it results from **(b)**].
(b) Denote $a_n = f^{(n)}(0)$. $f(x)e^{-x} = \sum_{n=0}^\infty \frac{a_n}{n!} x^n e^{-x}$;
$\sum_{n=0}^\infty \int_0^\infty \left| \frac{a_n}{n!} x^n e^{-x} \right| \mathrm{d}x = \sum_{n=0}^\infty |a_n| \int_0^\infty \frac{1}{n!} x^n e^{-x}\ \mathrm{d}x = \sum_{n=0}^\infty |a_n| < \infty$.
Using a corollary of the Lebesgue dominated convergence theorem $\implies x \mapsto f(x)e^{-x}$ is integrable over $[0, \infty)$ and $\int_0^\infty f(x)e^{-x}\,\mathrm{d}x = \sum_{n=0}^\infty a_n \int_0^\infty \frac{1}{n!} x^n e^{-x}\,\mathrm{d}x = \sum_{n=0}^\infty a_n$.

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