Seemous 2017, February 28 – March 5, 2017, Ohrid, Republic of Macedonia

Problems (time for work: 5 hours).

$\boxed{\mathbf{1.}}$ Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$ such that $a^2 + b^2 + c^2 + d^2 < \frac{1}{5}$.
Show that $I + A$ is invertible.

$\boxed{\mathbf{2.}}$ Let $A, B \in \mathcal{M}_n(\mathbb{R})$.
a) Show that there exists $a > 0$ such that for every $\varepsilon \in (-a, a)$, $\varepsilon \neq 0$, the matrix equation

$$AX + \varepsilon X = B, \quad X \in \mathcal{M}_n(\mathbb{R})$$

has a unique solution $X(\varepsilon) \in \mathcal{M}_n(\mathbb{R})$.
b) Prove that if $B^2 = I_n$, and A is diagonalizable then

$$\lim_{\varepsilon \to 0} \mathrm{Tr}(BX(\varepsilon)) = n - \mathrm{rank}(A).$$

$\boxed{\mathbf{3.}}$ Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that

$$\int_0^4 f(x(x-3)^2)\, \mathrm{d}x = 2 \int_1^3 f(x(x-3)^2)\, \mathrm{d}x$$

$\boxed{\mathbf{4.}}$ **a)** Let $n \geq 0$ be an integer. Calculate $\int_0^1 (1-t)^n e^t\, \mathrm{d}t$.
b) Let $k \geq 0$ be a fixed integer and let $(x_n)_{n \geq k}$ be a sequence defined by

$$x_n = \sum_{i=k}^{n} \binom{i}{k} \left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{i!} \right).$$

Prove that the sequence converges and find its limit.

Solutions

$\boxed{\mathbf{1.}}$ Obvious. $\|A\| < 1 \implies (I \pm A)$ are invertible. Or, prove directly: $\det(A + I) > 0$.
Note: $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$.

$\boxed{\mathbf{2.}}$ **a)** The equation is $(A + \varepsilon I)X = B$. Take $a = \min\{|\lambda| : \lambda \in \sigma_p(A) \setminus \{0\}\}$. Then $\exists (A + \varepsilon I)^{-1}$ for $0 < |\varepsilon| < a$.
b) $X(\varepsilon) = (A + \varepsilon I)^{-1} B \implies BX(\varepsilon) = B(A + \varepsilon I)^{-1} B = B^{-1}(A + \varepsilon I)^{-1} B \implies$
$\mathrm{Tr}(BX(\varepsilon)) = \mathrm{Tr}\left((A + \varepsilon I)^{-1}\right) = \sum_{k=1}^{n} \frac{1}{\lambda_k + \varepsilon}$, where λ_k are the eigenvalues (with multiplicities).
$\lim_{\varepsilon \to 0} \varepsilon\, \mathrm{Tr}(BX(\varepsilon)) = \lim_{\varepsilon \to 0} \sum_{k=1}^{n} \frac{\varepsilon}{\lambda_k + \varepsilon} = \mathrm{card}\{k \in \{1, \ldots, n\} : \lambda_k = 0\}$.
If A is diagonalizable this is exactly $n - \mathrm{rank}(A)$.
[It is enough for A to satisfy the condition that the algebraic and geometric multiplicity for 0 be the same]

$\boxed{\mathbf{3.}}$ Denote $p : [0, 4] \to [0, 4]$, $p(x) = x(x-3)^2$ and the intervals $J_1 = [0, 1]$, $J_2 = [1, 3]$ and $J_3 = [3, 4]$.
Let p_k be the restriction of p to J_k. p_k is bijective, and denote by $q_k : [0, 4] \to J_k$ its inverse.
Notice that $p_1' > 0, p_2' < 0, p_3' > 0$ in $\mathring{J}_k$, so $q_1' > 0, q_2' < 0, q_3' > 0$ in the open interval $(0, 4)$.
Changing the variables $p_k(x) = y$ we obtain:

$$\int_{J_k} f(p(x))\, \mathrm{d}x = \int_0^4 f(y)\, |q_k'(y)|\, \mathrm{d}y, \;\; k = 1 \ldots 3. \tag{*}$$

Now, $q_k(y)$ are the roots of the equation $p(x) = y$, i.e. $x^3 - 6x^2 + 9x - y = 0$.
It follows that $q_1 + q_2 + q_3 = 6$, hence $q_1' + q_2' + q_3' = 0$ in $(0, 4)$
Since only q_2' is negative: $|q_k'| - |q_k'| + |q_k'| = 0$ in $(0, 4)$
Using (*) it results:
$\int_{J_1} f(p(x))\,\mathrm{d}x - \int_{J_2} f(p(x))\,\mathrm{d}x + \int_{J_3} f(p(x))\,\mathrm{d}x = 0$
which is exactly the desired conclusion.

Note. The integrals in the right hand side of (*) are improper. If we want to avoid this, then we may apply the change of variables only for smaller intervals e.g. $J_k(\varepsilon) = p_k^{-1}([\varepsilon, 4-\varepsilon])$ and take $\varepsilon \to 0_+$ at the end.

$\boxed{\textbf{4.}}$ **a)** Denote by I_k the integral.
$I_0 = e - 1$. An integration by parts gives $I_n = -1 + nI_{n-1}$, $n \geq 1$.
$I_n/n! = -1/n! + I_{n-1}/(n-1)!$ After a telescoping summation we obtain
$I_n/n! - I_0 = -\sum_{k=1}^{n} 1/k!$

$$I_n = n!\left(e - \sum_{k=0}^{n} \frac{1}{k!}\right)$$

b) $x_n = \sum_{i=k}^{n} \binom{i}{k} I_i/i! = \sum_{i=k}^{n} \frac{I_i}{k!(i-k)!}$. The sequence is increasing, so its limit exists and
$\lim_{n\to\infty} x_n = \sum_{i=k}^{\infty} \frac{I_i}{k!(i-k)!} =$
$\frac{1}{k!}\sum_{i=0}^{\infty} \frac{I_{k+i}}{i!} =$
$\frac{1}{k!}\sum_{i=0}^{\infty} \frac{I_{k+i}}{i!} =$
$\frac{1}{k!}\sum_{i=0}^{\infty} \frac{1}{i!}\int_0^1 (1-t)^{k+i} e^t\,\mathrm{d}t =$ [by the theorem of monotone convergence]
$\frac{1}{k!}\int_0^1 \sum_{i=0}^{\infty} \frac{(1-t)^{k+i}}{i!} e^t\,\mathrm{d}t =$
$\frac{1}{k!}\int_0^1 (1-t)^k \sum_{i=0}^{\infty} \frac{(1-t)^i}{i!} e^t\,\mathrm{d}t =$
$\frac{1}{k!}\int_0^1 (1-t)^k e^{1-t} e^t\,\mathrm{d}t = \frac{e}{(k+1)!}$.

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