Exercise. Find the differential equation of the family of planar curves

\[ y = ax^2, \ a \in \mathbb{R} \]

and its orthogonal trajectories. We say that two intersecting planar curves are orthogonal if their directions in the point of intersection are orthogonal. We say that two simple families of planar curves are orthogonal if a curve from one family is orthogonal to each curve from the other family.

The Picard sequence of successive approximations.

Let \( U \subset \mathbb{R}^2 \) be a domain, \( (x_0, y_0) \in U \) and \( f : U \to \mathbb{R} \) be a continuous function. We consider the initial value problem

\[
\begin{align*}
  y' &= f(x, y) \\
  y(x_0) &= y_0.
\end{align*}
\]

We have that the initial value problem (1) is equivalent to the integral equation (called Volterra integral equation)

\[
y(x) = y_0 + \int_{x_0}^x f(s, y(s)) ds.
\]

Given an open real interval \( I \), for every continuous function \( y : I \to \mathbb{R} \) we associate a function \( v : I \to \mathbb{R} \) given by

\[
v(x) = y_0 + \int_{x_0}^x f(s, y(s)) ds.
\]

In this way we define a mapping \( A : C(I) \to C(I) \) by \( A(y) = v \). Here \( C(I) \) is the linear space of continuous functions on the interval \( I \). Using that (1) and (2) are equivalent, we have that a function \( \varphi \) is a solution of the initial value problem (1) if and only if

\[
\varphi = A(\varphi),
\]

which also means that \( \varphi \) is a fixed point of the mapping \( A \).

In general, given a metric space \( X \) and an operator \( A : X \to X \), we remind that the sequence of Picard iterations starting at \( \varphi_0 \in X \) of \( A \) is given by the recurrence relation

\[
\varphi_{n+1} = A(\varphi_n), \text{ for each } n \geq 0.
\]
In the case that the sequence \((\varphi_n)_{n \geq 0}\) converges to a fixed point \(\varphi^*\) of \(A\), we say that \(\varphi^*\) can be obtained by the method of successive approximations. Notice that when the operator \(A\) is continuous and the sequence \((\varphi_n)_{n \geq 0}\) converges, its limit must be a fixed point of \(A\).

Coming back to the operator \(A : C(I) \to C(I)\) defined above in the relation to the initial value problem (1), we see that its sequence of Picard iterations starting at \(\varphi_0 \in C(I)\) has the recurrence formula

\[
\varphi_{n+1}(x) = y_0 + \int_{x_0}^{x} f(s, \varphi_n(s)).
\]

When this sequence converges to a solution \(\varphi^*\) of (1), we say that \(\varphi^*\) can be obtained by the method of successive approximations starting with \(\varphi_0\).

**Example.** We consider the initial value problem

\[
\begin{align*}
y' &= y \\
y(0) &= 1.
\end{align*}
\]

We know that it has a unique solution \(\varphi^* : \mathbb{R} \to \mathbb{R}\) given by \(\varphi^*(x) = e^x\). The integral equation equivalent to this problem is

\[y(x) = 1 + \int_{0}^{x} y(s)ds.\]

The recurrence relation for the sequence of Picard iterations starting at \(\varphi_0 \in C(\mathbb{R})\) is

\[
\varphi_{n+1}(x) = 1 + \int_{0}^{x} \varphi_n(s)ds, \quad n \geq 0.
\]

Starting with the constant function \(\varphi_0 = 1\), we will find all the Picard iterations and we prove that this sequence converges uniformly on each compact real interval \([-a, a]\) to \(\varphi^*\), the unique solution of (4).

We have

\[
\begin{align*}
\varphi_1(x) &= 1 + \int_{0}^{x} \varphi_0(s)ds = 1 + \int_{0}^{x} ds = 1 + x, \\
\varphi_2(x) &= 1 + \int_{0}^{x} \varphi_1(s)ds = 1 + \int_{0}^{x} (1 + s)ds = 1 + x + \frac{x^2}{2}.
\end{align*}
\]

We will prove inductively that, for all \(n \in \mathbb{N}\),

\[
\varphi_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}.
\]
If (5) is valid for the integer \( n \), then
\[
\varphi_{n+1}(x) = 1 + \int_0^x \varphi_n(s) \, ds = 1 + \int_0^x \left(1 + s + \frac{s^2}{2} + \cdots + \frac{s^n}{n!}\right) \, ds = 
= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!}.
\]

By the induction principle, (5) is valid for all \( n \in \mathbb{N} \).

We notice that \( \varphi_n \) is the Taylor expansion polynomial of degree \( n \) around \( x = 0 \) of the function \( e^x \). It is known that the sequence \( (\varphi_n)_{n \geq 0} \) converges uniformly on each compact real interval \([-a, a]\) to \( e^x \).

In the above example we showed that the unique solution of (4) can be obtained by the method of successive approximations starting with the constant function 1. Generically, this is the case for any initial value problem. In the next two theorems we state precisely the hypotheses on which this holds true.

**Theorem 1** (Global existence and uniqueness) We assume that the domain \( U \subset \mathbb{R}^2 \) is such that \( B = [x_0 - a, x_0 + a] \times \mathbb{R} \subset U \) and that \( f \in C^1(U) \) is such that \( \frac{\partial f}{\partial y}(x, y) \) is bounded on \( B \). Then there exists a unique solution of the initial value problem (1), defined on \([x_0 - a, x_0 + a]\). Moreover, it can be obtained by the method of successive approximations starting from every continuous function \( \varphi_0 : [x_0 - a, x_0 + a] \rightarrow \mathbb{R} \).

**Theorem 2** (Local existence and uniqueness) We assume that \( f \in C^1(U) \). Let \( a, b > 0 \) such that \( D = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b] \) is included in \( U \). Let \( M > 0 \) such that \( |f(x, y)| \leq M \) for all \((x, y) \in D\) and \( \delta = \min \left\{ a, \frac{b}{M}\right\} \). Then the initial value problem (1) has a unique solution on \([x_0 - \delta, x_0 + \delta]\). Moreover, it can be obtained by the method of successive approximations starting from every continuous function \( \varphi_0 : [x_0 - \delta, x_0 + \delta] \rightarrow \mathbb{R} \) with \( |\varphi_0(x) - y_0| \leq b, \forall x \in [x_0 - \delta, x_0 + \delta] \).

The above theorems are not only important for the conclusion about the method of successive approximations, but, mainly, for proving the existence and uniqueness of the solution. This is the explanation of their names.

**Example.** We consider the initial value problem
\[
\begin{cases}
y' + p(x)y = q(x) \\
y(0) = y_0
\end{cases}
\]
where \( p, q \in C^1(\mathbb{R}) \) and \( y_0 \in \mathbb{R} \) are given. We prove that it has a unique solution \( \varphi^* : \mathbb{R} \to \mathbb{R} \) that can be obtained by the method of successive approximations.

We apply the Global Existence and Uniqueness Theorem. The function \( f \) is given by \( f(x, y) = -p(x)y + q(x) \) and it is easy to see that \( f : \mathbb{R}^2 \to \mathbb{R} \) is of class \( C^1(\mathbb{R}^2) \). We take an arbitrary \( a > 0 \). The partial derivative \( \frac{\partial f}{\partial y}(x, y) = -p(x) \) is bounded on \([-a, a] \times \mathbb{R}\).

Then there exists a unique solution \( \varphi^* : [-a, a] \to \mathbb{R} \) of our initial value problem, that can be obtained by the method of successive approximations starting with an arbitrary continuous function \( \varphi_0 : [-a, a] \to \mathbb{R} \). Since \( a \) is arbitrary, \( \varphi^* \) can be extended on the whole real line, i.e. we have \( \varphi^* : \mathbb{R} \to \mathbb{R} \).

Example. We prove that the initial value problem

\[
\begin{align*}
y' &= xy^2 + 1 \\
y(0) &= 0,
\end{align*}
\]

has a unique solution \( \varphi^* : [-1/2, 1/2] \to \mathbb{R} \) that can be obtained by the method of successive approximations starting with every continuous function \( \varphi_0 : [-1/2, 1/2] \to \mathbb{R} \) that satisfies \( |\varphi_0(x)| \leq 1 \) for all \( x \in [-1/2, 1/2] \).

The function \( f \) given by \( f(x, y) = xy^2 + 1 \) is \( C^1(\mathbb{R}^2) \). We take \( a = b = 1 \) and \( D = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2 \). We can prove that \( |f(x, y)| \leq 2 \), for all \((x, y) \in D\) and consider \( \delta = \min \{1, 1/2\} = 1/2 \). Then, indeed, this initial value problem has a unique solution which is defined at least on the interval \([-1/2, 1/2]\).

The integral equation equivalent to this initial value problem is

\[
y(x) = \int_0^x (sy^2(s) + 1) \, ds,
\]

and the sequence of Picard iterations is recurrently defined by

\[
\varphi_{n+1}(x) = \int_0^x (s\varphi_n^2(s) + 1) \, ds.
\]

The Local Existence and Uniqueness Theorem assures the convergence of this sequence if we start with some continuous function \( \varphi_0 : [-1/2, 1/2] \to \mathbb{R} \) whose graph is contained in \( D \). For example, we consider the constant function \( \varphi_0(x) = 0 \) and successively calculate by hand the next terms of the sequence. We obtain \( \varphi_1(x) = x, \varphi_2(x) = x + x^4/4, \varphi_3(x) = x + x^2/2 + x^6/24, \ldots \). We can use these polynomial functions as approximations of the solution of the given initial value problem.
Numerical methods.

Picard’s method of successive approximations suggests the idea of finding functions as close as possible to the solution of an initial value problem. Especially in practice, an approximate solution is better than no solution at all. This raises the issue of designing methods that lead to approximate solutions and of controlling the error involved. We will discuss first the Euler method, then the second-order Runge-Kutta method, and finally some ideas of error theory.

With Picard’s method we found a sequence of functions that approximate a solution. Here we will do something different. The estimate we obtain will be a collection of points \((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\), which track a curve that approximates the graph of the solution passing through \((x_0, y_0)\). So, we don’t end up with a function, but with a table of values.

0.0.1 Euler’s method.

This is the prototype of the numerical solvers for a differential equation, and it was proposed in 1768 by the Swiss mathematician Leonhard Euler. Finding a numerical solution of the initial value problem

\[
\begin{align*}
  y' &= f(x, y) \\
  y(x_0) &= y_0
\end{align*}
\]

means obtaining approximate values of the solution for a set of values of the independent variable \(x\) in some interval \([x_0, a]\), say, at \(x_0 < x_1 < \cdots < x_{n-1} < x_n = a\).

We work in the hypothesis that the initial value problem has a unique solution on the interval \([x_0, a]\).

The starting point of this method is \((x_0, y_0)\).

The slope of the tangent to the graph of the unknown solution \(\varphi\) is \(\varphi'(x_0) = f(x_0, y_0)\), so the equation of the line through \((x_0, y_0)\) and of slope \(f(x_0, y_0)\) is

\[
y = y_0 + f(x_0, y_0)(x - x_0).
\]

Using this equation and moving along the tangent line from \(x_0\) to \(x_1\), we obtain an approximate value \(y_1\) for \(\varphi(x_1)\):

\[
y_1 = y_0 + f(x_0, y_0)(x_1 - x_0).
\]

Next we take \(f(x_1, y_1)\) as an approximate value for the slope of the tangent to the solution’s graph at \((x_1, \varphi(x_1))\). We cannot obtain the exact value \(f(x_1, \varphi(x_1))\) of the
slope, but we use the value of the direction field in the approximate point \((x_1, y_1)\). In this way we obtain the approximation \(y_2\) for \(\varphi(x_2)\):

\[
y_2 = y_1 + f(x_1, y_1)(x_2 - x_1).
\]

Continuing this procedure we obtain Euler’s numerical formula:

\[
y_{k+1} = y_k + f(x_k, y_k)(x_{k+1} - x_k), \quad k = 0, n - 1.
\] (6)

If we divide the interval \([x_0, a]\) in equal parts, each of length \(h\), formula (6) becomes

\[
y_{k+1} = y_k + hf(x_k, y_k), \quad k = 0, n - 1.
\]

So, this method computes the approximate values \(y_1, \ldots, y_n\) instead of the exact values \(\varphi(x_1), \ldots, \varphi(x_n)\).

The larger value of \(n\), the better (or, equivalently, the smaller value of \(h\), the better).

**Example.** Let us apply Euler’s method in the interval \([0, 1.5]\) with step size 0.25 to solve numerically the initial value problem

\[
y' = 2xy + e^{x^2}, \quad y(0) = 1.
\]

For comparison we will use its exact solution \(\varphi(x) = (x + 1)e^{x^2}\).

**Step 1.** Notice first that \(f(x, y) = 2xy + e^{x^2}\), \(x_0 = 0\) and \(y_0 = 1\). We take as \([x_0, a]\) the interval \([0, 1.5]\) and divide it into six equal parts to have the step size \(h = 0.25\). This makes \(x_1 = 0.25\), \(x_2 = 0.5\), \(x_3 = 0.75\), \(x_4 = 1\), \(x_5 = 1.25\), and \(x_6 = 1.5\).

**Step 2.** \( y_1 = y_0 + hf(x_0, y_0) = 1 + 0.25 \cdot f(0, 1) = 1 + 0.25 \cdot 1 = 1.25. \)

**Step 3.** \( y_2 = y_1 + hf(x_1, y_1) = 1.25 + 0.25 \cdot f(0.25, 1.25) = 1.25 + 0.25 \cdot 1.689\ldots = 1.672\ldots. \)

**Step 4.** Analogously, we obtain \( y_3 = 2.411\ldots, y_4 = 3.754\ldots, y_5 = 6.311\ldots, \) and \( y_6 = 11.448\ldots. \)

These computations are summarized in the next Table. Compare the numerical values \(y_n\) with the approximation \(\varphi(x_n)\) of the exact ones.
What happens if we take a smaller step size (say, $\bar{h} = 0.1$) in the same interval $[0, 1.5]$? The points that divide the interval are $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, ..., $x_{15} = 1.5$. The corresponding approximations $y_1, y_2, ..., y_{15}$ as well as $\varphi(x_1), \varphi(x_2), ..., \varphi(x_{15})$ are given in the next Table.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_n$</th>
<th>$y_n$</th>
<th>$\varphi(x_n)$</th>
</tr>
</thead>
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<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
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<td>1.50</td>
<td>11.448</td>
<td>23.719</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_n$</th>
<th>$y_n$</th>
<th>$\varphi(x_n)$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1.00</td>
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<tr>
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<td>0.3</td>
<td>1.376</td>
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<tr>
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<td>1.567</td>
<td>1.642</td>
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<tr>
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<td>0.5</td>
<td>1.810</td>
<td>1.926</td>
</tr>
<tr>
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<td>0.6</td>
<td>2.120</td>
<td>2.293</td>
</tr>
<tr>
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<td>0.7</td>
<td>2.518</td>
<td>2.774</td>
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<td>3.033</td>
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<tr>
<td>15</td>
<td>1.5</td>
<td>16.889</td>
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</table>
Again notice that the smaller the step size, the better the approximations. There is
a limit to that tendency, though. If the step is too small, round-off errors lead to less
accurate results. A second observation is that the farther from the starting point, the
worse the approximation gets. This happens because the errors add up. In the last Table,
at \( n = 1 \) the error is approximately 0.01, whereas at \( n = 15 \) it is almost 7. So we expect
that approximations are better on shorter intervals.

The fundamental question concerning any numerical method is that of the algorithm’s
convergence. This is, when \( h \) approaches 0, does \( y_k \) approach \( \varphi(x_k) \) for every \( k = 0, n \)? If
the answer is negative, the method is worthless. It can be shown that Euler’s method is
convergent.

**The second-order Runge-Kutta method**

It was developed in 1895 by the German mathematician and physicist Carl Runge and
improved to higher orders in 1901 by another German mathematician, Wilhelm Kutta.

Instead of approximating \( f(x, y) \) by the value at the left end point of the interval, as
Euler’s procedure does, the second order Runge-Kutta method takes the average of the
approximate values of \( f(x, y) \) at both ends of the interval.

For intervals of equal size \( h \), the formula is

\[
y_{k+1} = y_k + \frac{h}{2} \left[ f(x_k, y_k) + f(x_k + hf(x_k, y_k), y_k) \right], \quad k = 0, n - 1,
\]

also called the Heun’s formula.

At the expense of doubling the amount of computations, the second-order Runge-
Kutta method offers a better approximation of the solution.

**Example.** Let us use the second–order Runge–Kutta method to obtain a numerical
solution for the initial value problem

\[
y' = 2xy + e^{x^2}, \quad y(0) = 1,
\]
in the interval \([0, 1.5]\), with step–size \( h = 0.1 \).

*Step 1.* Take \( x_0 = 0, x_1 = 0.1, \ldots, x_{15} = a = 1.5, y_0 = 1 \), and \( h = 0.1 \).

*Step 2.* \( y_1 = y_0 + \left( \frac{h}{2} \right) \left[ y_0 + f(x_0, y_0) + f(x_0, y_0 + hf(x_0, y_0)) \right] = 1.111 \ldots \).
Step 3. $y_2 = y_1 + (h/2) [y_1 + f(x_1, y_1) + f(x_1, y_1 + hf(x_1, y_1))] = 1.249$....

Step 4. Continuing the procedure we obtain the values in the next Table.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_n$</th>
<th>$y_n$</th>
<th>$\varphi(x_n)$</th>
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</tr>
</tbody>
</table>

Errors

Evaluating the errors of numerical methods is an important but difficult task. Errors occur because both the formulas and the values we substitute into the formulas are approximate. Let us describe different types of errors.

The global truncation error $E_k$ at step $k$ is the absolute value of the difference between the exact solution $\varphi(x_k)$ and the numerical solution $y_k$,

$$E_k = |\varphi(x_k) - y_k|,$$

assuming that the process of approximation has started at step 0 (i.e. the only exact value of the sequence $y_0, y_1, \ldots, y_n$ is $y_0$). In general, this error is difficult to determine; therefore we usually deal with the so-called local truncation error.
The local truncation error $e_k$ at step $k$ is the absolute value of the difference between the exact value $\varphi(x_k)$ and the approximate value $y_k$,

$$e_k = |\varphi(x_k) - y_k^e|$$

($e$ stand for ”exact”), but assuming that the process of approximation has started only at this step, i.e. $y_{k-1} = \varphi(x_{k-1})$. In other words, the data are considered exact, so the only source of error is the formula of the method.

The round-off error $r_k$ at step $k$ is the absolute value of the difference between the approximate value $y_k$, as given by the computer with a finite number of decimal places, and the theoretical approximation $Y_k$, supposed to have infinitely many digits. So

$$r_k = |y_k - Y_k|.$$  

The total error $T_k$ at step $k$ (or the theoretical error) is the absolute value of the difference between the exact value $\varphi(x_k)$ and the theoretical approximation $Y_k$,

$$T_k = |\varphi(x_k) - Y_k|.$$  

Since we have

$$T_k = |\varphi(x_k) - Y_k| = |\varphi(x_k) - y_k + y_k - Y_k| \leq |\varphi(x_k) - y_k| + |y_k - Y_k| = E_k + r_k,$$

we deduce that the total error is never greater than the sum of the global truncation error and the round-off error.

Errors can be expressed with respect to the step size $h$. For Euler’s method, $e_n$ is of order $h^2$, whereas for the second order Runge-Kutta method, $e_n$ is of order $h^3$. This explains why the Runge–Kutta method is better.

The equation of our example is linear. In general, linear differential equation raise fewer problems if treated numerically. Nonlinear differential equation are more troublesome, especially if the interval of approximation is large.