Workshop dedicated to the memory of Professor Gabriela Kohr
(3rd edition)
Geometric Function Theory in Several Complex Variables and Complex Banach Spaces

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Geometric Properties of Some Special Functions

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Abstract

A widely investigated homogeneous second-order differential equation is given by

$$z^2\omega''(z) + qz\omega'(z) + \left[rz^2 - p^2 + (1-q)p\right]\omega(z) = 0,$$

whose solutions are extensions of the *generalized Bessel function*, $p, q \in \mathbb{R}$ and $r \in \mathbb{C}$. The *generalized Bessel function of order* p is the particular solution of this differential equation which has the power series expansion

$$\omega_{p,q,r}(z) = \sum_{j=0}^{\infty} \frac{(-r)^j}{\Gamma(j+1)\Gamma\left(p+j+\frac{q+1}{2}\right)} \left(\frac{z}{2}\right)^{2j+p}.$$

It is worth mentioning that the above differential equation has a particular interest, and it allows us to know more information regarding the Bessel, modified Bessel, and spherical Bessel functions. This power series is convergent everywhere while it is not univalent in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. Bearing in mind also that special values of the parameters p, q and r will give us the well-known *Bessel, modified Bessel, and spherical Bessel functions.* One can observe that $\omega_{p,q,r}$ is not usually normalized, therefore we consider the following transformation

$$\mathrm{u}_{p,q,r}(z) := 2^p\, \Gamma\left(p + \frac{q+2}{2}\right)\, z^{-\frac{p}{2}}\, \omega_{p,q,r}\left(\sqrt{z}\right),$$

and the series expansion of $\mathrm{u}_{p,q,r}$ has the form

$$\mathrm{u}_{p,q,r}(z) = \sum_{j=0}^{\infty} \frac{(-r)^j}{4^j (1)_j \left(p+\frac{q+2}{2}\right)_j} z^j,$$

where $\sigma := p + (q+2)/2 \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, $r \in \mathbb{C} \setminus \{0\}$, and $(\rho)_n$ represents the *Pochhammer symbol.* For $p, q, r \in \mathbb{C}$ satisfying the previous conditions, the *normalization of the of generalized Bessel functions* $\mathrm{U}_{\sigma,r}$ is defined by

$$\mathrm{U}_{\sigma,r}(z) := z \cdot \mathrm{u}_{p,q,r}(z) = z + \sum_{j=1}^{\infty} \frac{(-r)^j}{4^j (1)_j (\sigma)_j} z^{j+1},\ z \in \mathbb{U}.$$

We establish geometric properties, such as starlikeness and convexity of order α $(0 \le \alpha < 1)$ in $\mathbb{U}$ for the *normalization of the generalized Bessel function.* In some previous paper we used for these purposes the L. Fejér (1936) and S. Ozaki's (1935) inequalities, while in the actual study we are using some properties of the gamma and digamma functions.

1. H. M. Zayed, T. Bulboacă, On some geometric properties for the combination of generalized Lommel-Wright function, J. Inequal. Appl. 2021, **158**(2021), 1–19, https://doi.org/10.1186/s13660-021-02690-z
2. H. M. Zayed, T. Bulboacă, Normalized generalized Bessel function and its geometric properties, J. Inequal. Appl. 2022, 158(2022), 1–26, https://doi.org/10.1186/s13660-022-02891-0
3. H. M. Zayed, T. Bulboacă, Geometric properties for the normalization of the generalized Bessel function, /submitted/