Adaptive Cubatures on Triangle

How to implement them

Radu T. Trîmbițaș

"Babeș-Bolyai" University

ROGER 2007, Königswinter

Outline

Problem

- Our problem: calculate the definite integral

$$If := \int_B f(x)dx$$

$$f : B \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}, \text{ given integrand, } B \text{ given region.}$$

Problem

- Our problem: calculate the definite integral

$$If := \int_B f(x)dx$$

$$f : B \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}, \text{ given integrand, } B \text{ given region.}$$

- The aim of constructing integration algorithm is to approximate *If* with a given error tolerance ε and as few function evaluations as possible.

What is an adaptive integration algorithm

- Adaptive algorithms decide dynamically how many function evaluations are needed. The information for such decisions is derived from numerical experiments based on integrand. In general, no a priori information about the decision process is available. The efficiency and reliability of such algorithms depends upon the subdivision strategy.
  - The decision as to whether or not a subregion has to be further subdivided is based on either local and global knowledge. This leads to local and global subdivision strategy respectively.
  - Local knowledge is based only on the considered subregion.
  - Global knowledge is based on knowledge about all subregions of the integration region.
  - In any case, the depth of the subdivision process is determined dynamically.

Notes

- For the 1D case the basic idea is as follows: Let $[a, b]$ be a bounded interval. In order to compute

$$I = \int_a^b f(x)\,\mathrm{d}x$$

we integrate f using two methods which provide us the approximations I_1 and I_2. If the difference of this two approximations is less than a given tolerance, we accept the better of them, say I_2, as approximate value of integral. Otherwise, we divide $[a, b]$ into two (or three) congruent parts; then proceed recursively on each part.

Notes

- For the 1D case the basic idea is as follows: Let $[a, b]$ be a bounded interval. In order to compute

$$I = \int_a^b f(x)\,\mathrm{d}x$$

  we integrate f using two methods which provide us the approximations I_1 and I_2. If the difference of this two approximations is less than a given tolerance, we accept the better of them, say I_2, as approximate value of integral. Otherwise, we divide $[a, b]$ into two (or three) congruent parts; then proceed recursively on each part.
- The idea is credited to Huygens, but in this form appear in [Davis, Rabinowitz 1984].

Negative results

- [deBoor 1971] it is impossible to construct a correct program that integrates each given function.
- Moreover, for a given program, it is possible to find a function f, which is not correctly integrated ([Kahan 1980]).
- Hence, the task of each implementer is to code programs which function correctly for a class of function as large as possible.

The Rice's meta algorithm

- The Rice's meta algorithm 2 [Rice 1975] is an abstract description of the mechanisms involved in adaptive integration.
- It can be used as a starting point for the development of adaptive integration algorithms based on a given formula Q_N with an error estimator E.
- We reproduced it here in the form given in [Überhuber 1995]

The Rice's meta algorithm

Meta algorithm for adaptive integration

Input: f, B, ε, Q_N, E.

Output: The approximate integral value q and the error estimation e.

$q := Q(f; B)$; $e := E(f, B)$;
insert (B, q, e) into the data structure;
while $e > \varepsilon$ **do**
choose an element of the data structure (with index s;)
Subdivide B_s into subregions B_ℓ, $\ell = 1, 2, \ldots, L$;
Calculate approximations for integrals over $B_1, \ldots, B_L$

$$q_\ell := Q_N(f; B_\ell), \quad \ell = 1, 2, \ldots, L;$$

Calculate corresponding error estimates;

$$e_\ell := E(f; B_\ell), \quad \ell = 1, 2, \ldots, L;$$

remove old data (B_s, q_s, E_s) from the data structure;
Insert $(B_1, q_1, e_1), \ldots (B_L, q_L, e_L)$ into the data structure;
$q := \sum_i q_i$; $e := \sum_i e_i$;
end while

The case of triangle

- As in the case of [Laurie 1982, Berntsen, Espelid 1992, Cools et al. 1997] our region B will be a collection of triangles.
- This allow:
  - A larger degree of generality
  - To restart the algorithm for refinement, performing the continuation of previous work.

Basic Elements

- A collection of triangles organized in a heap; M is the current number of triangles
- A quadrature rule Q to produce a local estimate to the integral over each triangle of the collection
- A procedure for error estimation E
- A strategy for picking the next triangle to be processed — in our case the triangle on the top of the heap
- A subdivision strategy

The algorithm

Initialize the triangle collection; $M := m$;
Compute $\hat{Q}_i$ and $\hat{E}_i$, $i = 1, 2, \ldots, m$
$\hat{Q} = \sum_{i=1}^{m} \hat{Q}_i$; $\hat{E} = \sum_{i=1}^{n} \hat{E}_i$;
while $\hat{E} > \varepsilon$ **do**
  {Control}
  Pick the triangle T_k on top of heap;
  {Subdivision}
  Divide T_k in p parts;
  {Process triangles}
  Compute $\hat{Q}_k^{(i)}, \hat{E}_k^{(i)}, i = 1, \ldots, p$;
  {Update}
  $\hat{Q} := \hat{Q} + \sum_{i=1}^{p} \hat{Q}_k^{(i)} - \hat{Q}_k$; $\hat{E} := \hat{E} + \sum_{i=1}^{p} \hat{E}_k^{(i)} - \hat{E}_k$;
  Replace triangle T_k by p new triangles;
  $M := M + p - 1$;
end while

Data structures

- A collection of triangles `Tri`, organized as an array of triangles. Information for each triangle:
  - `V1`, `V2`, `V3` - pointers to vertices (see `Vertex` bellow)
  - `VI` - approximate of the integral
  - `EE` - error estimation
- A collection of vertices, `Vertex`, organized as a matrix with two columns (coordinates)
- A heap of pointers to triangles, `Heap`. Ordered by `EE`. Triangle with maximum `EE` on top.

Cubature rules

- The user may choose the cubature rule. A procedures that initializes the nodes and the coefficients is specified at invocation.
- The rule must be given in fully symmetric form, as in [Stroud 71] or in Ronald Cools' Encyclopedia of cubature formula [Encyclopedia].
- A procedure evaluates the cubature formula given in fully symmetric form.
- Supported: a 37 point PI rule of degree 13 [Berntsen, Espelid 1990] and a seven point PI rule of degree 5, due to Radon

Error estimation

- Two methods:

Error estimation

- Two methods:
- Embedded cubature formulas.

Error estimation

- Two methods:
- Embedded cubature formulas.
- Null rules.

Embedded cubature formulas

- We shall use two cubature formulas

$$Q_j[f] = \sum_{i=1}^{N_j} f(x_{ij}, y_{ij}), \qquad j \in \{1, 2\},$$

  with degree of exactness d_j, $d_1 < d_2$, where N_j is the number of nodes for Q_j.

- In order to reduce the number of function evaluation (and so the amount of work) one tries to choose Q_1 and Q_2 such that

$$\{(x_{i1}, y_{i1} : i = 1, \dots, N_1\} \subset \{x_{i2}, y_{i2} : i = 1, \dots, N_2\}.$$

  A pair (Q_1, Q_2) having this property is called an embedded pair.

- The difference $|Q_1[f] - Q_2[f]|$ is used as an error estimation for Q_1.

Null rules

Definition

[Lyness 1965]A rule

$$N[f] = \sum_{i=0}^{n} u_i f(x_i) \tag{1}$$

is a null rule iff it has at least one nonzero weight, and in addition $\sum_{i=0}^{n} u_i = 0$. A null rule has the degree d if it integrates to zero all basic monomials of degree $\leq d$ and fail to do so for a monomial of degree $d+1$.

- A null rule of the form (1) has the degree at most $n-1$.
- Null rules may be used as estimations of error.
- An estimation based on a single null rule is sometimes unreliable; in practice one uses combination of null rules of various degrees.[Berntsen, Espelid 1991]

Error estimation using null rules

{Compute}
$e_j := N_j[f], j = 1, \ldots, 2k;$
$E_j := (e_{2j-1}^2 + e_{2j}^2)^{1/2},\ j = 1, \ldots, k;$
$r_j := E_j / E_{j+1},\ j = 1, \ldots, k-1;$
if $r > 1$ **then**
  $\hat{E} = 10 \max_j E_j$ {Nonasymptotic}
else if $1/2 \leq r$ **then**
  $\hat{E} := 10 r^1 E_1$ {Weakly-asymptotic}
else
  $E = 10 \cdot 4 r^3 E_1$ {Strongly-asymptotic}
end if

- Since cubature rules and null rules are based on the same set of nodes, they are evaluated simultaneously.
- Embedded cubatures are considered combination of a cubature rule Q_1 and a null rule $Q_1 - Q_2$.

Subdivision

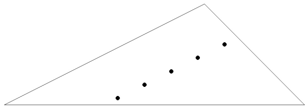
- The simplest subdivision is in four congruent triangles, determined by vertices and midpoints of edges
- A **more flexible method**
- Use subdivision directions parallel to the sides of the triangle
- 4th differences parallel to the sides are computed
- Let e be a unit vector along one side of triangle T_k, h the length of the side, C the barycenter. Define the measure of variation of f in direction e:

$$D(e) = h^q \left| f\left(C - \frac{4}{15}he\right) - 4f\left(C - \frac{2}{15}he\right) + 6f(C) - 4f\left(C + \frac{2}{15}he\right) + f\left(C - \frac{4}{15}he\right) \right| \tag{2}$$

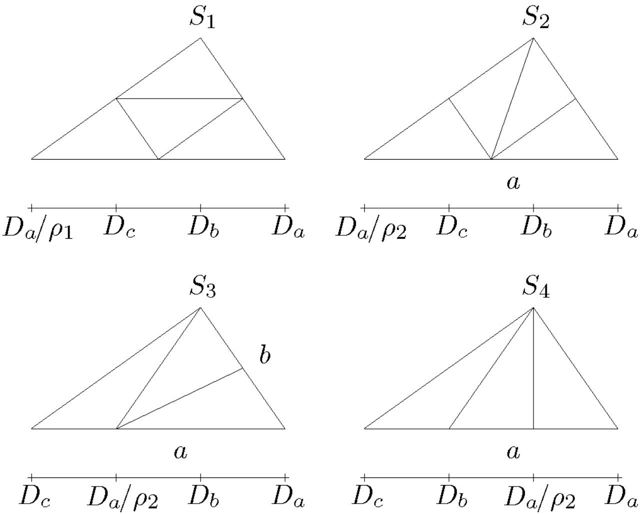
- Three heuristic constants q, ρ_1, ρ_2 are involved

Subdivision

- Define $D_a = D(a/\|a\|)$, $D_b = \ldots$ using (2)
- The triangle is divided into four or three triangles according to magnitude of D_a, D_b, D_c
- Requires 13 new function evaluations



Subdivision types



Algorithm — Choice of subdivision

1. Compute estimates D_a, D_b, D_c for sides a, b, c
2. Relabel the sides so that $D_a \geq D_b \geq D_c$
3. **If** $D_c \geq D_a/\rho_1$ **then** choose S_1;

   else if $D_b \geq D_a/\rho_2$ and $D_c \geq D_a/\rho_2$ **then** choose S_2;

   else if $D_b \geq D_a/\rho_2$ and $D_c < D_a/\rho_2$ **then** choose S_3;

   else choose S_4

   end if

MATLAB implementation - main function

- We code a flexible set of functions which allow to select the cubature formula and null rules. There exists also a restart facility that allows the refinement of previous result. The syntax of main function, `CubatureTriang` is

$$[\texttt{result}, \texttt{ee}, \texttt{stat}, \texttt{Tri}, \texttt{Vertex}, \texttt{VI}, \texttt{EE}] = \texttt{CubatureTriang}(\texttt{F}, \texttt{Tri}, \ldots$$
$$\texttt{Vertex}, \texttt{VI}, \texttt{EE}, \texttt{opt}, \texttt{varargin})$$

- Parameters:

`F` - function to be integrated

MATLAB implementation - main function

- We code a flexible set of functions which allow to select the cubature formula and null rules. There exists also a restart facility that allows the refinement of previous result. The syntax of main function, `CubatureTriang` is

$$[\mathtt{result}, \mathtt{ee}, \mathtt{stat}, \mathtt{Tri}, \mathtt{Vertex}, \mathtt{VI}, \mathtt{EE}] = \mathtt{CubatureTriang}(\mathtt{F}, \mathtt{Tri}, \ldots$$
$$\mathtt{Vertex}, \mathtt{VI}, \mathtt{EE}, \mathtt{opt}, \mathtt{varargin})$$

- Parameters:

`Tri` - collection of triangles

MATLAB implementation - main function

- We code a flexible set of functions which allow to select the cubature formula and null rules. There exists also a restart facility that allows the refinement of previous result. The syntax of main function, `CubatureTriang` is

$$[\texttt{result}, \texttt{ee}, \texttt{stat}, \texttt{Tri}, \texttt{Vertex}, \texttt{VI}, \texttt{EE}] = \texttt{CubatureTriang}(\texttt{F}, \texttt{Tri}, \ldots$$
$$\texttt{Vertex}, \texttt{VI}, \texttt{EE}, \texttt{opt}, \texttt{varargin})$$

- Parameters:

`Vertex` - collection of vertices

MATLAB implementation - main function

- We code a flexible set of functions which allow to select the cubature formula and null rules. There exists also a restart facility that allows the refinement of previous result. The syntax of main function, `CubatureTriang` is

$$[\mathtt{result}, \mathtt{ee}, \mathtt{stat}, \mathtt{Tri}, \mathtt{Vertex}, \mathtt{VI}, \mathtt{EE}] = \mathtt{CubatureTriang}(\mathtt{F}, \mathtt{Tri}, \ldots$$
$$\mathtt{Vertex}, \mathtt{VI}, \mathtt{EE}, \mathtt{opt}, \mathtt{varargin})$$

- Parameters:

`VI` - value of integral for a triangle

MATLAB implementation - main function

- We code a flexible set of functions which allow to select the cubature formula and null rules. There exists also a restart facility that allows the refinement of previous result. The syntax of main function, `CubatureTriang` is

$$[\mathtt{result}, \mathtt{ee}, \mathtt{stat}, \mathtt{Tri}, \mathtt{Vertex}, \mathtt{VI}, \mathtt{EE}] = \mathtt{CubatureTriang}(\mathtt{F}, \mathtt{Tri}, \ldots \\ \mathtt{Vertex}, \mathtt{VI}, \mathtt{EE}, \mathtt{opt}, \mathtt{varargin})$$

- Parameters:

`EE` - error estimation for a triangle

MATLAB implementation - main function

- We code a flexible set of functions which allow to select the cubature formula and null rules. There exists also a restart facility that allows the refinement of previous result. The syntax of main function, `CubatureTriang` is

$$[\mathtt{result}, \mathtt{ee}, \mathtt{stat}, \mathtt{Tri}, \mathtt{Vertex}, \mathtt{VI}, \mathtt{EE}] = \mathtt{CubatureTriang}(\mathtt{F}, \mathtt{Tri}, \ldots \\ \mathtt{Vertex}, \mathtt{VI}, \mathtt{EE}, \mathtt{opt}, \mathtt{varargin})$$

- Parameters:

`opt` - options: `errabs`, `errel`, `restart`, `initf`, `trace`, `nfev` -

MATLAB implementation - main function

- We code a flexible set of functions which allow to select the cubature formula and null rules. There exists also a restart facility that allows the refinement of previous result. The syntax of main function, `CubatureTriang` is

$$[\mathtt{result}, \mathtt{ee}, \mathtt{stat}, \mathtt{Tri}, \mathtt{Vertex}, \mathtt{VI}, \mathtt{EE}] = \mathtt{CubatureTriang}(\mathtt{F}, \mathtt{Tri}, \ldots$$
$$\mathtt{Vertex}, \mathtt{VI}, \mathtt{EE}, \mathtt{opt}, \mathtt{varargin})$$

- Parameters:

`initf` - initialization function; return cubature parameters: weights, nodes, null rulles type;
call: `[W,G,m,p]=initf`

MATLAB implementation - main function

- We code a flexible set of functions which allow to select the cubature formula and null rules. There exists also a restart facility that allows the refinement of previous result. The syntax of main function, `CubatureTriang` is

$$[\mathtt{result}, \mathtt{ee}, \mathtt{stat}, \mathtt{Tri}, \mathtt{Vertex}, \mathtt{VI}, \mathtt{EE}] = \mathtt{CubatureTriang}(\mathtt{F}, \mathtt{Tri}, \ldots$$
$$\mathtt{Vertex}, \mathtt{VI}, \mathtt{EE}, \mathtt{opt}, \mathtt{varargin})$$

- Parameters:

`result` - approximate of integral

MATLAB implementation - main function

- We code a flexible set of functions which allow to select the cubature formula and null rules. There exists also a restart facility that allows the refinement of previous result. The syntax of main function, `CubatureTriang` is

$$[\mathtt{result}, \mathtt{ee}, \mathtt{stat}, \mathtt{Tri}, \mathtt{Vertex}, \mathtt{VI}, \mathtt{EE}] = \mathtt{CubatureTriang}(\mathtt{F}, \mathtt{Tri}, \ldots$$
$$\mathtt{Vertex}, \mathtt{VI}, \mathtt{EE}, \mathtt{opt}, \mathtt{varargin})$$

- Parameters:

ee - error estimation

MATLAB implementation - main function

- We code a flexible set of functions which allow to select the cubature formula and null rules. There exists also a restart facility that allows the refinement of previous result. The syntax of main function, `CubatureTriang` is

$$[\mathtt{result}, \mathtt{ee}, \mathtt{stat}, \mathtt{Tri}, \mathtt{Vertex}, \mathtt{VI}, \mathtt{EE}] = \mathtt{CubatureTriang}(\mathtt{F}, \mathtt{Tri}, \ldots$$
$$\mathtt{Vertex}, \mathtt{VI}, \mathtt{EE}, \mathtt{opt}, \mathtt{varargin})$$

- Parameters:

`stat` - statistics: number of function evaluations, number of triangles, success/failure.

MATLAB implementation - cubature and null rules

- Cubature rule and null rules are given in fully symmetric form.
- The function `fselcub` approximate the integral and the error on the current triangle.
- The selection of cubature and null rules is performed via the `initf` parameter of `CubatTri`. Implemented:
  - Berntsen & Espelid 13 degree formula with eight null rules, function `BerntsenEspelid`
  - embedded 5-7 degree cubature formula, function `ecf57` [Cools, Haegemans 1988].
  - embedded 5-7 degree cubature formula, function `ecf58` [Laurie 1982].
- The user can code his own function if he/she obeys the call syntax.

MATLAB implementation - Data structures management

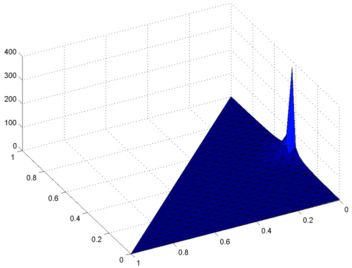
- Function NewVertex inserts a new vertex into the Vertex matrix
- Function NewTriangle inserts a triple of pointers (indices) to the vertices of triangle into the array Tri
- Function InsertIntoHeap takes a pointer to the current triangle and its error estimation and insert the pointer into heap at an appropriate place, udating the heap
- Function ExtractMaxFromHeap extract the top triangle from heap and update the data structures.

Examples and tests - Test families

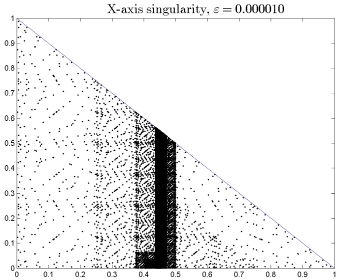
	Test family	Attributes
1	$f_1(x,y) = (\lvert x-\beta_1\rvert + y)^{d1}$	X-axis singularity
2	$f_2(x,y) = \begin{cases} 1 & \sqrt{(x-\beta_1)^2+(y-\beta_2)^2} < d2 \\ 0 & \text{otherwise} \end{cases}$	Discontinuous
3	$f_3(x,y) = \exp(-\alpha_1\lvert x-\beta_1\rvert - \alpha_2\lvert y-\beta_2\rvert)$	C_0 function
4	$f_4(x,y) = \exp(-\alpha_1^2(x-\beta_1)^2 - \alpha_2^2(y-\beta_2)^2)$	Gaussian
5	$f_5(x,y) = (\alpha_1^{-2}+(x-\beta_1)^2)^{-1}(\alpha_2^{-2}+y^2)^{-1}$	X-axis peak
6	$f_6(x,y) = (\alpha_1^{-2}+(x-\beta_1)^2)^{-1}(\alpha_2^{-2}+(y-\beta_2)^2)^{-1}$	Internal peak
7	$f_7(x,y) = \cos(2\pi\beta_1+\alpha_1 x+\alpha_2 y)$	Oscillatory

- d_j - difficulty parameters, $j = 1, \ldots, 7$
- α_1, α_2, β_1, β_2 - random parameters uniformly distributed on $[0,1]$.
- α_1, α_2 scaled such that $\alpha_1 + \alpha_2 = d_j$

Test - family 1



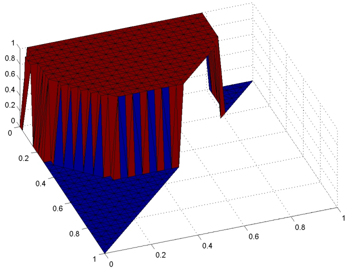
(a) Graph of f_1



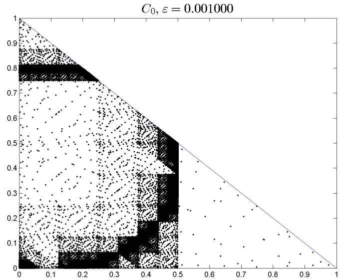
(b) Evaluation points

Figure: Test for family 1

Test - family 2



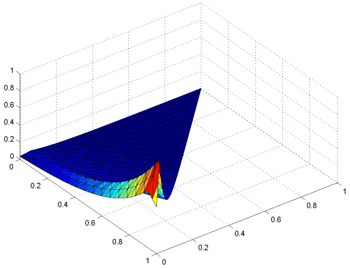
(a) Graph of f_2



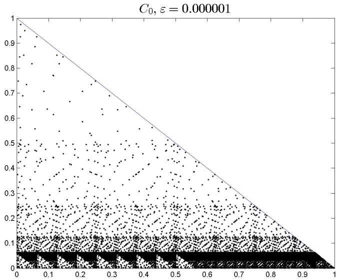
(b) Evaluation points

Figure: Test for family 2

Test - family 3



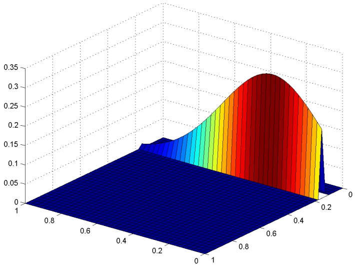
(a) Graph of f_3



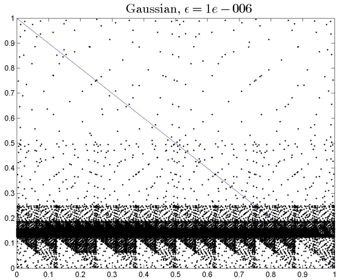
(b) Evaluation points

Figure: Test for family 3

Test - family 4



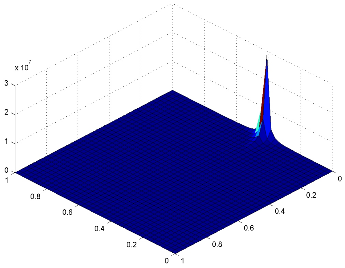
(a) Graph of f_4



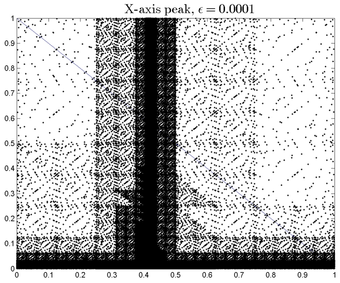
(b) Evaluation points

Figure: Test for family 4

Test - family 5



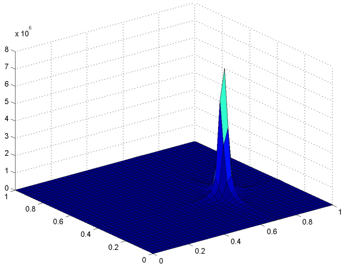
(a) Graph of f_5



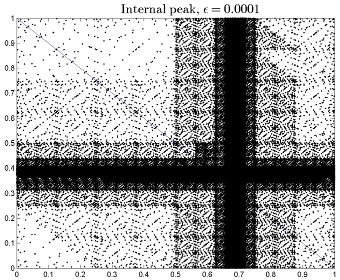
(b) Evaluation points

Figure: Test for family 5

Test - family 6



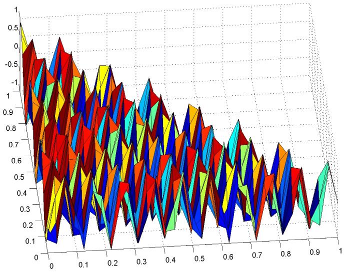
(a) Graph of f_6



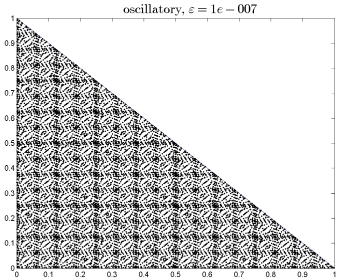
(b) Evaluation points

Figure: Test for family 6

Test - family 7



(a) Graph of f_7

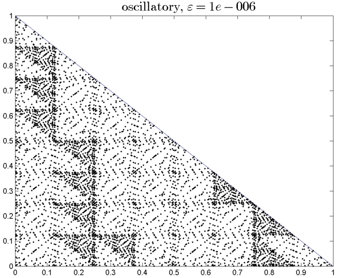


(b) Evaluation points

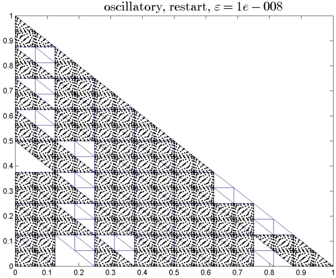
Figure: Test for family 7

Test with restart

Family 7, first call for `errabs=1e-6`, then restart with `errabs=1e-8`.



(a) First step

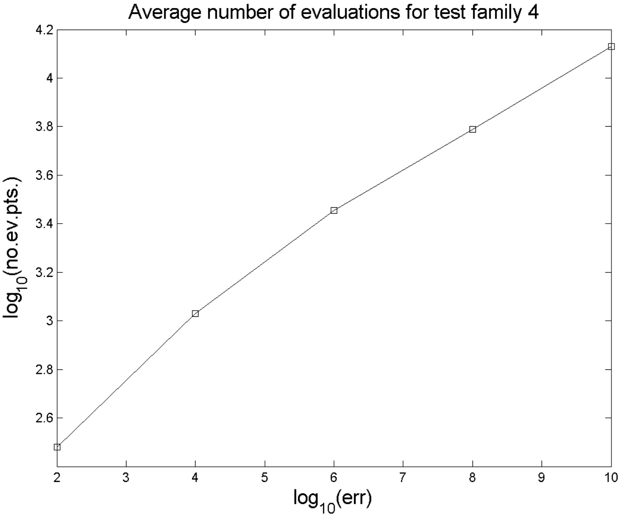


(b) Second step, restart

Figure: Test for family 7 with restart

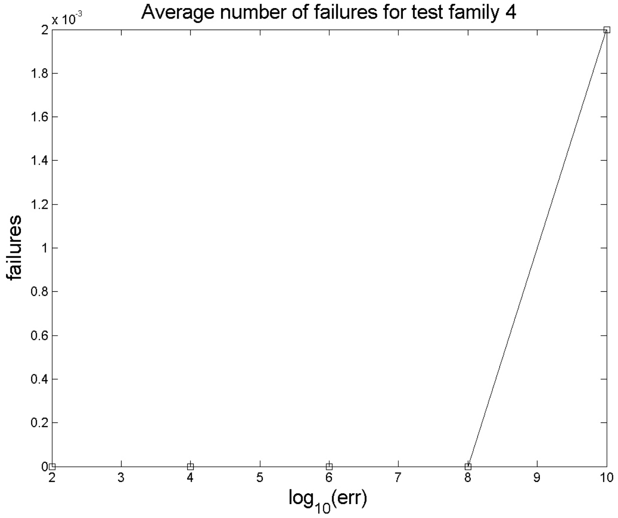
Number of function evaluation - family 4

$\varepsilon = 10^{-2}, 10^{-4}, \ldots, 10^{-10}$, 500 samples for each error



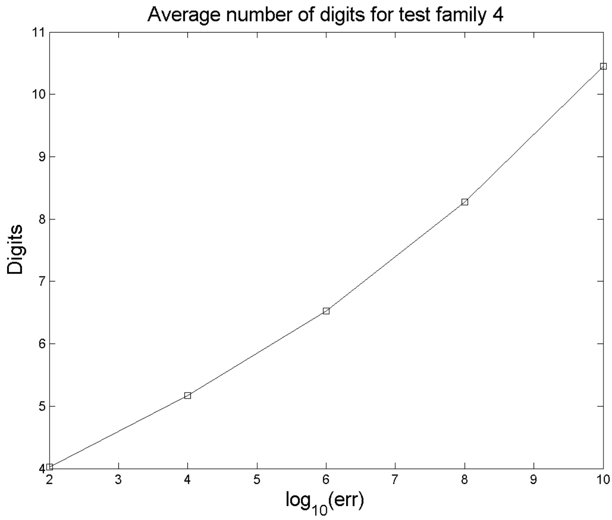
Number of failures - family 4

$\varepsilon = 10^{-2}, 10^{-4}, \ldots, 10^{-10}$, 500 samples for each error



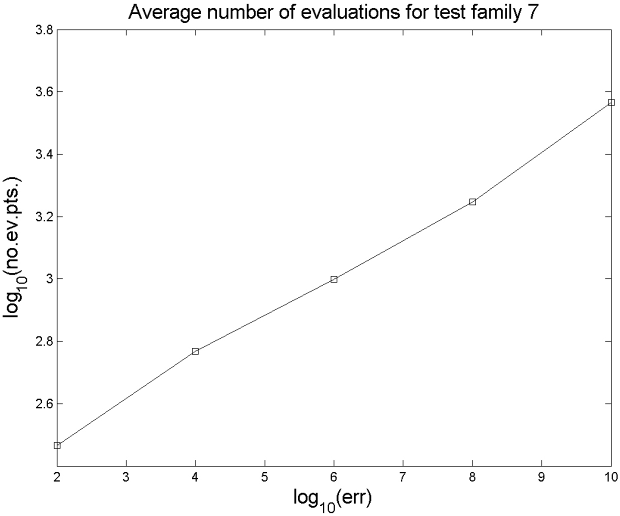
Number of correct digits - family 4

$\varepsilon = 10^{-2}, 10^{-4}, \ldots, 10^{-10}$, 500 samples for each error



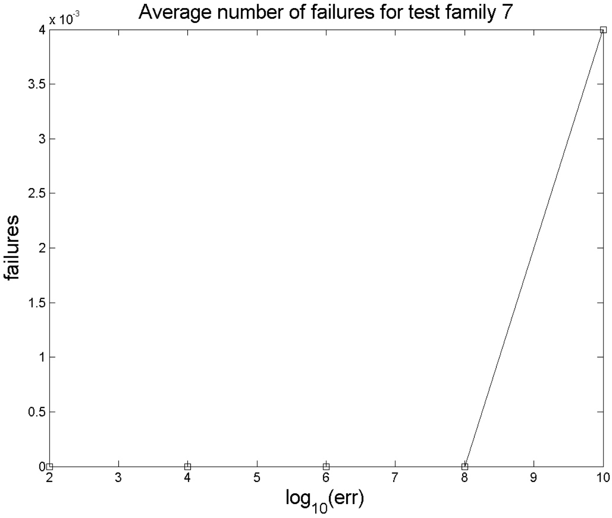
Number of function evaluation - family 7

$\varepsilon = 10^{-2}, 10^{-4}, \dots, 10^{-10}$, 500 samples for each error



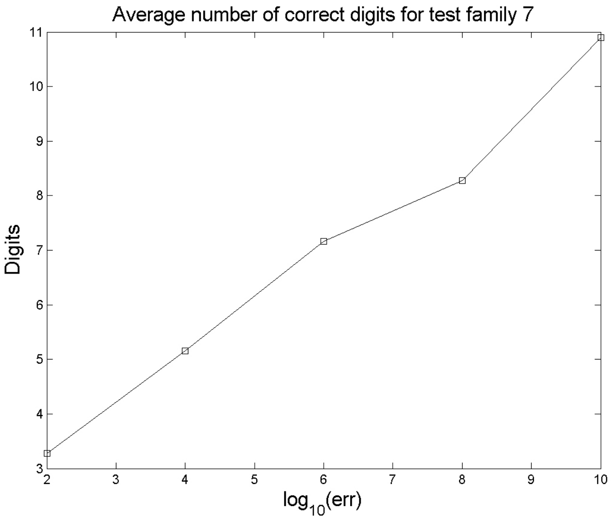
Number of failures - family 7

$\varepsilon = 10^{-2}, 10^{-4}, \ldots, 10^{-10}$, 500 samples for each error



Number of correct digits - family 7

$\varepsilon = 10^{-2}, 10^{-4}, \ldots, 10^{-10}$, 500 samples for each error



References I

- J. Berntsen, T. O. Espelid:
  Algorithm 706: DCUTRI: An algorithm for Adaptive Cubature Over a Collection of Triangles,
  ACM TOMS, vol. 18, No. 3, 1992, pp. 329–342.
- J. Berntsen, T. O. Espelid:
  Error Estimation in Automatic Quadrature Routines,
  ACM TOMS, vol. 17, No. 2, 1991, pp. 233–252.
- J. Berntsen, T. O. Espelid:
  Degree 13 Symmetric Quadrature Rules for the Triangle,
  Report No. 44, Bergen University, Norway, 1990.

References II

T. O. Espelid, A. Genz:
On the Subdivison Strategy in Adaptive Cubature Algorithms for Triangular Regions,
Report No. 44, Bergen University, Norway, 1990.

Carl de Boor:
On writing automatic integration algorithm,
in: *Mathematical Software*, John R. Rice ed., Academic Press, New York, pp. 201–209, 1971.

R. Cools, A. Haegemans:
An embedded pair of cubature formulae of degree 5 and 7 for the triangle,
BIT 28, 1988, pp. 357–359.

References III

R. Cools, D. P. Laurie, L. Pluym:
Cubpack++: A C++ package for Authomatic Two-dimensional Cubature,
ACM TOMS, vol. 23, No. 1, 1997, pp. 1–15.

Philip J. Davis, Philip Rabinowitz:
Methods of Numerical Integration, 2nd edition,
Academic Press, Orlando, 1984.

William M. Kahan:
Handheld calculator evaluates integrals,
Hewlett-Packard Journal **31**(8), 23–32, 1980.

D. P. Laurie:
Algorithm 584: CUBTRI: Automatic Cubature over a triangle,
ACM TOMS, vol. 8, No. 2, 1982, pp. 210–218.

References IV

J. Rice:
A metaalgorithm for adaptive quadrature,
JACM, **22**(1975), pp. 61–82.

I. Somogyi, R. Trîmbiţas:
The study of an adaptive algorithm for some cubature formulas on triangle,
Studia UBB.

I. Somogyi, R. Trîmbiţas:
On an adaptive algorithm based on embedded cubature formulas on triangle,
MACS6, Pécs, 2006.

A. H. Stroud:
Approximate Calculation of Multiple Integrals,
Englewood Cliffs, N.J. Prentice-Hall, Inc.1971.

References V

Cr. Überhuber,
Computer Numerik, Band II,
Springer, 1995.

Ronald Cools:
Encyclopedia of Cubature Formulae
www.cs.kuleuven.be/cwis/research/nines/research/ecf

J. Nievergelt, H. Hinterberger, K. C. Sevcik:
The grid file: An adaptable, symmetric, multikey file structure,
ACM Trans. on DB Syst. 9(1), 1984.

H. Samet:
The Design and Analysis of Spatial Data Structure,
Addison -Wesley, Reading, Ma., 1991.

References VI

M. G. Trîmbițaș, R. T. Trîmbițaș:
Adaptive cubatures on triangle and spatial data structures,
Proceedings NAAT 2006, pp. 401–409.

J. N. Lyness:
Symmetric integration rules for hypercubes III. Construction of integration rules using null rules.
Math. Comput. 19, 1965, pp. 625–637.