MATHEMATICA, 65 (88), N° 2, 2023, pp. 275–285

EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR NON-LINEAR WAVE EQUATIONS OF KIRCHHOFF TYPE WITH VISCOELASTICITY

MOHAMED MELLAH and ALI HAKEM

Abstract. In this paper we deal with the initial boundary value problem of the following non-linear wave equation of Kirchhoff type

$$|u_t|^\rho u_{tt} - M\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s - \Delta u_t = 0,$$

where M is a continuous function on $[0,+\infty)$ such that $M(s) \geq m_0 > 0$ for all $s \geq 0$. By assuming $\rho > 0$ is such that $H_0^1(\Omega) \hookrightarrow L^{\rho+2}(\Omega)$ and $g > 0$ is exponentially decreasing, we discuss the global existence and asymptotic behavior of solutions.

MSC 2020. 35J05.

Key words. Viscoelastic, Kirchhoff-type, global existence, asymptotic behavior.

1. INTRODUCTION

In this paper, we investigate the existence and asymptotic behavior of solutions for the following non-linear Kirchhoff type problem

$$(1)\qquad \begin{aligned} &|u_t|^\rho u_{tt} - M\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u - \Delta u_{tt} \\ &+ \int_0^t g(t-s)\Delta u(s)\mathrm{d}s - \Delta u_t = 0, x \in \Omega, t > 0, \\ &u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad x \in \Omega, \end{aligned}$$

where M is a continuous function on $[0,+\infty)$ such that $M(s) \geq m_0 > 0$ for all $s \geq 0$, Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$, and ρ is a positive constant. Here, g represents the kernel of the memory term which is assumed to decay exponentially (see assumption **(A2)**).

The authors thank the referee for his helpful comments and suggestions.
Corresponding author: Ali Hakem.

DOI: 10.24193/mathcluj.2023.2.13

Viscoelasticity problems have been handled carefully in several papers, and other results relating the global existence and decay of the global solution have been found. For example, Cavalcanti et al. [1] studied the following problem

$$
\text{(2)} \qquad \begin{aligned}
&|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} \\
&+ \int_0^t g(t-s)\Delta u(s)\mathrm{d}s - \gamma \Delta u_t = 0 \quad (x,t) \in \Omega \times \mathbb{R}^+, \\
&u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad x \in \Omega,
\end{aligned}
$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and ρ is a positive real number. By assuming $\rho > 0$ is such that $H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega)$ and $g > 0$ is exponentially decreasing, they established global existence in the case $\gamma \geq 0$ and obtained exponential decay of the energy in the case $\gamma > 0$. Cavalcanti et al. [2] considered this model and proved intrinsic decays for large classes of relaxation kernels described by the inequality $g' + H(g) \leq 0$ with convex function H. Replacing strong damping by weak damping in (2), several authors have studied the energy decay rates of the related problems like

$$
\text{(3)} \qquad \begin{aligned}
&|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} \\
&+ \int_0^t g(t-s)\Delta u(s)\mathrm{d}s + h(u_t) = 0, \quad (x,t) \in \Omega \times \mathbb{R}^+, \\
&u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad x \in \Omega.
\end{aligned}
$$

When $h(u_t) = u_t$, Han and Wang [4] investigated the global existence and exponential stability of the energy for solutions for (3). When $h(u_t) = |u_t|^m u_t$ ($m > 0$), the general decay of energy was investigated by the same authors [3]. Later, Park and Park [8] established the general decay for (2) with general nonlinear weak damping.

Messaoudi and Tatar ([6, 7]) considered (2) only with integral dissipation, namely

$$
\begin{aligned}
&|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s = 0, \quad (x,t) \in \Omega \times \mathbb{R}^+, \\
&u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+, \\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad x \in \Omega.
\end{aligned}
$$

Under some assumptions on g, they obtained exponential and polynomial decay rates. Motivated by the above contributions, in the present work we will study the initial-boundary value problem (1). Under suitable assumptions, we prove the existence of a global solution by means of the Galerkin method. Further, the asymptotic behavior of solution is established.

2. PRELIMINARIES

In this section, we begin with some notations and assumptions used throughout this article. For the Sobolev space $H_0^1(\Omega)$ we consider the norm $\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_2$, where $\|\cdot\|_p$ denotes the standard norm in $L^p(\Omega)$. The inner product in L^p is denoted by $(\cdot,\cdot)$. If $u = u(t,x)$ is a function in $L^2(0,T;H_0^1(\Omega))$ and g is continuous, we put

$$(g \circ u)(t) = \int_0^t g(t-s)|\nabla u(t) - \nabla u(s)|^2 \mathrm{d}s.$$

Now, we make the following assumptions on problem (1):

(A1) Assumption on $M(s)$.
We assume that $M(s) \in C([0,\infty),\mathbb{R})$ satisfying

$$M(s) \geq m_0 > 0, \quad M(s)s \geq \int_0^s M(\tau)\mathrm{d}\tau,$$

for all $s \geq 0$. For example $M(s) = m_0 + s^r$, $r \geq 1$.

(A2) Assumption on g.
We assume that $g : [0,\infty) \to (0,\infty)$ is a bounded C^1 function satisfying

$$g(0) > 0, \quad m_0 - \int_0^\infty g(s)\mathrm{d}s = l > 0,$$

such that there exist positive constants ξ_1 and ξ_2 satisfying

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad \text{for all } t \geq 0.$$

(A3) For the nonlinear term $|u_t|^\rho u_{tt}$, we further assume

$$0 < \rho < \infty, \text{ when } n \leq 2, \qquad \text{and} \qquad 0 < \rho \leq \frac{4}{n-2}, \text{ when } n \geq 3.$$

Definition 2.1 (Weak solution). A function $u(x,t)$ is called a weak solution of (1) on the interval $\Omega \times [0,T)$, with $0 < T \leq +\infty$ being the maximal existence time, if

$$u \in L^\infty(0,T;H_0^1(\Omega)), \quad u_t \in L^\infty(0,T;H_0^1(\Omega)) \quad \text{and} \quad u_{tt} \in L^\infty(0,T;H_0^1(\Omega))$$

satisfies the following conditions:

(i) for any $\phi \in H_0^1(\Omega)$ and a.e. $t \in [0,T)$, we have

$$(|u_t|^\rho u_{tt}, \phi) + \left(M\left(\|\nabla u\|_2^2\right)\nabla u, \nabla\phi\right) + (\nabla u_{tt}, \nabla\phi) + (\nabla u_t, \nabla\phi) - \left(\int_0^t g(t-s)\nabla u(s)\mathrm{d}s, \nabla\phi\right) = 0,$$

(ii) $u(x,0) = u_0(x)$ in $H_0^1(\Omega)$, $u_t(x,0) = u_1(x)$ in $H_0^1(\Omega)$.

REMARK 2.2. Since $0 < \rho < \infty$, when $n \leq 2$, and $0 < \rho \leq \frac{4}{n-2}$, when $n \geq 3$, according to the Sobolev embedding theorem, we have

$$H_0^1(\Omega) \hookrightarrow L^{\rho+2}(\Omega) \quad \text{and} \quad \|\phi\|_{\rho+2} \leq C_{\rho+2}\|\nabla\phi\|_2, \quad \text{for all} \quad \phi \in H_0^1(\Omega),$$

where $C_{\rho+2}$ is the optimal embedding constant from the Sobolev space $H_0^1(\Omega)$ to $L^{\rho+2}(\Omega)$. Noting that

$$\frac{\rho}{\rho+2} + \frac{1}{\rho+2} + \frac{1}{\rho+2} = 1,$$

by the Hölder inequality we can see that the nonlinear term

$$\int_\Omega |u_t|^\rho u_{tt}\phi \, \mathrm{d}x$$

makes sense.

3. GLOBAL EXISTENCE OF SOLUTIONS

The main goal in this section is devoted to discuss the existence of global weak solutions for the problem (1) by using the Galerkin approximation.

THEOREM 3.1. *Assume that* **(A1)**-**(A3)** *hold. Let* $u_0(x), u_1(x) \in H_0^1(\Omega)$, *then the problem* (1) *admits at least a global weak solution* $u : \Omega \to \mathbb{R}$ *such that*

$$u \in L^\infty(0,\infty; H_0^1(\Omega)), \quad u_t \in L^\infty(0,\infty; H_0^1(\Omega)), \quad u_{tt} \in L^\infty(0,\infty; H_0^1(\Omega)).$$

Proof. To establish the existence of a solution to problem (1), we use the Faedo-Galerkin approximations. Let $\{\omega_j(x)\}$ be a complete orthogonal basis in $H_0^1(\Omega)$. Then we construct the approximate solutions u_k for the problem (1) in the form

$$u_k(t) = \sum_{j=1}^{k} \delta_{kj}(t)\omega_j(x), \quad k = 1, 2, ...,$$

which satisfies

$$\begin{aligned} &(|u_{kt}|^\rho u_{ktt}, \omega_j) + \left(M\left(\|\nabla u_k\|_2^2\right)\nabla u_k, \nabla\omega_j\right) + (\nabla u_{ktt}, \nabla\omega_j) + (\nabla u_{kt}, \nabla\omega_j) \\ &\qquad - \left(\int_0^t g(t-s)\nabla u_k(s)\mathrm{d}s, \nabla\omega_j\right) = 0, \end{aligned} \tag{4}$$

and

$$\begin{cases} u_k(x,0) = \sum_{j=1}^{k} \delta_{kj}(0)\omega_j(x) \to u_0(x) \text{ in } H_0^1(\Omega), \quad k \to \infty, \\ u_{kt}(x,0) = \sum_{j=1}^{k} \delta'_{kj}(0)\omega_j(x) \to u_1(x) \text{ in } H_0^1(\Omega), \quad k \to \infty. \end{cases} \tag{5}$$

Multiplying (4) by $\delta'_{kj}(t)$ and summing for $j = 1, ..., k$, we obtain

$$(|u_{kt}|^{\rho} u_{ktt}, u_{kt}) + \left(M\left(\|\nabla u_k\|_2^2\right)\nabla u_k, \nabla u_{kt}\right) + (\nabla u_{ktt}, \nabla u_{kt}) + (\nabla u_{kt}, \nabla u_{kt}) - \left(\int_0^t g(t-s)\nabla u_k(s)\mathrm{d}s, \nabla u_{kt}\right) = 0.$$

By a direct calculation, it follows that

$$\begin{aligned}&\frac{1}{\rho+2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{kt}\|_{\rho+2}^{\rho+2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_{kt}\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\|\nabla u_k\|_2^2} M(s)\mathrm{d}s \\ &- \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t g(s)\mathrm{d}s\|\nabla u_k(t)\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(g\circ\nabla u_k)(t) \\ &= -\|\nabla u_{kt}\|_2^2 - \frac{1}{2}g(t)\|\nabla u_k\|_2^2 + \frac{1}{2}(g'\circ\nabla u_k)(t) \le 0,\end{aligned}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_k(t) = -\|\nabla u_{kt}\|_2^2 + \frac{1}{2}(g'\circ\nabla u_k)(t) - \frac{1}{2}g(t)\|\nabla u_k\|_2^2 \le 0, \tag{6}$$

where

$$\begin{aligned}E_k(t) = E(u_k, u_{kt}) = &\frac{1}{\rho+2}\|u_{kt}\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_{kt}\|_2^2 \\ &+ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{\|\nabla u_k\|_2^2} M(s)\mathrm{d}s - \frac{1}{2}\int_0^t g(s)\mathrm{d}s\|\nabla u_k(t)\|_2^2 \\ &+ \frac{1}{2}(g\circ\nabla u_k)(t).\end{aligned} \tag{7}$$

Integrating (6) over $(0, t)$, we obtain

$$E_k(t) \le E_k(0). \tag{8}$$

On the other hand, from **(A1)** and **(A2)**, we get

$$\begin{aligned}E_k(t) \ge &\frac{1}{\rho+2}\|u_{kt}\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_{kt}\|_2^2 + \frac{1}{2}\left(m_0 - \int_0^t g(s)\mathrm{d}s\right)\|\nabla u_k\|_2^2 \\ &+ \frac{1}{2}(g\circ\nabla u_k)(t) \ge \frac{1}{\rho+2}\|u_{kt}\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_{kt}\|_2^2 + \frac{l}{2}\|\nabla u_k\|_2^2 \\ &+ \frac{1}{2}(g\circ\nabla u_k)(t) \ge 0.\end{aligned} \tag{9}$$

Combining (8) and (9), and using (5), we infer

$$\begin{aligned}&\|u_{kt}\|_{\rho+2}^{\rho+2} + \|\nabla u_{kt}\|_2^2 + \|\nabla u_k\|_2^2 + (g\circ\nabla u_k)(t) \\ &\le \frac{E_k(0)}{\min\left\{\frac{1}{\rho+2}, \frac{1}{2}, \frac{l}{2}\right\}} \le C_1.\end{aligned} \tag{10}$$

Here and in the sequel C_i, $i = 1, 2, ...$, we will denote various constants independent of k and t.

Also, integrating (8) from 0 to t, there appears the relation

$$\int_0^t g(t-s)\|\nabla u_k(s)\|_2^2 \mathrm{d}s = 2(E_k(0) - E_k(t)) + \int_0^t (g' \circ u_k)(s)\mathrm{d}s - 2\int_0^t \|\nabla u_{kt}\|_2^2 \mathrm{d}s,$$

which together with (9) and **(A2)** yields that

$$\int_0^t g(t-s)\|\nabla u_k(s)\|_2^2 \mathrm{d}s \le 2E_k(0) \le C_2. \tag{11}$$

Multiplying (4) by $\delta''_{kj}(t)$ and summing for $j = 1, ..., k$, one has

$$\begin{aligned} &(|u_{kt}|^\rho u_{ktt}(t), u_{ktt}) + (M(\|\nabla u_k\|_2^2)\nabla u_k, \nabla u_{ktt}) + (\nabla u_{ktt}, \nabla u_{ktt}) \\ &+ (\nabla u_{kt}, \nabla u_{ktt}) - \left(\int_0^t g(t-s)\nabla u_k(s)\mathrm{d}s, \nabla u_{ktt}\right) = 0. \end{aligned} \tag{12}$$

Applying the Young and Hölder inequalities, we get from (12) and **(A1)** that

$$\begin{aligned} \int_\Omega |u_{kt}|^\rho |u_{ktt}|^2 \mathrm{d}x + \|\nabla u_{ktt}\|_2^2 &= -(M(\|\nabla u_k\|_2^2)\nabla u_k, \nabla u_{ktt}) \\ &+ (\nabla u_{kt}, \nabla u_{ktt}) - \left(\int_0^t g(t-s)\nabla u_k(s)\mathrm{d}s, \nabla u_{ktt}\right) \\ &\le 3\eta\|\nabla u_{ktt}\|_2^2 + \frac{1}{4\eta}\|\nabla u_{kt}\|_2^2 + \frac{m_0^2}{4\eta}\|\nabla u_k\|_2^2 \\ &+ \frac{1}{4\eta}\int_0^t g(s)\mathrm{d}s \int_0^t g(t-s)\|\nabla u_k(s)\|_2^2 \mathrm{d}s, \ \forall \eta > 0. \end{aligned} \tag{13}$$

Let us take η small enough such that $1 - 3\eta > 0$. So, by a simple calculation, (13) becomes

$$\begin{aligned} &\int_\Omega |u_{kt}|^\rho |u_{ktt}|^2 \mathrm{d}x + (1-3\eta)\|\nabla u_{ktt}\|_2^2 \le \frac{1}{4\eta}\|\nabla u_{kt}\|_2^2 \\ &+ \frac{m_0^2}{4\eta}\|\nabla u_k\|_2^2 + \frac{1}{4\eta}\int_0^t g(s)\mathrm{d}s \int_0^t g(t-s)\|\nabla u_k(s)\|_2^2 \mathrm{d}s. \end{aligned} \tag{14}$$

Using (10) and (11), we easily obtain from (14) the following inequality

$$\|\nabla u_{ktt}\|_2^2 \le C_3, \quad 0 \le t < \infty. \tag{15}$$

Using Hölder inequality, we have for $0 \le t < \infty$

$$(|u_{kt}|^\rho u_{ktt}, u_{ktt}) \le \|u_{kt}\|_{\rho+2}^\rho \|u_{ktt}\|_{\rho+2}^2 \le [(\rho+2)C_1]^{\frac{\rho}{\rho+2}} \|u_{ktt}\|_{\rho+2}^2. \tag{16}$$

The estimates (10), (15) and (16) allow us to get a subsequence of $\{u_k\}$, which from now on will be denoted by $\{u_k\}$; and functions u, μ, χ such that:

$$u_k \to u \;\text{ weak star in }\; L^\infty(0,\infty;H_0^1(\Omega)), \quad k\to\infty, \tag{17}$$

$$u_{kt} \to u_t \;\text{ weak star in }\; L^\infty(0,\infty;H_0^1(\Omega)), \quad k\to\infty, \tag{18}$$

$$u_{kt} \to u_t \;\text{ weak star in }\; L^\infty(0,\infty;L^{\rho+2}(\Omega)), \quad k\to\infty, \tag{19}$$

$$u_{ktt} \to u_{tt} \;\text{ weakly }\; L^\infty(0,\infty;H_0^1(\Omega)), \quad k\to\infty, \tag{20}$$

$$|u_{kt}|^\rho u_{kt} \to \mu \;\text{ weak star in }\; L^\infty(0,\infty;L^{\frac{\rho+2}{\rho+1}}(\Omega)), \quad k\to\infty, \tag{21}$$

$$|u_{kt}|^\rho u_{ktt} \to \chi \;\text{ weakly in }\; L^\infty(0,\infty;L^{\frac{\rho+2}{\rho+1}}(\Omega)), \quad k\to\infty. \tag{22}$$

Since $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, by the Aubin-Lions theorem, we deduce that

$$u_k \to u \;\text{ strongly in }\; L^\infty(0,\infty;L^2(\Omega)), \quad k\to\infty, \tag{23}$$

$$u_{kt} \to u_t \;\text{ strongly in }\; L^\infty(0,\infty;L^2(\Omega)), \quad k\to\infty, \tag{24}$$

$$u_{ktt} \to u_{tt} \;\text{ strongly in }\; L^\infty(0,\infty;L^2(\Omega)), \quad k\to\infty, \tag{25}$$

and further using Lemma 1.3 in [5], we obtain easily

$$|u_{kt}|^\rho u_{kt} \to \mu = |u_t|^\rho u_t \;\text{ weak star in }\; L^\infty(0,\infty;L^{\frac{\rho+2}{\rho+1}}(\Omega)), \quad k\to\infty, \tag{26}$$

$$|u_{kt}|^\rho u_{ktt} \to \chi = |u_t|^\rho u_{tt} \;\text{ weakly in }\; L^\infty(0,\infty;L^{\frac{\rho+2}{\rho+1}}(\Omega)), \quad k\to\infty. \tag{27}$$

Taking $k\to\infty$ in (4) and then making use of (17) − (20) and (26) − (27), we arrive at

$$\begin{aligned}&(|u_t|^\rho u_{tt},\omega_j) + (M(\|\nabla u\|_2^2)\nabla u,\nabla\omega_j) + (\nabla u_{tt},\nabla\omega_j) + (\nabla u_t,\nabla\omega_j)\\ &\qquad - \left(\int_0^t g(t-s)\nabla u(s)\mathrm{d}s,\nabla\omega_j\right) = 0.\end{aligned} \tag{28}$$

Considering that the basis $\{\omega_j(x)\}_{j=1}^\infty$ is dense in $H_0^1(\Omega)$, we choose a function $\phi\in H_0^1(\Omega)$ having the form $\phi = \sum_{j=1}^k \delta_j\omega_j(x)$, where $\{\delta_j\}_{j=1}^\infty$ are given functions.

Multiplying (28) by δ_j and then summing for $j = 1, \ldots..$, it follows that

$$\begin{aligned}&(|u_t|^\rho u_{tt},\phi) + (M(\|\nabla u\|_2^2)\nabla u,\nabla\phi) + (\nabla u_{tt},\nabla\phi) + (\nabla u_t,\nabla\phi)\\ &- \left(\int_0^t g(t-s)\nabla u(s)\mathrm{d}s,\nabla\phi\right) = 0, \quad \forall\phi\in H_0^1(\Omega).\end{aligned} \tag{29}$$

Hence, the proof of Theorem 3.1 is completed. □

4. ASYMPTOTIC BEHAVIOR OF SOLUTIONS

In this section we consider the asymptotic behavior of solutions to (1).

THEOREM 4.1. *Suppose that the assumptions* **(A1)**-**(A3)** *hold and* $u = u(x,t)$ *be the global solution to problem* (1) *obtained in Theorem* 3.1. *For* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *a increasing* C^2 *function such that*

$$\varphi(0) = 0, \qquad \varphi'(0) > 0, \qquad \lim_{t\to+\infty} \varphi(t) = +\infty, \qquad \varphi''(t) < 0 \quad , \forall t \geq 0,$$

we have for $\kappa > 0$

$$E(t) \leq E(0)e^{-\kappa\varphi(t)}, \quad \forall t \geq 0.$$

Proof. Let $\phi = u_t$ in equation (29), then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\int_0^{\|\nabla u\|_2^2} M(s)\mathrm{d}s\right] \\
&- \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\int_0^t g(s)\mathrm{d}s\|\nabla u(t)\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t)\right] \\
&+ \|\nabla u_t\|_2^2 + \frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}(g'\circ\nabla u)(t) = 0,
\end{aligned}$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \|\nabla u_t\|_2^2 + \frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}(g'\circ\nabla u)(t) = 0,$$

where $E(t)$ is defined in (7),

$$\begin{aligned}
E(t) = &\frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\int_0^{\|\nabla u\|_2^2} M(s)\mathrm{d}s \\
&- \frac{1}{2}\left(\int_0^t g(s)\mathrm{d}s\right)\|\nabla u\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t).
\end{aligned} \tag{30}$$

Then, in view of assumption **(A2)**, one has

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t) &= -\|\nabla u_t\|_2^2 - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}(g'\circ\nabla u)(t) \\
&\leq -\|\nabla u_t\|_2^2 - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \frac{1}{2}\xi_2(g\circ\nabla u)(t) \leq 0.
\end{aligned} \tag{31}$$

This means that the energy $E(t)$ is uniformly bounded (by $E(0)$) and is decreasing in t.

Before proving Theorem 4.1 we need to state some technical lemmas.

LEMMA 4.2. *For any* $t \geq 0$*, the energy* $E(t)$ *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq -\|\nabla u_t\|_2^2 - \frac{1}{2}\xi_2(g\circ u)(t) - \frac{1}{2}\left[g(0) - \xi_1\|g\|_{L^1(0,\infty)}\right]\|\nabla u\|_2^2. \tag{32}$$

Proof. From assumption **(A2)** and since

$$\int_0^t g'(s)\mathrm{d}s = g(t) - g(0),$$

we obtain

$$\begin{aligned} -\frac{1}{2}g(t)\|\nabla u\|_2^2 &= -\frac{1}{2}g(0)\|\nabla u\|_2^2 - \frac{1}{2}\left(\int_0^t g'(s)\mathrm{d}s\right)\|\nabla u\|_2^2 \\ &\leq -\frac{1}{2}g(0)\|\nabla u\|_2^2 + \frac{\xi_1}{2}\|g\|_{L^1(0,\infty)}\|\nabla u\|_2^2. \end{aligned} \tag{33}$$

Combining (31) and (33) we conclude that for all $t \geq 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq -\|\nabla u_t\|_2^2 - \frac{1}{2}\xi_2(g\circ\nabla u)(t) - \frac{1}{2}\left[g(0) - \xi_1\|g\|_{L^1(0,\infty)}\right]\|\nabla u\|_2^2 \leq 0.$$

□

LEMMA 4.3. *The energy $E(t)$ satisfies*

$$E(t) \leq \beta\|\nabla u_t\|_2^2 + \widetilde{m_0}\|\nabla u\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t), \quad \textit{for all } t \geq 0, \tag{34}$$

where

$$\beta = \left(\frac{1}{\rho+2}C_{\rho+2}^{\rho+2}(2E(0)^{\frac{\rho}{2}} + \frac{1}{2}\right),$$

and

$$0 < \widetilde{m_0} = \frac{1}{2}\max\{M(s),\ s \in [0, C_1]\} < \infty.$$

Proof. First we note that similarly to (9) one has

$$\begin{aligned} E(0) \geq E(t) &= \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_t\|_2^2 \\ &+ \frac{1}{2}\int_0^{\|\nabla u\|_2^2} M(s)\mathrm{d}s - \frac{1}{2}\int_0^t g(s)\mathrm{d}s\|\nabla u(t)\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t) \\ &\geq \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\left(m_0 - \int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2 \\ &+ \frac{1}{2}(g\circ\nabla u)(t) \geq \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_t\|_2^2 \\ &+ \frac{l}{2}\|\nabla u\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t) \geq 0. \end{aligned}$$

Since $E(t)$ is decreasing, from the Sobolev embedding theorem we have

$$\begin{aligned} \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} &\leq \frac{1}{\rho+2}C_{\rho+2}^{\rho+2}\|\nabla u_t\|_2^{\rho+2} = \frac{1}{\rho+2}C_{\rho+2}^{\rho+2}\|\nabla u_t\|_2^{\rho}\|\nabla u_t\|_2^2 \\ &\leq \frac{1}{\rho+2}C_{\rho+2}^{\rho+2}(2E(0))^{\frac{\rho}{2}}\|\nabla u_t\|_2^2. \end{aligned} \tag{35}$$

From (30), (35), **(A1)** and since $\int_0^t g(s)\mathrm{d}s > 0$, we deduce

$$\begin{aligned} E(t) &\leq \left(\frac{1}{\rho+2} C_{\rho+2}^{\rho+2} (2E(0))^{\frac{\rho}{2}} + \frac{1}{2} \right) \|\nabla u_t\|_2^2 \\ &+ \frac{1}{2} \int_0^{\|\nabla u\|_2^2} M(s)\mathrm{d}s + \frac{1}{2}(g \circ \nabla u)(t) \\ &\leq \left(\frac{1}{\rho+2} C_{\rho+2}^{\rho+2} (2E(0))^{\frac{\rho}{2}} + \frac{1}{2} \right) \|\nabla u_t\|_2^2 \\ &+ \frac{1}{2} M(\|\nabla u\|_2^2)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) \\ &\leq \left(\frac{1}{\rho+2} C_{\rho+2}^{\rho+2} (2E(0))^{\frac{\rho}{2}} + \frac{1}{2} \right) \|\nabla u_t\|_2^2 + \widetilde{m_0}\|\nabla u\|_2^2 \\ &+ \frac{1}{2}(g \circ \nabla u)(t). \end{aligned}$$

Let $\beta = \left(\frac{1}{\rho+2} C_{\rho+2}^{\rho+2} (2E(0))^{\frac{\rho}{2}} + \frac{1}{2} \right)$, then we get (34). □

Multiplying (32) by $e^{\kappa\varphi(t)}$ (where $\kappa > 0$) and utilizing Lemma 4.2, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\kappa\varphi(t)}E(t)\right) &\leq -e^{\kappa\varphi(t)}\|\nabla u_t\|_2^2 - \frac{1}{2}\left[g(0) - \xi_1\|g\|_{L^1(0,\infty)}\right] e^{\kappa\varphi(t)}\|\nabla u\|_2^2 \\ &- \frac{1}{2}\xi_2 (g\circ\nabla u)(t)e^{\kappa\varphi(t)} + \kappa\varphi'(t)e^{\kappa\varphi(t)}E(t) \\ &\leq -\left[1 - \kappa\beta\varphi'(t)\right] e^{\kappa\varphi(t)}\|\nabla u_t\|_2^2 \\ &- \frac{1}{2}\left[\xi_2 - \kappa\varphi'(t)\right] e^{\kappa\varphi(t)}(g\circ\nabla u)(t) \\ &- \frac{1}{2}\left[g(0) - \xi_1\|g_1\|_{L^1(0,\infty)} - 2\kappa\widetilde{m_0}\varphi'(t)\right] e^{\kappa\xi(t)}\|\nabla u\|_2^2. \end{aligned}$$

Using the fact that φ' is decreasing we arrive at

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\kappa\varphi(t)}E(t)\right) &\leq -\left[1 - \kappa\beta\varphi'(0)\right] e^{\kappa\varphi(t)}\|\nabla u_t\|_2^2 \\ &- \frac{1}{2}\left[\xi_2 - \kappa\varphi'(0)\right] e^{\kappa\varphi(t)}(g\circ\nabla u)(t) \\ &- \frac{1}{2}\left[g(0) - \xi_1\|g_1\|_{L^1(0,\infty)} - 2\kappa\widetilde{m_0}\varphi'(0)\right] e^{\kappa\xi(t)}\|\nabla u\|_2^2. \end{aligned} \tag{36}$$

Choosing $\|g\|_{L^1(0,\infty)}$ sufficiently small so that

$$g(0) - \xi_1\|g_1\|_{L^1(0,\infty)} = B > 0,$$

and defining

$$\kappa_0 = \min\left\{ \frac{1}{\beta\varphi'(0)}, \frac{\xi_2}{\varphi'(0)}, \frac{B}{2\widetilde{m_0}\varphi'(0)} \right\},$$

we conclude by taking $\kappa \in (0, \kappa_0]$ in (36) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\kappa\varphi(t)}E(t)\right) \leq 0, \quad t > 0. \tag{37}$$

Integrating (37) over $(0, t)$, it follows that

$$E(t) \leq E(0)e^{-\kappa\varphi(t)}, \quad t > 0.$$

This completes the proof of Theorem 4.1. □

EXAMPLE 4.4. For $\varphi(t) = t + \frac{t}{t+1}$, we get the following exponential decay rate

$$E(t) \leq E(0)e^{-\kappa t}, \quad \text{for all } t \geq 0.$$

For $\zeta(t) = \ln(1+t)$, we get the following polynomial decay rate

$$E(t) \leq E(0)(1+t)^{-\kappa}, \quad \text{for all } t \geq 0.$$

REFERENCES

[1] M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. Ferreira, *Existence and uniform decay for a non-linear viscoelastic equation with strong damping*, Math. Methods Appl. Sci., **24** (2001), 1043–1053.

[2] M. M. Cavalcanti, V. N. Domingos Cavalcanti, I. Lasiecka and C. M. Webler, *Intrinsic decay rates for the energy of a nonlinear viscoelastic equation modeling the vibrations of thin rods with variable density*, Adv. Nonlinear Anal., **6** (2016), 121–145.

[3] X. Han and M. Wang, *General decay of energy for a viscoelastic equation with nonlinear damping*, Math. Methods Appl. Sci., **32** (2009), 346–358.

[4] X. Han and M. Wang, *Global existence and uniform decay for a nonlinear viscoelastic equation with damping*, Nonlinear Anal., **70** (2009), 3090–3098.

[5] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod Gauthier-Villars, Paris, 1969.

[6] S. A. Messaoudi and N.-E. Tatar, *Exponential or polynomial decay for a quasilinear viscoelastic equation*, Nonlinear Anal., **68** (2008), 785–793.

[7] S. A. Messaoudi and N.-E. Tatar, *Exponential decay for a quasilinear viscoelastic equation*, Math. Nachr., **282** (2009), 1443–1450.

[8] J. Y. Park and S. H. Park, *General decay for quasilinear viscoelastic equations with nonlinear damping*, J. Math. Phys., **50** (2009), 1–10.

Received May 10, 2022
Accepted October 29, 2022

Hassiba Benbouali University
Faculty of Exact Sciences and Computer Science
Chlef, Algeria
E-mail: m.mellah@univ-chlef.dz

Djillali Liabes University
Laboratory ACEDP
Sidi Bel Abbes, Algeria
E-mail: hakemali@yahoo.com
https://orcid.org/0000-0001-6145-4514