MATHEMATICA, 65 (88), N° 2, 2023, pp. 215–218

STRONG INNER AND STRONG REFLEXIVE INVERSES IN SEMIPRIME RINGS

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Abstract. Motivated by some recent work on von Neumann regular elements in semiprime rings, we study how strongly regular elements of semiprime rings are related in terms of their sets of strong inner inverses and strong reflexive inverses.

MSC 2020. 16U90, 16E50, 16N60.

Key words. Strongly regular element, strong inner inverse, strong reflexive inverse, semiprime ring.

1. INTRODUCTION

An important subclass of von Neumann regular rings [8] consists of strongly regular rings, which were introduced by Arens and Kaplansky [2]. Recall that a ring R with identity is called *von Neumann regular* if every element $a \in R$ has an *inner* (or *generalized) inverse* $b \in R$ in the sense that $a = aba$. Also, a ring R with identity is called *strongly regular* if for every $a \in R$ there is $b \in R$ such that $a = a^2b$, and this definition turns out to be left-right symmetric.

In terms of elements, $a \in R$ is called *strongly regular* if $a \in a^2R \cap Ra^2$ (e.g., see [6]). Note that if a is strongly regular with $a = a^2u = va^2$ for some $u, v \in R$, then one may choose $w = au^2$ and one has $a = a^2w = wa^2$ by an argument of Azumaya [3, Lemma 1]. Hence $a \in R$ is strongly regular if and only if there is $w \in R$ such that $a = a^2w = wa^2$ and $aw = wa$. In this case w is called a *strong inner* (or *strong generalized) inverse* of a. An element $u \in R$ is called a *strong reflexive inverse* of $a \in R$ if u is a strong inner inverse of a and a is a strong inner inverse of u. Note also that $a \in R$ is strongly regular if and only if it has a unique reflexive inverse [7, Proposition 3.4].

An interesting problem on generalized inverses in a ring is to relate elements in terms of their sets of generalized inverses. In this direction we mention the recent work of Alahmadi, Jain and Leroy [1], and Lee [4, 5], who studied inner and reflexive inverses in semiprime rings. In this paper we consider the case of strong inner inverses and strong reflexive inverses in semiprime rings.

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DOI: 10.24193/mathcluj.2023.2.07

First, we describe the sets $S(a)$ of strong inner inverses and $\mathrm{SRef}(a)$ of strong reflexive inverses of a strongly regular element a of a ring R. In the main part of the paper we consider a semiprime ring R and strongly regular elements $a, b \in R$. We prove that if $S(a) \cap S(b) \neq \emptyset$, then $S(a) \subseteq S(b)$ if and only if $b^2 = ab = ba$, and in particular, $S(a) = S(b)$ if and only if $a = b$. Also, we deduce that $\mathrm{SRef}(a) \cap \mathrm{SRef}(b) \neq \emptyset$ if and only if $a = b$.

2. THE SETS OF STRONG INNER AND STRONG REFLEXIVE INVERSES

Throughout the paper R will be a ring with identity. We denote by $S(a)$ the set of strong inner inverses, and by $\mathrm{SRef}(a)$ the set of strong reflexive inverses of $a \in R$. We completely describe their elements as follows.

LEMMA 2.1. *Let $a \in R$ be a strongly regular element and let $w \in S(a)$. Then:*

$$S(a) = \{w + x - wax \mid x \in R\} = \{w + x - xaw \mid x \in R\}.$$

Proof. First, let $v \in S(a)$. Then $v = w + x - wax$, where $x = v - w + aw^2$. Indeed, we have:

$$\begin{aligned} w + x - wax &= v + aw^2 - wa(v - w + aw^2) \\ &= v + aw^2 - wav + waw - wa^2w^2 \\ &= v + aw^2 - wav + aw^2 - aw^2 = v, \end{aligned}$$

because

$$wav = awv = a^2w^2v = w^2a^2v = w^2a = aw^2.$$

Now let $v = w + x - wax$ for some $x \in R$. Then we have:

$$a^2v = a^2(w + x - wax) = a^2w + a^2x - a^2wax = a^2w + a^2x - a^2x = a.$$

Hence $v \in S(a)$. Consequently, $S(a) = \{w + x - wax \mid x \in R\}$.

Similarly, one shows that $S(a) = \{w + x - xaw \mid x \in R\}$. $\square$

As a consequence, we may deduce the following result. Recall that R is called a *semiprime* ring if for every $a \in R$ such that $aRa = 0$, one has $a = 0$. For instance, every von Neumann regular ring is semiprime.

PROPOSITION 2.2. *Let R be a semiprime ring, and let $a, b \in R$ such that a is strongly regular. Then $bS(a)b$ is a singleton if and only if $b \in Ra \cap aR$.*

Proof. Let us first assume that $bS(a)b = \{bwb\}$ for some $w \in S(a)$. By Lemma 2.1, it follows that

$$b(w + x - xaw)b = bwb$$

for every $x \in R$. Then $bxb - bxawb = 0$, whence we get $bx(1 - aw)b = 0$, and further

$$(1 - aw)bx(1 - aw)b = 0$$

for every $x \in R$. Since R is semiprime, we have $b(1 - aw) = 0$, which implies that $b = baw = bwa \in Ra$. Similarly, one shows that $b \in aR$.

Conversely, assume that $b \in Ra \cap aR$. Hence $b = ua = av$ for some $u, v \in R$. Now let $w \in S(a)$. By Lemma 2.1, it follows that:

$$\begin{aligned} b(w + x - wax)b &= (bw + bx - bawx)b = (uaw + uax - uaawx)b \\ &= (uaw + uax - uax)b = uawb = uawav = uav, \end{aligned}$$

for every $x \in R$. Hence $bS(a)b = \{uav\}$ is a singleton. □

Proposition 2.3. *Let $a \in R$ be a strongly regular element. Then:*

$$\mathrm{SRef}(a) = S(a)aS(a).$$

Proof. First, let $w \in \mathrm{SRef}(a)$. In particular, we have $w = w^2a = waw \in S(a)aS(a)$.

Now let $w \in S(a)aS(a)$. Then $w = cad$ for some $c, d \in S(a)$. It follows that:

$$\begin{aligned} &a^2w = a^2cad = aca^2d = aca = a^2c = a, \\ &w^2a = cadcada = c^2a^2dca^2d = c^2aca^2d = c^2a^2d = cad = w, \end{aligned}$$

hence $w \in \mathrm{SRef}(a)$. □

3. MAIN RESULTS

Now we study when the sets of strong inner inverses of strongly regular elements of a semiprime ring are comparable.

Theorem 3.1. *Let R be a semiprime ring, and let $a, b \in R$ be strongly regular elements such that $S(a) \cap S(b) \neq \emptyset$. Then $S(a) \subseteq S(b)$ if and only if $b^2 = ab = ba$.*

Proof. Let $w \in S(a) \cap S(b)$.

Assume first that $S(a) \subseteq S(b)$. By Lemma 2.1, we have $w + x - xaw \in S(a) \subseteq S(b)$ for every $x \in R$. It follows that $(w + x - xaw)b^2 = b$, that is, $wb^2 + xb^2 - xawb^2 = b$. Since $wb^2 = b$, we have $x(b - a)b = 0$. Hence

$$(b - a)bx(b - a)b = 0$$

for every $x \in R$, whence the semiprimeness of R implies that $b^2 = ab$. By Lemma 2.1, we also have $w + x - wax \in S(a) \subseteq S(b)$, and one deduces that $b^2 = ba$ in a similar way as above.

Conversely, assume that $b^2 = ab = ba$. For every $x \in R$, we have:

$$(w + x - xaw)b^2 = wb^2 + xb^2 - xawb^2 = b + xb^2 - xab = b,$$

and similarly, $b^2(w + x - wax) = b$. Using Lemma 2.1, we deduce that $S(a) \subseteq S(b)$. □

Remark 3.2. Under the conditions of Theorem 3.1, non-commuting elements must have non-comparable sets of strong inner inverses.

COROLLARY 3.3. *Let R be a semiprime ring, and let $a, b \in R$ be strongly regular elements such that $S(a) \cap S(b) \neq \emptyset$. Then $S(a) = S(b)$ if and only if $a = b$.*

Proof. For the non-trivial implication, suppose that $S(a) = S(b)$ and let $w \in S(a) \cap S(b)$. By Theorem 3.1, we have $a^2 = ab = ba$ and $b^2 = ab = ba$. Hence $a^2 = b^2$, which yields $a = a^2w = b^2w = b$. □

Finally, we may deduce the following result on the equality of the sets of strong reflexive inverses of strongly regular elements of a semiprime ring.

THEOREM 3.4. *Let R be a semiprime ring, and let $a, b \in R$ be strongly regular elements. Then* $\mathrm{SRef}(a) \cap \mathrm{SRef}(b) \neq \emptyset$ *if and only if $a = b$.*

Proof. For the non-trivial implication, assume that $w \in \mathrm{SRef}(a) \cap \mathrm{SRef}(b)$. Then we have:

$$b = wb^2 = aw^2b^2 = awwb^2 = awb = a^2wwb = a^2w^2b,$$

which implies

$$b^2 = a^2w^2b^2 = a^2wwb^2 = ab.$$

Similarly, one shows that $b^2 = ba$. Thus, we have $S(a) \subseteq S(b)$ by Theorem 3.1. By symmetry, we also deduce that $S(b) \subseteq S(a)$, and consequently, $S(a) = S(b)$. Finally, Corollary 3.3 implies $a = b$. □

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Received February 11, 2023
Accepted April 28, 2023

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