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SOME RESULTS ON DERIVATIONS AND GENERALIZED DERIVATIONS IN RINGS

ABDELLAH MAMOUNI, LAHCEN OUKHTITE, and MOHAMMED ZERRA

Abstract. The purpose of this paper is to study derivations and generalized derivations in prime rings satisfying certain differential identities. Some well-known results characterizing commutativity of prime rings have been generalized. Moreover, we provide examples to show that the assumed restrictions cannot be relaxed.

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Key words. Prime ring, Prime ideal, commutativity, derivations, generalized derivations.

1. INTRODUCTION

Throughout this paper R will represent an associative ring with center $Z(R)$. Recall that a proper ideal P of R is said to be prime if for any $x, y \in R$, $xRy \subseteq P$ implies that $x \in P$ or $y \in P$. The ring R is a prime ring if and only if (0) is a prime ideal of R. For any $x, y \in R$ the symbol $[x, y]$ will denote the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. An additive mapping $d : R \longrightarrow R$ is a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let $a \in R$ be a fixed element. A map $d : R \longrightarrow R$ defined by $d(x) = [a, x] = ax - xa$, $x \in R$, is a derivation on R, which is called *inner derivation* defined by a. Many results in literature indicate how the global structure of a ring R is often tightly connected to the behaviour of additive mappings defined on R (for example, see [1] and [11]). A well known result of Posner [12] states that if d is a derivation of the prime ring R such that $[d(x), x] \in Z(R)$, for any $x \in R$, then either $d = 0$ or R is commutative. In [9] Lanski generalizes the result of Posner to a Lie ideal.

More recently several authors consider similar situation in the case the derivation d is replaced by a generalized derivation. More specifically an additive map $F : R \longrightarrow R$ is said to be a generalized derivation if there exists a derivation d of R such that, for all $x, y \in R$, $F(xy) = F(x)y + xd(y)$. Basic examples of generalized derivations are the usual derivations on R and left R-module mappings from R into itself. An important example is a map of

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Corresponding author: Lahcen Oukhtite.

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the form $F(x) = ax + xb$, for some $a, b \in R$; such generalized derivations are called *inner*. Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting (see for example [7]and [8]).

During the last two decades, many authors have studied commutativity of prime and semiprime rings admitting suitably constrained additive mappings acting on appropriate subsets of the rings. Moreover, many of obtained results extend other ones proven previously just for the action of the considered mapping on the entire ring. In this direction, the recent literature contains numerous results on commutativity in prime and semi-prime rings admitting suitably constrained derivations and generalized derivations, and several authors have improved these results by considering rings with involution (for example, see [10]). In the present paper we continue this line of investigation and study the structure of a prime ring admitting a derivations and generalized derivations satisfying more specific algebraic identities.

2. SOME RESULTS FOR DERIVATIONS

In [12] Posner's first theorem states that; If R is a 2-torsion free prime ring and d_1, d_2 are derivations of R such that the iterate d_1d_2 is also a derivation of R, then either $d_1 = 0$ or $d_2 = 0$. The purpose of the following theorem is to give an improved version of this result, our conclusion is of different kind.

THEOREM 2.1. *Let R be a ring and P be a prime ideal of R. If d_1 and d_2 are two derivations of R such that*

$$d_1d_2(xy) - d_1d_2(x)y - xd_1d_2(y) \in P \ \textit{for all} \ x, y \in R.$$

1) *If* $\mathrm{char}(R/P) \neq 2$, *then* $d_1(R) \subseteq P$ *or* $d_2(R) \subseteq P$.

2) *If* $\mathrm{char}(R/P) = 2$ *and* $\overline{d_1(x)} \neq \overline{0}$, *then* $\overline{d_1(x)} = \lambda\overline{d_2(x)}$ *for some* λ *in the extended centroid of* R/P.

Proof.

$$d_1d_2(xy) - d_1d_2(x)y - xd_1d_2(y) \in P \ \text{for all} \ x, y \in R. \tag{1}$$

Because of d_2 is a derivation of R, the last expression reduces to

$$d_2(x)d_1(y) + d_1(x)d_2(y) \in P \ \text{for all} \ x, y \in R. \tag{2}$$

Putting ry instead of y in (2) and using it, we can see that

$$d_2(x)rd_1(y) + d_1(x)rd_2(y) \in P \ \text{for all} \ r, x, y \in R \tag{3}$$

in such a way that

$$\overline{d_2(x)}\overline{r}\overline{d_1(x)} + \overline{d_1(x)}\overline{r}\overline{d_2(x)} = \overline{0} \ \text{for all} \ r, x \in R. \tag{4}$$

Suppose that the characteristic of the ring R/P is different from 2, then according to [4, Lemma 1.1], we get

$$\overline{d_1(x)}\overline{r}\overline{d_2(x)} = \overline{0} \ \text{for all} \ r, x \in R. \tag{5}$$

In light of primeness of R/P, (5) assures that either $d_1(R) \subseteq P$ or $d_2(R) \subseteq P$.

Now suppose that the characteristic of R/P is 2, then the relation (2) becomes

$$d_2(x)rd_1(y) - d_1(x)rd_2(y) \in P \text{ for all } r, x, y \in R. \tag{6}$$

This may be restated as

$$\overline{d_2(x)}\overline{r}\overline{d_1(x)} = \overline{d_1(x)}\overline{r}\overline{d_2(x)} \text{ for all } r, x \in R. \tag{7}$$

Invoking [6, Lemma 7.42], there exists λ in the extended centroid of R/P such that $\overline{d_1(x)} = \lambda\overline{d_2(x)}$. □

If the ring R is prime, then (0) is a prime ideal of R. In this case, application of Theorem 2.1, yields an improved version of [12, Theorem 1] as follows.

Corollary 2.2. *Let R be a prime ring and d_1, d_2 two derivations of R such that the composition d_1d_2 is a derivation of R.*
1) *If* $\mathrm{char}(R) \neq 2$, *then* $d_1 = 0$ *or* $d_2 = 0$.
2) *If* $\mathrm{char}(R) = 2$, *then* $d_1 = \lambda d_2$ *for some* λ *in the extended centroid of* R.

In [5, Theorem 2.1] it is proved that if a prime ring R admits derivations d, g and h such that $d(x) = ag(x) + h(x)b$ for all $x \in R$, where $a, b \notin Z(R)$, then there exists λ in the extended centroid of R such that $d(x) = [\lambda ab, x]$, $g(x) = [\lambda b, x]$ and $h(x) = [\lambda a, x]$ for all $x \in R$.

Our purpose in the following theorem is to study the more general case

$$d(x) - ag(x) - h(x)b \in Z(R) \text{ for all } x \in R$$

under the hypothesis $d(Z(R)) \neq (0)$. However, we will provide a counterexample which proves that the additional assumption "$d(Z(R)) \neq (0)$" is not superfluous.

Theorem 2.3. *Let R be a prime ring and d a derivation of R such that $d(Z(R)) \neq (0)$. If $a, b \notin Z(R)$, then there are no derivations g and h of R satisfying*

$$d(x) - ag(x) - h(x)b \in Z(R) \quad \textit{for all } x \in R.$$

Proof. Suppose there exist derivations g and h of R such that

$$d(x) - ag(x) - h(x)b \in Z(R) \quad \text{for all } x \in R. \tag{8}$$

Substituting xz for x in (8), where $z \in Z(R)\backslash\{0\}$, we obtain

$$d(x)z + xd(z) - ag(x)z - axg(z) - h(x)zb - xh(z)b \in Z(R) \quad \text{for all } x \in R. \tag{9}$$

This may be restated as

$$(d(x) - ag(x) - h(x)b)z + x(d(z) - ag(z) - h(z)b) + [x, a]g(z) \in Z(R) \tag{10}$$

for all $x \in R$. Combining equations (10) with (8) , we arrive at

$$x(d(z) - ag(z) - h(z)b) + [x, a]g(z) \in Z(R) \quad \text{for all } x \in R \tag{11}$$

which leads to $[[x,a],x]g(z)=0$, in such a way that

$$[[x,a],x]Rg(z)=0 \quad \text{for all } x\in R. \tag{12}$$

In light of primeness, the relation (12) implies that either $[[x,a],x]=0$ or $g(z)=0$.

Assume that

$$[[x,a],x]=0 \quad \text{for all } x\in R. \tag{13}$$

Linearizing (13), we can see that

$$[[x,a],y]+[[y,a],x]=0 \quad \text{for all } x,y\in R. \tag{14}$$

Substituting yx for y in (14), we obtain

$$[[x,a],y]x+[y,x][x,a]+[[y,a],x]x=0 \quad \text{for all } x,y\in R. \tag{15}$$

Invoking (14), the last expression reduces to $[y,x][x,a]=0$ and therefore

$$[y,x]R[x,a]=0 \quad \text{for all } x,y\in R. \tag{16}$$

In particular, we get $[x,a]R[x,a]=0$ for all $x\in R$. Once again using the primeness, we get $[x,a]=0$ for all $x\in R$ and thus $a\in Z(R)$, a contradiction. Accordingly, $g(z)=0$ for all $z\in Z(R)\backslash\{0\}$.

Taking $x=z$ in (8), we obviously get

$$d(z)-h(z)b\in Z(R) \tag{17}$$

which, because of $d(z)\in Z(R)$, forces $h(z)b\in Z(R)$. Accordingly,

$$h(z)R[b,x]=0. \tag{18}$$

Since R is prime and $b\notin Z(R)$, the last equation reduces to $h(z)=0$.

Now suppose that $h(z)=0$ for all $z\in Z(R)$, then (11) becomes

$$xd(z)\in Z(R) \quad \text{for all } x\in R \tag{19}$$

and therefore

$$[x,t]Rd(z)=0 \quad \text{for all } x,t\in R. \tag{20}$$

Since R is prime, equation (20) combined with the fact that $d(Z(R))\neq(0)$ yields that

$$[x,t]=0 \quad \text{for all } x,t\in R$$

proving that R is a commutative and therefore $a\in Z(R)$, a contradiction. □

The following example proves that the condition "$d(Z(R))\neq(0)$" imposed in the hypotheses of Theorem 2.3 is necessary.

Example 2.4. Let us consider $R=M_2(\mathbb{Z})$ and $d\begin{pmatrix} x & y \\ z & t\end{pmatrix}=\begin{pmatrix} 0 & y \\ -z & 0\end{pmatrix}$. It is straightforward to check that R is a prime ring and d is a derivation of R such that $d(X)=0$ for all $X\in Z(R)$. Define two derivations g and h on R

by setting $g = h = d$. Clearly, $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \notin Z(R)$ and $b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin Z(R)$. Furthermore, for $X = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in R$ we have

$$\begin{aligned} d(X) - ag(X) - h(X)b &= \begin{pmatrix} 0 & y \\ -z & 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & y \\ -z & 0 \end{pmatrix} - \begin{pmatrix} 0 & y \\ -z & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \end{aligned}$$

which proves that $d(X) - ag(X) - h(X)b \in Z(R)$ for all $X \in R$.

The following example shows that the primeness hypothesis in Theorem 2.3 is not superfluous. In particular, our theorem cannot be extended to semi-prime rings.

Example 2.5. Let us consider $R = \mathbb{Q}[X] \times M_2(\mathbb{Z})$. It is straightforward to check that R is a noncommutative semi-prime ring. Let d, g and h be derivations of R such that

$$d(P, M) = g(P, M) = h(P, M) = (P', 0) \quad \text{for all} \quad (P, M) \in \mathbb{Q}[X] \times M_2(\mathbb{Z}).$$

Let us set $a = \left(-X, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\right) \notin Z(R)$, $b = \left(X + 1, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \notin Z(R)$.

It is easy to see that d, g and h satisfy the condition

$$d(P, M) - ag(P, M) - h(P, M)b \in Z(R) \quad \text{for all} \quad (P, M) \in \mathbb{Q}[X] \times M_2(\mathbb{Z}).$$

As an application of Theorem 2.3, we get the following result.

Proposition 2.6. *Let d, g and h be derivations of a prime ring R such that*

$$d(x) = ag(x) + h(x)b \quad \textit{for all} \ x \in R.$$

If $d(Z(R)) \neq (0)$, then either $a \in Z(R)$ or $b \in Z(R)$.

Proof. Suppose that $a \notin Z(R)$ and $b \notin Z(R)$, we have

$$d(x) - ag(x) - h(x)b = 0 \in Z(R)$$

and

$$d(Z(R)) \neq (0).$$

Using the proof of Theorem 2.3, it follows that R is an integral domain and we can again employ the argument of [5, Theorem 2.1], there exists $\lambda \in C$ such that $d(x) = [\lambda ab, x]$ so that $d = 0$, a contradiction. Consequently, $a \in Z(R)$ or $b \in Z(R)$. □

3. SOME RESULTS FOR GENERALIZED DERIVATIONS

In [5, Theorem 3.2] it is proved that a prime ring R must be an integral domain if it admits derivations d and g such that $d(x)x-xg(x) \in Z(R)$ for all x in a nonzero left ideal U of R and $d \neq 0$. In [3, Theorem 3.1], without 2-torsion freeness hypothesis, it is proved that if a prime ring R admits two generalized derivations F and G associated with derivations f and g respectively, such that $F(x)x - xG(x) = 0$ for all x in a nonzero Jordan ideal J, then R is an integral domain and $F = G$ or G is a left multiplier and $F = G + f$.

Motivated by these results, our fundamental aim is to consider the more general identity $F(x)x - xG(x) \in Z(R)$ for all $x \in R$ under the hypothesis "$f(Z(R)) \neq (0)$ or $g(Z(R)) \neq (0)$". Furthermore, we will provide a counter-example which shows that this restriction is not superfluous.

THEOREM 3.1. *Let R be a 2-torsion free prime ring. Let F and G two generalized derivations of R associated with derivations f and g, respectively, such that*

$$F(x)x - xG(x) \in Z(R) \quad \textit{for all } x \in R.$$

If either $f(Z(R)) \neq (0)$ or $g(Z(R)) \neq (0)$, then R is an integral domain.

We will need the following lemma.

LEMMA 3.2 ([3, Lemma 3.1]). *Let R be a 2-torsion free prime ring and two generalized derivations F and G associated with f and g, respectively. If $F(x)x - xG(x) = 0$ for all $x \in R$, then one of the following hold:*
(1) *R is commutative and $F = G$;*
(2) *G is a left multiplier and $F = G + f$.*

Now we are in a position to prove our result.

Proof of Theorem 3.1. If $Z(R) = (0)$, then R is not commutative and our hypothesis becomes $F(x)x - xG(x) = 0$ for all $x \in R$. Invoking Lemma 3.2, it follows that $F = G + f$ and G is a left multiplier. Therefore $g = 0$ which contradicts the fact that $g(Z(R)) \neq (0)$. Consequently, $Z(R) \neq (0)$ we are given that

$$F(x)x - xG(x) \in Z(R) \quad \text{for all } x \in R. \tag{21}$$

Linearizing (21), we can see that

$$F(x)y + F(y)x - xG(y) - yG(x) \in Z(R) \quad \text{for all } x, y \in R. \tag{22}$$

Substituting yz instead of y in (22), where $z \in Z(R)\backslash\{0\}$, we get

$$F(x)yz + F(y)zx + yf(z)x - xG(y)z - xyg(z) - yzG(x) \in Z(R), \tag{23}$$

for all $x, y \in R$.

Combining equation (23) with (22), we arrive at

$$yf(z)x - xyg(z) \in Z(R) \quad \text{for all } x, y \in R. \tag{24}$$

Once again replacing y by x in (24), we obtain

$$x^2(f(z) - g(z)) \in Z(R)$$

and the primeness of R yields that

$$x^2 \in Z(R) \quad \text{for all} \quad x \in R \quad \text{or} \quad f(z) = g(z) \quad \text{for all} \quad z \in Z(R).$$

If $x^2 \in Z(R)$ for all $x \in R$, then R is commutative. Therefore we assume henceforth that $f(z) = g(z)$ for all $z \in Z(R)$, then relation (24) will be

$$[y, x]f(z) \in Z(R) \quad \text{for all} \quad x, y \in R. \tag{25}$$

Since R is prime, the fact that $f(Z(R)) \neq (0)$ forces $[y, x] \in Z(R)$ for all $x, y \in R$. Which yields that R is commutative, and this completes the proof of our theorem. □

The following example proves that the condition "$f(Z(R)) \neq (0)$ or $g(Z(R)) \neq (0)$" is necessary in Theorem 3.1.

EXAMPLE 3.3. Let us consider $R = M_2(\mathbb{Z})$, and $F(x) = ax + xb$ for all $x \in R$, where

$$a = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

F is a generalized derivation associated with the inner derivation $f(x) = [x, b]$, which clearly satisfies $f(Z(R)) = (0)$. Let us consider the generalized derivation G defined by $G(x) = (a + b)x$ for all $x \in R$, and associated with the zero derivation. Since

$$\begin{aligned} F(x)x - xG(x) &= (ax + xb)x - x(a + b)x \\ &= ax^2 + xbx - xax - xbx \\ &= (ax - xa)x = [a, x]x \\ &= 0 \in Z(R) \end{aligned}$$

then F and G satisfy the hypothesis of Theorem 3.1, however R is a non commutative ring.

The following example proves that the primeness hypothesis in Theorem 3.1 is not superfluous. In particular, our theorem cannot be extended to semi-prime rings.

EXAMPLE 3.4. Let us consider the noncommutative semi-prime ring $R = \mathbb{Q}[X] \times M_2(\mathbb{Z})$. Define $F(P, M) = (P', 0)$ and $G(P, M) = (P, 0)$ for all $(P, M) \in R$. Then F and G are generalized derivations of R associated with derivations $f = F$ and $g = 0$, respectively. Moreover, F and G satisfy the conditions of Theorem 3.1, but R is a noncommutative ring.

In [14] Rehman proves that if R is a 2-torsion free prime ring provided with a generalized derivation associated with a nonzero derivation d such that $F([x, y]) - [x, y] = 0$ for all x, y in a square closed Lie ideal U, then $U \subseteq Z(R)$.

Motivated by this results, Quadri et al. [13] with no further assumption on the characteristic, established that a prime ring R must be commutative if it admits a generalized derivation F associated with a derivation d such that $F([x,y]) - [x,y] = 0$ for all x, y in a nonzero ideal I of R.

Our goal in the following theorem is to investigate a more general context of differential identity involving a prime ideal P by omitting the primeness assumption imposed on the considered ring R. This approach allows us to generalize the preceding results in two direction. First of all, we will assume that for all $x, y \in R$ the expression $F([x,y]) - [x,y]$ belongs to a prime ideal P rather than $F([x,y]) - [x,y] = 0$. Secondly, we will investigate the behavior of the more general expression $F(xy) - G(yx)$ involving two generalized derivations F and G associated with derivations f and g respectively, instead of the expression $F([x,y]) - [x,y]$. Moreover, our result are of more specific interest because we will characterize not only the structure of the ring R/P, but we will also prove that the derivations f and g are with range in the prime ideal P. More precisely we will prove the following result.

Theorem 3.5. *Let R be a ring and P a prime ring of R. If F and G are two generalized derivations of R, associated with derivations f and g respectively, such that*

$$F(xy) - G(yx) \in P \quad \textit{for all } x, y \in R,$$

then one of the following assertions hold:

1) *R/P is an integral domain or (F and G have their images in P);*
2) *R/P is an integral domain and $(F - G)(R) \subseteq P$.*

Proof. We are given that

$$F(xy) - G(yx) \in P \quad \text{for all } x, y \in R. \tag{26}$$

Substituting ry for y in (26), we get

$$F(xry) - G(ryx) \in P \quad \text{for all } r, x, y \in R. \tag{27}$$

On the other hand replacing x by xr in (26), we obtain

$$F(xry) - G(yxr) \in P \quad \text{for all } r, x, y \in R. \tag{28}$$

Subtracting relation (28) from (27), we arrive at

$$G([r, yx]) \in P \quad \text{for all } r, x, y \in R. \tag{29}$$

Now putting xr instead of x in (29), we can see that

$$G([r, yx])r + [r, yx]g(r) \in P \quad \text{for all } r, x, y \in R. \tag{30}$$

Once again employing the fact that $G([r, yx]) \in P$ by (29), we get

$$[r, y]xg(r) + y[r, x]g(r) \in P \quad \text{for all } r, x, y \in R. \tag{31}$$

Writing sy instead of y in the above expression and using it, we obtain

$$[r, s]yxg(r) \in P \quad \text{for all } r, s, x, y \in R. \tag{32}$$

As a special case of the last relation, setting $y = yg(r)$, we may write

$$[r, s]yg(r)xg(r) \in P \quad \text{for all} \;\; r, s, x, y \in R. \tag{33}$$

In light of primeness of P, we get either $g(r) \in P$ or $[r, R] \subseteq P$ for all $r \in R$. Consequently, R is a union of two additive subgroups G_1 and G_2, where

$$G_1 = \{r \in R \,|\, [r, R] \subseteq P\} \;\text{ and }\; G_2 = \{r \in R \,|\, g(r) \in P\}.$$

Since a group cannot be a union of two of its proper subgroups, we are forced to conclude that $R = G_1$ or $R = G_2$. If $R = G_2$, then our hypothesis becomes

$$F(xy) - G(y)x \in P \quad \text{for all} \;\; x, y \in R. \tag{34}$$

Substituting yx for x in (34) and applying it, we arrive at

$$xyf(x) \in P \quad \text{for all} \;\; x, y \in R.$$

By view of primeness of P, we conclude that $f(R) \subseteq P$. Now our hypothesis becomes $F(x)y - yG(x) \in P$ and replacing y by yr and using the last expression we obtain $G(x)R[r, x] \subseteq P$, then $G(R) \subseteq P$ or R/P is an integral domain. So the identity reduces to $F(x)y \in P$, hence $F(R) \subseteq P$.

Now suppose that $R = G_1$, then $\overline{R/P \text{ is an integral domain}}$, in this direction replacing $\underline{y \text{ by } yx \text{ in}}$ (26), we arrive at $\overline{xy(f(x) - g(x))} = \overline{0}$ for all $x, y \in R$, accordingly $\overline{f(x)} = \overline{g(x)}$. On the other hand our hypothesis yields

$$\overline{F(x)y + xf(y) - G(y)x - yg(x)} \in P \;\text{ for all }\; x, y \in R.$$

Putting xy instead of y in this relation, we obviously obtain

$$\overline{F(x)xy + xf(x)y + x^2 f(y) - G(x)yx - xg(y)x - xyg(x)} = \overline{0},$$

for all $x, y \in R$.

So that

$$\overline{(F(x) - G(x))yx} = \overline{0} \text{ for all } x, y \in R.$$

Finally we conclude that

$$(F - G)(R) \subseteq P.$$

$\square$

Now if R is a prime ring, then (0) is a prime ideal. Hence we have the following corollary which is a generalization of [13, Theorem 2.1].

Corollary 3.6. *Let R be a prime ring. If F and G are two generalized derivations of R associated with nonzero derivations f and g respectively, such that $F(xy) - G(yx) = 0$ for all $x, y \in R$, then R is an integral domain and $F = G$*

The next corollary extended the results of Quadri et al. [13, Theorem 2.1] and Rehman [14, Theorem 3.3] to semi-prime rings.

Corollary 3.7. *Let R be a semi-prime ring. If F and G are two generalized derivations of R associated with nonzero derivations f and g respectively, such that*

$$F(xy) - G(yx) = 0 \textit{ for all } x, y \in R,$$

then R contains a nonzero central ideal.

Proof. Assume that $F(xy) - G(yx) = 0$ for all $x, y \in R$. Since the ring R is semi-prime then there exists a family $\mathcal{P}$ of prime ideals P such that $\bigcap_{P\in\mathcal{P}} P = (0)$. Therefore

$$F(xy) - G(yx) \in P \text{ for all } x, y \in R \text{ and for all } P \in \mathcal{P}.$$

Using the proof of Theorem 3.5, it follows from relation (33) that

$$[r, s]yg(r)xg(r) \in P \quad \text{for all } r, s, x, y \in R \text{ and for all } P \in \mathcal{P}. \tag{35}$$

Replacing x by xr in this expression and combining it with (35), we arrive at

$$[r, s]yg(r)x[r, g(r)] \in P \quad \text{for all } P \in \mathcal{P}. \tag{36}$$

Now putting yr instead of y in (36), we get

$$[r, s]yrg(r)x[r, g(r)] \in P \quad \text{for all } P \in \mathcal{P}. \tag{37}$$

On the other hand replacing x by rx in (36), we obtain

$$[r, s]yg(r)rx[r, g(r)] \in P \quad \text{for all } P \in \mathcal{P}. \tag{38}$$

Subtracting (38) from (37), we find that

$$[r, s]y[r, g(r)]x[r, g(r)] \in P \quad \text{for all } P \in \mathcal{P}. \tag{39}$$

In particular we have

$$[r, g(r)]R[r, g(r)]R[r, g(r)] \subseteq P \quad \text{for all } r \in R \text{ and for all } P \in \mathcal{P}. \tag{40}$$

This means that

$$[g(r), r]R[g(r), r]R[g(r), r] \subseteq \bigcap_{P\in\mathcal{P}} P = (0)\,,$$

for all $r \in R$. Hence the semiprimeness hypothesis forces that $[g(r), r] = 0$ for all $r \in R$, then we can again employ the argument of [2, Theorem 3], we conclude that R contains a nonzero central ideal. □

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University Moulay Ismail
Faculty of Sciences
Department of Mathematics
Meknes, Morocco
E-mail: a.mamouni.fste@gmail.com
https://orcid.org/0000-0001-7994-1564

University Sidi Mohamed Ben Abdellah
Faculty of Sciences and Technology
Department of Mathematics
Fez, Morocco
E-mail: lahcen.oukhtite@usmba.ac.ma
https://orcid.org/0000-0003-4408-4421

University Sidi Mohamed Ben Abdellah
Faculty of Sciences and Technology
Department of Mathematics
Fez, Morocco
E-mail: mohamed.zerra@gmail.com
https://orcid.org/0000-0003-2711-2857