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ON A GENERALIZATION OF GRADED 2-ABSORBING SUBMODULES

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Abstract. Let G be a group with identity e. Let R be a G-graded commutative ring with nonzero identity and M a graded R-module. In this paper, we introduce the concept of graded $G2$-absorbing submodule as a new generalization of a graded 2-absorbing submodule on the one hand and a generalization of a graded primary submodule on other hand. We give a number of results concerning these classes of graded submodules and their homogeneous components. In fact, our objective is to investigate graded $G2$-absorbing submodules, and we examine in particular when graded submodules are graded $G2$-absorbing submodules. For example, we give a characterization of graded $G2$-absorbing submodules. We also study the behaviour of graded $G2$-absorbing submodules under graded homomorphisms and under localization.

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1. INTRODUCTION AND PRELIMINARIES

Throughout this paper all rings are commutative with identity and all modules are unitary.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [12–15] for these basic properties and more information on graded rings and modules.

Let G be a multiplicative group and e denote the identity element of G. A ring R is called a graded ring (or G-graded ring) if there exist additive subgroups R_α of R indexed by the elements $\alpha \in G$ such that $R = \bigoplus_{\alpha\in G} R_\alpha$ and $R_\alpha R_\beta \subseteq R_{\alpha\beta}$ for all α, $\beta \in G$. The elements of R_α are called homogeneous of degree α and all the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \cup_{\alpha\in G} R_\alpha$. If $r \in R$, then r can be written uniquely as $\sum_{\alpha\in G} r_\alpha$, where r_α is called a homogeneous component of r in R_α. Moreover, R_e is a subring of R and $1 \in R_e$. Let $R = \bigoplus_{\alpha\in G} R_\alpha$ be a G-graded ring. An ideal I of R is said to be a graded ideal if $I = \bigoplus_{\alpha\in G}(I \cap R_\alpha) := \bigoplus_{\alpha\in G} I_\alpha$. Let $R =$

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$\bigoplus_{\alpha\in G} R_\alpha$ be a G-graded ring. A Left R-module M is said to be *a graded R-module* (or *G-graded R-module*) if there exists a family of additive subgroups $\{M_\alpha\}_{\alpha\in G}$ of M such that $M = \bigoplus_{\alpha\in G} M_\alpha$ and $R_\alpha M_\beta \subseteq M_{\alpha\beta}$ for all $\alpha, \beta \in G$. Also if an element of M belongs to $\cup_{\alpha\in G} M_\alpha = h(M)$, then it is called a homogeneous. Note that M_α is an R_e-module for every $\alpha \in G$. So, if $I = \bigoplus_{\alpha\in G} I_\alpha$ is a graded ideal of R, then I_α is an R_e-module for every $\alpha \in G$. Let $R = \bigoplus_{\alpha\in G} R_\alpha$ be a G-graded ring. A submodule N of M is said to be *a graded submodule of* M if $N = \bigoplus_{\alpha\in G}(N \cap M_\alpha) := \bigoplus_{\alpha\in G} N_\alpha$. In this case, N_α is called the α-component of N. Moreover, M/N becomes a G-graded R-module with α-component $(M/N)_\alpha := (M_\alpha + N)/N$ for $\alpha \in G$. Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fraction $S^{-1}R$ is a graded ring which is called the graded ring of fractions. Indeed, $S^{-1}R = \bigoplus\limits_{\alpha\in G} (S^{-1}R)_\alpha$ where $(S^{-1}R)_\alpha = \{r/s : r \in R, s \in S \text{ and } \alpha = (\deg s)^{-1}(\deg r)\}$. Let M be a graded module over a G-graded ring R and $S \subseteq h(R)$ be a multiplicatively closed subset of R. The module of fractions $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called the module of fractions, if $S^{-1}M = \bigoplus\limits_{\alpha\in G} (S^{-1}M)_\alpha$ where $(S^{-1}M)_\alpha = \{m/s : m \in M, s \in S \text{ and } \alpha = (\deg s)^{-1}(\deg m)\}$. We write $h(S^{-1}R) = \bigcup\limits_{\alpha\in G} (S^{-1}R)_\alpha$ and $h(S^{-1}M) = \bigcup\limits_{\alpha\in G} (S^{-1}M)_\alpha$. Consider the graded homomorphism $\eta : M \to S^{-1}M$ defined by $\eta(m) = m/1$. For any graded submodule N of M, the submodule of $S^{-1}M$ generated by $\eta(N)$ is denoted by $S^{-1}N$. Also, $S^{-1}N = \left\{\beta \in S^{-1}M : \beta = m/s \text{ for } m \in N \text{ and } s \in S\right\}$ and $S^{-1}N \neq S^{-1}M$ if and only if $S \cap (N :_R M) = \phi$ (see [15].)

Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. Then $(N :_R M)$ is defined as $(N :_R M) = \{r \in R | rM \subseteq N\}$. It is shown in [7, Lemma 2.1] that if N is a graded submodule of M, then $(N :_R M) = \{r \in R : rN \subseteq M\}$ is a graded ideal of R.

The *graded radical* of a graded ideal I of R, denoted by $\mathrm{Gr}(I)$, is the set of all $x = \sum_{g\in G} x_g \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in \mathrm{Gr}(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$ (see [18].) The *graded radical* of a graded submodule N of M, denoted by $Gr_M(N)$, is defined to be the intersection of all graded prime submodules of M containing N. If N is not contained in any graded prime submodule of M, then $Gr_M(N) = M$ (see [8].)

The concept of graded 2-absorbing ideals was introduced in [3, 16] as a generalization of the notion of graded prime ideals. Recall from [3] that a proper graded ideal I of R is said to be *a graded 2-absorbing ideal of* R if whenever $r_g, s_h, t_\lambda \in h(R)$ with $r_g s_h t_\lambda \in I$, then $r_g s_h \in I$ or $r_g t_\lambda \in I$ or $s_h t_\lambda \in I$. Also, a graded ideal $I = \bigoplus_{g\in G} I_g$ of a graded ring R is said to be a g-2-absorbing ideal of R if $I_g \neq R_g$ and whenever $r_g, s_g, t_g \in R_g$ with $r_g s_g t_g \in I$, then $r_g s_g \in I$ or $r_g t_g \in I$ or $s_g t_g \in I$.

Al-Zoubi and Abu-Dawwas in [2] extended graded 2-absorbing ideals to graded 2-absorbing submodules. Recall from [2] that a proper graded submodule N of a graded R-module M is said to be *a graded 2-absorbing submodule* of M if whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ with $r_g s_h m_\lambda \in N$, then $r_g s_h \in (N :_R M)$ or $r_g m_\lambda \in N$ or $s_h m_\lambda \in N$.

The concept of graded primary ideals was introduced by Refai and Al-Zoubi in [18] as a generalization of the notion of graded prime ideals. Recall from [18] that a proper graded ideal P of a graded ring R is said to be *a graded primary ideal* if whenever $r_g, s_h \in h(R)$ with $r_g s_h \in P$, then either $r_g \in P$ or $s_h \in \mathrm{Gr}(P)$.

Atani and Farzalipour in [8] extended graded primary ideals to graded primary submodules and studied in [1, 5, 9, 17]. Recall from [8] that a proper graded submodule N of a graded R-module M is said to be *a graded primary submodule* if whenever $r_g \in h(R)$ and $m_h \in h(M)$ with $r_g m_h \in N$, then either $m_h \in N$ or $r_g \in \mathrm{Gr}((N :_R M))$.

Al-Zoubi and sharafat in [6] introduced the concept of graded 2-absorbing primary ideal of a graded commutative rings that is a generalization of the concept of graded primary ideal. Recall from [6] that a graded ideal $I = \bigoplus_{g\in G} I_g$ of a graded ring R is said to be *a graded 2-absorbing primary ideal of* R if $I \neq R$ and whenever $r_g, s_h, t_\lambda \in h(R)$ with $r_g s_h t_\lambda \in I$, then $r_g s_h \in I$ or $r_g t_\lambda \in \mathrm{Gr}(I)$ or $s_h t_\lambda \in \mathrm{Gr}(I)$. Also, a graded ideal $I = \bigoplus_{g\in G} I_g$ of a graded ring R is said to be a g-2-absorbing primary ideal of R if $I_g \neq R_g$ and whenever $r_g, s_g, t_g \in R_g$ with $r_g s_g t_g \in I$, then $r_g s_g \in I$ or $r_g t_g \in \mathrm{Gr}(I)$ or $s_g t_g \in \mathrm{Gr}(I)$.

Celikel in [10] extended graded 2-absorbing primary ideals to graded 2-absorbing primary submodules. Recall from [10] that a proper graded submodule N of a graded R-module M is said to be *a graded 2-absorbing primary submodule of* M if whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ with $r_g s_h m_\lambda \in N$, then $r_g s_h \in (N :_R M)$ or $r_g m_\lambda \in Gr_M(N)$ or $s_h m_\lambda \in Gr_M(N)$.

Al-Zoubi and Al-Azaizeh in [4] introduced the concept of graded weakly 2-absorbing primary submodule as a new generalization of graded 2-absorbing primary submodule. Recall from [4] that a proper graded submodule N of a graded R-module M is said to be a *graded weakly 2-absorbing primary submodule* of M if $N \neq M$; and whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ with $0 \neq r_g s_h m_\lambda \in N$, then either $r_g s_h \in (N :_R M)$ or $r_g m_\lambda \in Gr_M(N)$ or $s_h m_\lambda \in Gr_M(N)$.

Dubey and Aggarwal in [11] introduced the concept of 2-absorbing primary submodule over a commutative ring with nonzero identity as a new generalization of primary submodule. As it defined in [11], a proper submodule N of a R-module M is said to be a 2-absorbing primary submodule of M if whenever $r, s \in R$, $m \in M$ and $rsm \in N$, then $rm \in N$ or $sm \in N$ or $rs \in \sqrt{(N : M)}$.

The scope of this paper is devoted to the theory of graded modules over graded commutative rings. One use of rings and modules with gradings is in describing certain topics in algebraic geometry. Here, we introduce the concept of graded $G2$-absorbing submodule as a new generalization of a graded 2-absorbing submodule on the one hand and a generalization of a graded primary submodule on other hand. A number of results concerning of these classes of graded submodules and their homogeneous components are given.

2. RESULTS

Definition 2.1. Let R be a G-graded ring, M a graded R-module, $N = \bigoplus_{h\in G} N_h$ a graded submodule of M and $h \in G$.

(i) We say that N_h is *a h-$G2$-absorbing submodule of the R_e-module M_h* if $N_h \neq M_h$; and whenever $r_e, s_e \in R_e$ and $m_h \in M_h$ with $r_e s_e m_h \in N_h$, then either $r_e s_e \in \mathrm{Gr}((N_h :_{R_e} M_h))$ or $r_e m_h \in N_h$ or $s_e m_h \in N_h$.

(ii) We say that N is *a graded $G2$-absorbing submodule of M* if $N \neq M$; and whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ with $r_g s_h m_\lambda \in N$, then either $r_g s_h \in \mathrm{Gr}((N :_R M))$ or $r_g m_\lambda \in N$ or $s_h m_\lambda \in N$.

It is clear that every graded 2-absorbing submodule is a graded $G2$-absorbing submodule. The following example shows that the converse is not true in general.

Example 2.2. Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$ be a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}_{16}$ be a graded R-module with $M_0 = \mathbb{Z}_{16}$ and $M_1 = \{0\}$. Now, consider a graded submodule $N = (8)$ of M. Then N is not a graded 2-absorbing submodule of M since $2 \cdot 2 \cdot 2 \in N$ and neither $2 \cdot 2 \in N$ nor $2 \cdot 2 \in (N :_R M) = 8\mathbb{Z}$. However an easy computation shows that N is a graded $G2$-absorbing submodule of M.

It is easy to see that every graded primary submodule is a graded $G2$-absorbing submodule. The following example shows that the converse is not true in general.

Example 2.3. Let $G = \mathbb{Z}_2$, then $R = \mathbb{Z}$ is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}$ be a graded R-module with $M_0 = \mathbb{Z}$ and $M_1 = \{0\}$. Now, consider a graded submodule $N = 6\mathbb{Z}$ of M. Then N is not a graded primary submodule of M since $3 \cdot 2 \in N = 6\mathbb{Z}$ but neither $2 \in 6\mathbb{Z}$ nor $3 \in \mathrm{Gr}((6\mathbb{Z} :_R \mathbb{Z}))$. However an easy computation shows that N is a graded $G2$-absorbing submodule of M.

Theorem 2.4. *Let R be a G-graded ring, M a graded R-module, $N = \bigoplus_{h\in G} N_h$ a graded submodule of M and $h \in G$. If N_h is a h-$G2$-absorbing submodule of the R_e-module M_h, then $(N_h :_{R_e} M_h)$ is an e-2-absorbing primary ideal of R.*

Proof. Let $r_e, s_e, t_e \in R_e$ such that $r_e s_e t_e \in (N_h :_{R_e} M_h)$. Assume that $r_e s_e \notin (N_h :_{R_e} M_h)$ and $s_e t_e \notin \mathrm{Gr}((N_h :_{R_e} M_h))$. It follows that $r_e s_e \notin (N_h :_{R_e} M_h)$ and $s_e t_e \notin (N_h :_{R_e} M_h)$. Then there exist $m_h, m'_h \in M_h$ such that $r_e s_e m_h \notin N_h$ and $s_e t_e m'_h \notin N_h$. Since $s_e m_h + s_e m'_h \in M_h$, $r_e t_e(s_e m_h + s_e m'_h) \in$

N_h and so we have either $r_et_e \in \mathrm{Gr}((N_h :_{R_e} M_h))$ or $r_e(s_em_h + s_em'_h) \in N_h$ or $t_e(s_em_h + s_em'_h) \in N_h$ as N_h is a h-$G2$-absorbing submodule of the R_e-module M_h. If $r_et_e \in \mathrm{Gr}((N_h :_{R_e} M_h))$, then we are done. If $r_e(s_em_h + s_em'_h) \in N_h$, then $r_es_em'_h \notin N_h$ since $r_es_em_h \notin N_h$. Since N_h is a h-$G2$-absorbing submodule of the R_e-module M_h, $r_es_et_em'_h \in N_h$, $r_es_em'_h \notin N_h$ and $s_et_em'_h \notin N_h$, we get $r_et_e \in \mathrm{Gr}((N_h :_{R_e} M_h))$. Similarly, if $t_e(s_em_h + s_em'_h) \in N_h$, then we get $t_es_em_h \notin N_h$. Since N_h is a h-$G2$-absorbing submodule of the R_e-module M_h, $r_es_et_em_h \in N_h, r_es_em_h \notin N_h$ and $s_et_em_h \notin N_h$, we get $r_et_e \in \mathrm{Gr}((N_h :_{R_e} M_h))$. Hence $(N_h :_{R_e} M_h)$ is an e-2-absorbing primary ideal of R. □

The following example shows that the converse of Theorem 2.4 is not true in general.

Example 2.5. Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$ be a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z} \times \mathbb{Z}$ be a graded R-module with $M_0 = \mathbb{Z} \times \mathbb{Z}$ and $M_1 = \{(0,0)\}$. Now, consider a graded submodule $N = (0) \times 6\mathbb{Z}$ of M. Then $(N :_R M) = \{0\}$ is a 2-absorbing primary ideal of R since R is a graded integral domain. But N is not a graded $G2$-absorbing submodule since $2 \cdot 3 \cdot (0,1) \in N$ but neither $2 \cdot (0,1) \in N$ nor $3 \cdot (0,1) \in N$ nor $2 \cdot 3 \in \mathrm{Gr}((N :_R M)) = \{0\}$.

Theorem 2.6. *Let R be a G-graded ring, M a graded R-module, $N = \oplus_{h\in G} N_h$ a graded submodule of M and $h \in G$. If N_h is a h-$G2$-absorbing submodule of the R_e-module M_h, then $\mathrm{Gr}((N_h :_{R_e} M_h))$ is an e-2-absorbing ideal of R.*

Proof. Let N_h be a h-$G2$-absorbing submodule of the R_e-module M_h. Then by Theorem 2.4, $(N_h :_{R_e} M_h)$ is an e-2-absorbing primary ideal of R. So by [6, Theorem 2.3], we have $\mathrm{Gr}((N_h :_{R_e} M_h))$ is an e-2-absorbing ideal of R. □

Theorem 2.7. *Let R be a G-graded ring, M a graded R-module, $N = \bigoplus_{h\in G} N_h$ a graded submodule of M and $h \in G$. If N_h is a h-$G2$-absorbing submodule of the R_e-module M_h, then $(N_h :_{R_e} m_h)$ is an e-2-absorbing primary ideal of R for each $m_h \in M_h \backslash N_h$.*

Proof. Let $m_h \in M_h\backslash N_h$, then $(N_h :_{R_e} m_h)$ is a proper ideal of R_e. Now, let $r_e, s_e, t_e \in R_e$ such that $r_es_et_e \in (N_h :_{R_e} m_h)$. Then $r_es_et_em_h \in N_h$, and so we get either $r_em_h \in N_h$ or $s_et_em \in N_h$ or $r_es_et_e \in \mathrm{Gr}((N_h :_{R_e} M_h))$ as N_h is a h-$G2$-absorbing submodule of the R_e-module M_h. If $r_em_h \in N_h$ or $s_et_em \in N_h$, then either $r_es_e \in (N_h :_{R_e} m_h)$ or $s_et_e \in (N_h :_{R_e} m_h)$. If $r_es_et_e \in \mathrm{Gr}((N_h :_{R_e} M_h))$, then either $r_es_e \in \mathrm{Gr}((N_h :_{R_e} M_h)) \subseteq \mathrm{Gr}((N_h :_{R_e} m_h))$ or $s_et_e \in \mathrm{Gr}((N_h :_{R_e} M_h)) \subseteq \mathrm{Gr}((N_h :_{R_e} m_h))$ or $r_et_e \in \mathrm{Gr}((N_h :_{R_e} M_h)) \subseteq \mathrm{Gr}((N_h :_{R_e} m_h))$ as $\mathrm{Gr}((N_h :_{R_e} M_h))$ is an e-2-absorbing ideal of R_e by Theorem 2.6. Therefore, $(N_h :_{R_e} m_h)$ is an e-2-absorbing primary ideal of R. □

Let R be a G-graded ring, M be a graded R-module and $t_\alpha \in h(R)$. The graded submodule $\{m \in M : t_\alpha m \in N\}$ will be denoted by $(N :_M t_\alpha)$.

Theorem 2.8. *Let R be a G-graded ring and M a graded R-module. If N is a graded $G2$-absorbing submodule M, then $(N :_M t_\alpha)$ is a graded $G2$-absorbing submodule of M for all $t_\alpha \in h(R)\backslash(N :_R M)$.*

Proof. Let $t_\alpha \in h(R)\backslash(N :_R M)$. Then $(N :_M t_\alpha)$ is a proper graded submodule of M. Now, let $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ such that $r_g s_h m_\lambda \in (N :_M t_\alpha)$. So, $r_g s_h t_\alpha m_\lambda \in N$, then either $s_h t_\alpha m_\lambda \in N$ or $r_g t_\alpha m_\lambda \in N$ or $r_g s_h \in \mathrm{Gr}((N :_R M))$ as N is a graded $G2$-absorbing submodule of M. Either $s_h m_\lambda \in (N :_M t_\alpha)$ or $r_g m_\lambda \in (N :_M t_\alpha)$ or $(r_g s_h)^n M \subseteq N$. If $(r_g s_h)^n M \subseteq N$, then $(r_g s_h)^n M \subseteq (N :_M t_\alpha)$ and hence $r_g s_h \in \mathrm{Gr}(((N :_M t_\alpha) :_R M))$. So, we get either $s_h m_\lambda \in (N :_M t_\alpha)$ or $r_g m_\lambda \in (N :_M t_\alpha)$ or $r_g s_h \in \mathrm{Gr}(((N :_M t_\alpha) :_R M))$. So, $(N :_M t_\alpha)$ is a graded $G2$-absorbing submodule of M. □

Recall from [18] that a proper graded submodule N of a graded module M is said to be *a graded irreducible* if N cannot be expressed as the intersection of two strictly larger graded submodules of M.

Theorem 2.9. *Let R be a G-graded ring, M a graded R-module and N a proper graded irreducible submodule of M. Then the following statements are equivalent:*

(i) *N is a graded $G2$-absorbing submodule of M.*
(ii) *$(N :_M r_g) = (N :_M r_g^2)$, for all $r_g \in h(R)\backslash\mathrm{Gr}((N :_R M))$.*

Proof. (i) ⇒ (ii) Suppose that N is a graded $G2$-absorbing submodule of M. Let $r_g \in h(R)\backslash\mathrm{Gr}((N :_R M))$, then it is clear that $(N :_M r_g) \subseteq (N :_M r_g^2)$. Now, let $m_h \in h(M) \cap (N :_M r_g^2)$, hence $r_g^2 m_h \in N$. This yields that either $r_g m_h \in N$ or $r_g^2 \in \mathrm{Gr}((N :_R M))$ as N is a graded $G2$-absorbing submodule of M. If $r_g^2 \in \mathrm{Gr}((N :_R M))$, then $r_g^{2k} M \subseteq N$ for some $k \in \mathbb{Z}^+$ and hence $r_g \in \mathrm{Gr}((N :_R M))$ which is a contradiction. Thus $r_g m_h \in N$, and so $m_h \in (N :_M r_g)$. Hence $(N :_M r_g^2) \subseteq (N :_M r_g)$. Therefore, $(N :_M r_g) = (N :_M r_g^2)$.

(ii) ⇒ (i) Let $r_{g_1}, r_{g_2} \in h(R)$ and $m_h \in h(M)$ such that $r_{g_1} r_{g_2} m_h \in N$ and $r_{g_1} r_{g_2} \notin \mathrm{Gr}((N :_R M))$. Since $r_{g_1} r_{g_2} \notin \mathrm{Gr}((N :_R M))$, we have $r_{g_1} \notin \mathrm{Gr}((N :_R M))$ and $r_{g_2} \notin \mathrm{Gr}((N :_R M))$. Hence by (ii) we get $(N :_M r_{g_1}) = (N :_M r_{g_1}^2)$ and $(N :_M r_{g_2}) = (N :_M r_{g_2}^2)$. It is clear that $N \subseteq (N + Rr_{g_1} m_h) \cap (N + Rr_{g_2} m_h)$. For the reverse inclusion, let $n \in ((N + Rr_{g_1} m_h) \cap (N + Rr_{g_2} m_h)) \cap h(M)$. Then $n = n_{\lambda_1} + r_{g_3} r_{g_1} m_h = n_{\lambda_2} + r_{g_4} r_{g_2} m_h$ where $n_{\lambda_1}, n_{\lambda_2} \in N \cap h(M)$ and $r_{g_3}, r_{g_4} \in h(R)$. Now, $r_{g_1} n = r_{g_1} n_{\lambda_1} + r_{g_3} r_{g_1}^2 m_h = r_{g_1} n_{\lambda_2} + r_{g_4} r_{g_1} r_{g_2} m_h \in N$, which yields that $r_{g_3} r_{g_1}^2 m_h \in N$ and hence $r_{g_3} m_h \in (N :_M r_{g_1}^2) = (N :_M r_{g_1})$. Hence $r_{g_3} r_{g_1} m_h \in N$ and so $n \in N$. Thus $(N + Rr_{g_1} m_h) \cap (N + Rr_{g_2} m_h) \subseteq N$. Therefore, $(N + Rr_{g_1} m_h) \cap (N + Rr_{g_2} m_h) = N$. Since N is a graded irreducible submodule of M, then we get either $(N + Rr_{g_1} m_h) = N$ or $(N + Rr_{g_2} m_h) = N$. Hence either $r_{g_1} m_h \in N$ or $r_{g_2} m_h \in N$. Therefore, N is a graded $G2$-absorbing submodule of M. □

Recall from [18] that a proper graded ideal P of R is said to be a graded prime ideal if whenever $r_g s_h \in P$, we have $r_g \in P$ or $s_h \in P$, where $r_g, s_h \in h(R)$).

THEOREM 2.10. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M such that $(N :_R M)$ is a graded prime ideal of R. Then the following statements are equivalent:*

(i) *N is a graded $G2$-absorbing submodule of M.*

(ii) *For each $m_{g_1}, m_{g_2} \in h(M)$ with $(N :_R m_{g_1})\backslash((N :_R m_{g_2}) \cup \mathrm{Gr}((N :_R M))) \neq \emptyset$, then $N = (N + Rm_{g_1}) \cap (N + Rm_{g_2})$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that N is a graded $G2$-absorbing submodule of M. Let $m_{g_1}, m_{g_2} \in h(M)$ such that $(N :_R m_{g_1})\backslash((N :_R m_{g_2}) \cup \mathrm{Gr}((N :_R M))) \neq \emptyset$. Then there exists $s_h \in ((N :_R m_{g_1}) \cap h(R))\backslash((N :_R m_{g_2}) \cup \mathrm{Gr}((N :_R M)))$. This yields that $s_h m_{g_1} \in N$, $s_h m_{g_2} \notin N$ and $s_h \notin \mathrm{Gr}((N :_R M))$. Now, it is clear that $N \subseteq (N + Rm_{g_1}) \cap (N + Rm_{g_2})$. For the reverse inclusion, let $n \in ((N + Rm_{g_1}) \cap (N + Rm_{g_2})) \cap h(M)$. Then $n = n_{\lambda_1} + r_{\alpha_1} m_{g_1} = n_{\lambda_2} + r_{\alpha_2} m_{g_2}$, where $n_{\lambda_1}, n_{\lambda_2} \in N \cap h(M)$ and $r_{\alpha_1}, r_{\alpha_2} \in h(R)$. Now, $s_h n = s_h n_{\lambda_1} + s_h r_{\alpha_1} m_{g_1} = s_h n_{\lambda_2} + s_h r_{\alpha_2} m_{g_2}$, it follows that $s_h r_{\alpha_2} m_{g_2} \in N$. Hence either $s_h r_{\alpha_2} \in \mathrm{Gr}((N :_R M))$ or $r_{\alpha_2} m_{g_2} \in N$ as N is a graded $G2$-absorbing submodule of M and $s_h m_{g_2} \notin N$. If $s_h r_{\alpha_2} \in \mathrm{Gr}((N :_R M))$, then $(s_h r_{\alpha_2})^k = (s_h)^k (r_{\alpha_2})^k \in (N :_R M)$ for some $k \in \mathbb{Z}^+$. Since $(N :_R M)$ is a graded prime ideal of R and $s_h \notin (N :_R M) \subseteq \mathrm{Gr}((N :_R M))$, we get $r_{\alpha_2} \in (N :_R M)$ and so $r_{\alpha_2} m_{g_2} \in N$. In both the cases, we get $n = n_{\lambda_2} + r_{\alpha_2} m_{g_2} \in N$. Hence $(N + Rm_{g_1}) \cap (N + Rm_{g_2}) \subseteq N$. Therefore, $N = (N + Rm_{g_1}) \cap (N + Rm_{g_2})$.

(ii) $\Rightarrow$ (i) Let $m_g \in h(M)$ and $s_{h_1}, s_{h_2} \in h(R)$ such that $s_{h_1} s_{h_2} m_g \in N$, $s_{h_1} m_g \notin N$ and $s_{h_1} s_{h_2} \notin \mathrm{Gr}((N :_R M))$. Hence, $s_{h_1} \in (N :_R s_{h_2} m_g)\backslash((N :_R m_g) \cup \mathrm{Gr}((N :_R M)))$. By (ii) we get $N = (N + Rs_{h_2} m_g) \cap (N + Rm_g)$ and so $s_{h_2} m_g \in N$. So, N is a graded $G2$-absorbing submodule of M. □

THEOREM 2.11. *Let R be a G-graded ring, M a graded R-module, N a graded $G2$-absorbing submodule of M and $L = \bigoplus_{\lambda \in G} L_\lambda$ a graded submodule of M. Then for every $r_h, s_g \in h(R)$ and $\lambda \in G$ with $r_h s_g L_\lambda \subseteq N$, either $r_h L_\lambda \subseteq N$ or $s_g L_\lambda \subseteq N$ or $r_h s_g \in \mathrm{Gr}((N :_R M))$.*

Proof. Let $r_h, s_g \in h(R)$ and $\lambda \in G$ with $r_h s_g L_\lambda \subseteq N$. Assume that $r_h s_g \notin \mathrm{Gr}((N :_R M))$, $r_h L_\lambda \nsubseteq N$ and $s_g L_\lambda \nsubseteq N$. Then there exist $l_\lambda, l'_\lambda \in L_\lambda$ such that $r_h l_\lambda \notin N$ and $s_g l'_\lambda \notin N$. Since $r_h s_g l_\lambda \in N$, $r_h l_\lambda \notin N$ and $r_h s_g \notin \mathrm{Gr}((N :_R M))$, we get $s_g l_\lambda \in N$ as N is a graded $G2$-absorbing submodule of M. Similarly, since $r_h s_g l'_\lambda \in N$, $s_g l'_\lambda \notin N$ and $r_h s_g \notin \mathrm{Gr}((N :_R M))$, we get $r_h l'_\lambda \in N$. Since $l_\lambda + l'_\lambda \in L_\lambda$, $r_h s_g (l_\lambda + l'_\lambda) \in N$. Then either $r_h(l_\lambda + l'_\lambda) \in N$ or $s_g(l_\lambda + l'_\lambda) \in N$ as N is a graded $G2$-absorbing submodule of M and $r_h s_g \notin \mathrm{Gr}((N :_R M))$. If $r_h(l_\lambda + l'_\lambda) \in N$, then $r_h l_\lambda \in N$ since $r_h l'_\lambda \in N$, a contradiction. Similarly, if $s_g(l_\lambda + l'_\lambda) \in N$, then $s_g l'_\lambda \in N$ since $s_g l_\lambda \in N$, a contradiction. Therefore, either $r_h s_g \in \mathrm{Gr}((N :_R M))$ or $r_h L_\lambda \subseteq N$ or $s_g L_\lambda \subseteq N$. □

Theorem 2.12. *Let R be a G-graded ring, M a graded R-module and N a graded $G2$-absorbing submodule of M. Let $I = \bigoplus_{g\in G} I_g$ be a graded ideal of R and $L = \bigoplus_{h\in G} L_h$ be a graded submodule of M. Then for every $r_\alpha \in h(R)$ and $g, h \in G$ with $r_\alpha I_g L_h \subseteq N$, either $I_g L_h \subseteq N$ or $r_\alpha L_h \subseteq N$ or $r_\alpha I_g \subseteq$ $\mathrm{Gr}((N :_R M))$.*

Proof. Let $r_\alpha \in h(R)$ and $g, h \in G$ with $r_\alpha I_g L_h \subseteq N$ and $I_g L_h \nsubseteq N$. We show that either $r_\alpha L_h \subseteq N$ or $r_\alpha I_g \subseteq \mathrm{Gr}((N :_R M))$. On contrary, we assume that $r_\alpha L_h \nsubseteq N$ and $r_\alpha I_g \nsubseteq \mathrm{Gr}((N :_R M))$. Then there exist $i_g \in I_g$ and $l_h \in L_h$ such that $r_\alpha i_g \notin \mathrm{Gr}((N :_R M))$ and $r_\alpha l_h \notin N$. Since $r_\alpha i_g l_h \in N$, we get $i_g l_h \in N$ as N is a graded $G2$-absorbing submodule of M. Now, since $I_g L_h \nsubseteq N$, there exist $i'_g \in I_g$ and $l'_h \in L_h$ such that $i'_g l'_h \notin N$ but $r_\alpha i'_g l'_h \in N$, then either $r_\alpha l'_h \in N$ or $r_\alpha i'_g \in \mathrm{Gr}((N :_R M))$ as N is a graded $G2$-absorbing submodule of M. Now, we have the following three cases:

Case 1. Suppose $r_\alpha l'_h \notin N$ and $r_\alpha i'_g \in \mathrm{Gr}((N :_R M))$. Since $r_\alpha i_g l'_h \in N$, $r_\alpha i_g \notin \mathrm{Gr}((N :_R M))$ and $r_\alpha l'_h \notin N$, then $i_g l'_h \in N$. Now, since $r_\alpha i'_g \in$ $\mathrm{Gr}((N :_R M))$ and $r_\alpha i_g \notin \mathrm{Gr}((N :_R M))$, we have $r_\alpha(i_g + i'_g) \notin \mathrm{Gr}((N :_R M))$. Since $i_g + i'_g \in I_g$, we get $r_\alpha(i_g + i'_g)l'_h \in N$. Again, since $r_\alpha(i_g + i'_g)l'_h \in N$ and $r_\alpha l'_h \notin N$, we have $(i_g + i'_g)l'_h = i_g l'_h + i'_g l'_h \in N$, it follows that $i'_g l'_h \in N$ since $i_g l'_h \in N$, a contradiction.

Case 2. Suppose $r_\alpha l'_h \in N$ and $r_\alpha i'_g \notin \mathrm{Gr}((N :_R M))$. Since $r_\alpha i'_g l_h \in N$, we get $i'_g l_h \in N$ as N is a graded $G2$-absorbing submodule. Since $i'_g l_h \in N$ and $i'_g l'_h \notin N$, then $i'_g(l_h + l'_h) \notin N$. By $l_h + l'_h \in L_h$, we get $r_\alpha i'_g(l_h + l'_h) \in N$. This yields that $r_\alpha(l_h + l'_h) \in N$. But $r_\alpha l'_h \in N$, so $r_\alpha l_h \in N$, a contradiction.

Case 3. Suppose $r_\alpha l'_h \in N$ and $r_\alpha i'_g \in \mathrm{Gr}((N :_R M))$. Since $r_\alpha l'_h \in N$ and $r_\alpha l_h \notin N$, we get $r_\alpha(l'_h + l_h) \notin N$. Similarly, since $r_\alpha i'_g \in \mathrm{Gr}((N :_R M))$ and $r_\alpha i_g \notin \mathrm{Gr}((N :_R M))$, we get $r_\alpha(i'_g + i_g) \notin \mathrm{Gr}((N :_R M))$. Since $l'_h + l_h \in L_h$, $r_\alpha i_g(l'_h + l_h) \in N$. This yields that $i_g(l'_h + l_h) \in N$ and then $i_g l'_h \in N$ since $i_g l_h \in N$. Similarly, consider $r_\alpha(i_g + i'_g)l_h \in N$, this yields that $(i_g + i'_g)l_h \in N$ and then $i'_g l_h \in N$ since $i_g l_h \in N$. Now, consider $r_\alpha(i_g + i'_g)(l_h + l'_h) \in N$, then we get $(i_g + i'_g)(l_h + l'_h) \in N$, but $i_g l_h \in N, i_g l'_h \in N$ and $i'_g l_h \in N$, we get $i'_g l'_h \in$ N, which is a contradiction. So, either $r_\alpha L_h \subseteq N$ or $r_\alpha I_g \subseteq \mathrm{Gr}((N :_R M))$. □

The next theorem gives a characterization of graded $G2$-absorbing submodules.

Theorem 2.13. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. Let $I = \bigoplus_{g\in G} I_g$, $J = \bigoplus_{h\in G} J_h$ be two graded ideals of R and $L = \bigoplus_{\lambda\in G} L_\lambda$ be a graded submodule of M. Then the following statements are equivalent:*

(i) *N is a graded $G2$-absorbing submodule of M.*
(ii) *For every h, g, $\lambda \in G$ with $J_h I_g L_\lambda \subseteq N$, either $I_g L_\lambda \subseteq N$ or $J_h L_\lambda \subseteq N$ or $J_h I_g \subseteq \mathrm{Gr}((N :_R M))$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that N is a graded $G2$-absorbing submodule of M and let $g,\ h,\ \lambda \in G$ with $J_h I_g L_\lambda \subseteq N$ and $I_g L_\lambda \nsubseteq N$. We show that either $J_h L_\lambda \subseteq N$ or $J_h I_g \subseteq \mathrm{Gr}((N :_R M))$. By Theorem 2.12, for all $j_h \in J_h$, either $j_h L_\lambda \subseteq N$ or $j_h I_g \subseteq \mathrm{Gr}((N :_R M))$. If $j_h L_\lambda \subseteq N$ for all $j_h \in J_h$, then $J_h L_\lambda \subseteq N$. Similarly, if $j_h I_g \subseteq \mathrm{Gr}((N :_R M))$ for all $j_h \in J_h$, then $J_h I_g \subseteq \mathrm{Gr}((N :_R M))$. Assume that there exist $j_h,\ j'_h \in J_h$ such that $j_h L_\lambda \nsubseteq N$ and $j'_h I_g \nsubseteq \mathrm{Gr}((N :_R M))$. Since $j_h I_g L_\lambda \subseteq N$, $I_g L_\lambda \nsubseteq N$ and $j_h L_\lambda \nsubseteq N$, by Theorem 2.12, we get $j_h I_g \subseteq \mathrm{Gr}((N :_R M))$. Also, since $j'_h I_g L_\lambda \subseteq N$, $I_g L_\lambda \nsubseteq N$ and $j'_h I_g \nsubseteq \mathrm{Gr}((N :_R M))$, by Theorem 2.12, we get $j'_h L_\lambda \subseteq N$. Since $j_h + j'_h \in J_h$, $(j_{h_1} + j'_h) I_g L_\lambda \subseteq N$. By Theorem 2.12, we get either $(j_h + j'_h) I_g \subseteq \mathrm{Gr}((N :_R M)))$ or $(j_h + j'_h) L_\lambda \subseteq N$. If $(j_h + j'_h) I_g \subseteq \mathrm{Gr}((N :_R M))$, then $j'_h I_g \subseteq \mathrm{Gr}((N :_R M))$ since $j_h I_g \subseteq \mathrm{Gr}((N :_R M))$, which is a contradiction. Similarly, if $(j_h + j'_h) L_\lambda \subseteq N$, then $j_h L_\lambda \subseteq N$ since $j'_h L_\lambda \subseteq N$, which is a contradiction. Therefore, either $J_h L_\lambda \subseteq N$ or $J_h I_g \subseteq \mathrm{Gr}((N :_R M))$.

(ii) $\Rightarrow$ (i) Assume that (ii) holds. Let $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ such that $s_h r_g m_\lambda \in N$. Let $J = (s_h)$ and $I = (r_g)$ be a graded ideals of R generated by s_h, r_g, respectively. And let $L = (m_\lambda)$ be a graded submodule of M generated by m_λ. Hence $J_h I_g L_\lambda \subseteq N$, and by our assumption we get either $I_g L_\lambda \subseteq N$ or $J_h L_\lambda \subseteq N$ or $J_h I_g \subseteq \mathrm{Gr}((N :_R M))$. It follows that either $s_h m_\lambda \in N$ or $r_g m_\lambda \in N$ or $s_h r_g \in \mathrm{Gr}((N :_R M))$. Therefore, N is a graded $G2$-absorbing submodule of M. $\square$

Recall from [2] that a graded zero-divisor on a graded R-module M is an element $r_g \in h(R)$ for which there exists $m_h \in h(M)$ such that $m_h \neq 0$ but $r_g m_h = 0$. The set of all graded zero-divisors on M is denoted by $G\text{-}Zdv_R(M)$.

The following result studies the behavior of graded $G2$-absorbing submodules under localization.

Theorem 2.14. *Let R be a G-graded ring, M a graded R-module and $S \subseteq h(R)$ be a multiplicatively closed subset of R.*

(i) *If N is a graded $G2$-absorbing submodule of M with $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a graded $G2$-absorbing submodule of $S^{-1}M$.*
(ii) *If $S^{-1}N$ is a graded $G2$-absorbing submodule of $S^{-1}M$ with $S \cap G\text{-}Zdv_R(M/N) = \emptyset$, then N is a graded $G2$-absorbing submodule of M.*

Proof. (i) Since $(N :_R M) \cap S = \emptyset$, $S^{-1}N$ is a proper graded submodule of $S^{-1}M$. Let $\frac{r_{g_1}}{s_{\lambda_1}}, \frac{r_{g_2}}{s_{\lambda_2}} \in h(S^{-1}R)$ and $\frac{m_h}{l_\alpha} \in h(S^{-1}M)$ such that $\frac{r_{g_1}}{s_{\lambda_1}} \frac{r_{g_2}}{s_{\lambda_2}} \frac{m_h}{l_\alpha} \in S^{-1}N$. Then there exists $s_{\lambda_3} \in S$ such that $s_{\lambda_3} r_{g_1} r_{g_2} m_h \in N$. Then either $s_{\lambda_3} r_{g_1} m_h \in N$ or $s_{\lambda_3} r_{g_2} m_h \in N$ or $r_{g_1} r_{g_2} \in \mathrm{Gr}((N :_R M))$ as N is a graded $G2$-absorbing submodule of M. This yields that either $\frac{r_{g_1}}{s_{\lambda_1}} \frac{m_h}{l_\alpha} = \frac{s_{\lambda_3} r_{g_1} m_h}{s_{\lambda_3} s_{\lambda_1} l_\alpha} \in S^{-1}N$ or $\frac{r_{g_2}}{s_{\lambda_2}} \frac{m_h}{l_\alpha} = \frac{s_{\lambda_3} r_{g_2} m_h}{s_{\lambda_3} s_{\lambda_2} l_\alpha} \in S^{-1}N$ or $\frac{r_{g_1}}{s_{\lambda_1}} \frac{r_{g_2}}{s_{\lambda_2}} = \frac{r_{g_1} r_{g_2}}{s_{\lambda_1} s_{\lambda_2}} \in S^{-1}\mathrm{Gr}((N :_R M)) = \mathrm{Gr}((S^{-1}N :_{S^{-1}R} S^{-1}M))$. Therefore, $S^{-1}N$ is a graded $G2$-absorbing submodule of $S^{-1}M$.

(ii) Let $r_{g_1}, r_{g_2} \in h(R)$ and $m_h \in h(M)$ such that $r_{g_1} r_{g_2} m_h \in N$. Then $\frac{r_{g_1}}{1_e}\frac{r_{g_2}}{1_e}\frac{m_h}{1_e} \in S^{-1}N$. Since $S^{-1}N$ is a graded $G2$-absorbing submodule of $S^{-1}M$, either $\frac{r_{g_1}}{1_e}\frac{m_h}{1_e} \in S^{-1}N$ or $\frac{r_{g_2}}{1_e}\frac{m_h}{1_e} \in S^{-1}N$ or $\frac{r_{g_1}}{1_e}\frac{r_{g_2}}{1_e} \in \mathrm{Gr}((S^{-1}N :_{S^{-1}R} S^{-1}M))$.

If $\frac{r_{g_1}}{1_e}\frac{m_h}{1_e} \in S^{-1}N$, then there exists $s_\lambda \in S$ such that $s_\lambda r_{g_1} m_h \in N$. This yields that $r_{g_1} m_h \in N$ since $S \cap G\text{-}Zdv_R(M/N) = \emptyset$. Similarly, we can show that if $\frac{r_{g_2}}{1_e}\frac{m_h}{1_e} \in S^{-1}N$, then $r_{g_2} m_h \in N$.

Now, if $\frac{r_{g_1}}{1_e}\frac{r_{g_2}}{1_e} \in \mathrm{Gr}((S^{-1}N :_{S^{-1}R} S^{-1}M)) = S^{-1}\mathrm{Gr}((N :_R M))$, then there exists $t_\lambda \in S$ such that $(t_\lambda r_{g_1} r_{g_2})^n M \subseteq N$ for some $n \in \mathbb{Z}^+$ and hence $r_{g_1} r_{g_2} \in \mathrm{Gr}((N :_R M))$ since $S \cap G\text{-}Zdv_R(M/N) = \emptyset$. So, N is a graded $G2$-absorbing submodule of M. □

Let M and M' be two graded R-modules. A homomorphism of graded R-modules $f : M \to M'$ is a homomorphism of R-modules verifying $f(M_g) \subseteq M'_g$ for every $g \in G$ (see [15].)

The following result studies the behavior of graded $G2$-absorbing submodules under graded homomorphism.

THEOREM 2.15. *Let R be a G-graded ring, M and M' be two graded R-modules and $f : M \to M'$ be a graded epimorphism. Then the following statements hold.*

(i) *If N is a graded $G2$-absorbing submodule of M with $ker(f) \subseteq N$, then $f(N)$ is a graded $G2$-absorbing submodule of M'.*

(ii) *If N' is a graded $G2$-absorbing submodule of M', then $f^{-1}(N')$ is a graded $G2$-absorbing submodule of M.*

Proof. (i) Suppose that N is a graded $G2$-absorbing submodule of M. It is easy to see that $f(N)$ is a proper graded submodule of M'.

Now, let $r_g, s_h \in h(R)$ and $m'_\lambda \in h(M')$ such that $r_g s_h m'_\lambda \in f(N)$, then there exists $m_\lambda \in h(M)$ such that $m'_\lambda = f(m_\lambda)$ as f is a graded epimorphism, so $r_g s_h m'_\lambda = f(r_g s_h m_\lambda) \in f(N)$. Then there exists $n_\alpha \in N \cap h(M)$ such that $f(r_g s_h m_\lambda) = f(n_\alpha)$. This yields that $r_g s_h m_\lambda - n_\alpha \in ker(f) \subseteq N$ and then $r_g s_h m_\lambda \in N$. Hence we get either $r_g m_\lambda \in N$ or $s_h m_\lambda \in N$ or $(r_g s_h)^k M \subseteq N$ for some $k \in \mathbb{Z}^+$ as N is a graded $G2$-absorbing submodule of M. Thus $r_g m'_\lambda \in f(N)$ or $s_h m'_\lambda \in f(N)$ or $(r_g s_h)^k M' \subseteq f(N)$.

Therefore, $f(N)$ is a graded $G2$-absorbing submodule of M'.

(ii) Suppose that N' is a graded $G2$-absorbing submodule of M'. It is easy to see that $f^{-1}(N')$ is a proper graded submodule of M.

Now, let $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ such that $r_g s_h m_\lambda \in f^{-1}(N')$, hence $r_g s_h f(m_\lambda) \in N'$. Thus either $r_g f(m_\lambda) = f(r_g m_\lambda) \in N'$ or $s_h f(m_\lambda) = f(s_h m_\lambda) \in N'$ or $(r_g s_h)^k M' = f((r_g s_h)^k M) \subseteq N'$ for some $k \in \mathbb{Z}^+$. This yields that either $r_g m_\lambda \in f^{-1}(N')$ or $s_h m_\lambda \in f^{-1}(N')$ or $(r_g s_h)^k M \subseteq f^{-1}(N')$. Therefore, N' is a graded $G2$-absorbing submodule of M. □

As an immediate consequence of Theorem 2.15 we have the following corollary.

Corollary 2.16. *Let R be a G-graded ring, M a graded R-module and N, K be two graded submodules of M with $K \subseteq N$. Then N is a graded $G2$-absorbing submodule of M if and only if N/K is a graded $G2$-absorbing submodule of M/K.*

Let R be a G-graded ring. A graded R-module M is said to be *a graded cancellative module* if whenever $r_g m_{h_1} = r_g m_{h_2}$ where $m_{h_1}, m_{h_2} \in h(M)$ and $r_g \in h(R)$, implies $m_{h_1} = m_{h_2}$ (see [15].) A graded submodule N of M is said to be *a graded pure submodule* if $r_g N = N \cap r_g M$ for every $r_g \in h(R)$ (see [8].)

Theorem 2.17. *Let R be a G-graded ring, M a graded cancellative R-module and N a proper graded submodule of M. If N is a graded pure submodule of M, then N is a graded $G2$-absorbing submodule of M with* $\mathrm{Gr}((N :_R M)) = \{0\}$.

Proof. Suppose N is a graded pure submodule of M. Let $r_{g_1}, r_{g_2} \in h(R)$, $m_{h_1} \in h(M)$ be such that $r_{g_1} r_{g_2} m_{h_1} \in N$ and $r_{g_1} r_{g_2} \notin \mathrm{Gr}((N :_R M))$. Then we get $r_{g_1} r_{g_2} m_{h_1} \in r_{g_1} r_{g_2} M \cap N = r_{g_1} r_{g_2} N$, so $r_{g_1} r_{g_2} m_{h_1} = r_{g_1} r_{g_2} n_\lambda$ for some $n_\lambda \in N \cap h(M)$. Then $r_{g_2} m_{h_1} = r_{g_2} n_\lambda \in N$ as M is a graded cancellative module. Thus N is a graded $G2$-absorbing submodule of M.

Now, assume that $\mathrm{Gr}((N :_R M)) \neq \{0\}$, so there exists $0 \neq r_{g_3} \in h(R)$ such that $r_{g_3}^k M \subseteq N$ for some $k \in \mathbb{Z}^+$. Since $N \neq M$, there exists $m_{h_2} \in h(M) \backslash N$ such that $r_{g_3}^k m_{h_2} \in r_{g_3}^k M \cap N = r_{g_3}^k N$. So, there exists $m_{h_3} \in N \cap h(M)$ such that $r_{g_3}^k m_{h_2} = r_{g_3}^k m_{h_3}$, so $m_{h_2} = m_{h_3} \in N$, a contradiction. So, $\mathrm{Gr}((N :_R M))=\{0\}$. □

Theorem 2.18. *Let R be a G-graded ring, M_1 and M_2 be two graded R-modules and N_1 a graded submodule of M_1. Let $M = M_1 \times M_2$. Then $K = N_1 \times M_2$ is a graded $G2$-absorbing submodule of M if and only if N_1 is a graded $G2$-absorbing submodule of M_1.*

Proof. Suppose that $K = N_1 \times M_2$ is a graded $G2$-absorbing submodule of M. Let $r_g, s_h \in h(R)$ and $m_\lambda \in h(M_1)$ such that $r_g s_h m_\lambda \in N_1$, $r_g m_\lambda \notin N_1$ and $s_h m_\lambda \notin N_1$. This yields that $r_g s_h (m_\lambda, 0) = (r_g s_h m_\lambda, 0) \in K$, $r_g(m_\lambda, 0) = (r_g m_\lambda, 0) \notin K$ and $s_h(m_\lambda, 0) = (s_h m_\lambda, 0) \notin K$. Then $r_g s_h \in \mathrm{Gr}((K :_R M))$ as K is a graded $G2$-absorbing submodule of M, It follows that $(r_g s_h)^n M \subseteq K$ for some $n \in \mathbb{Z}^+$. So $(r_g s_h)^n M_1 \subseteq N_1$. Therefore, N_1 is a graded $G2$-absorbing submodule of M_1.

Conversely, suppose that N_1 is a graded $G2$-absorbing submodule of M_1. Let $r_g, s_h \in h(R)$ and $(m_\lambda, x_\alpha) \in h(M)$ such that $r_g s_h (m_\lambda, x_\alpha) = (r_g s_h m_\lambda, r_g s_h x_\alpha) \in K$, $r_g(m_\lambda, x_\alpha) = (r_g m_\lambda, r_g x_\alpha) \notin K$ and $s_h(m_\lambda, x_\alpha) = (s_h m_\lambda, s_h x_\alpha) \notin K$. Hence, $r_g m_\lambda \notin N_1$ and $s_h m_\lambda \notin N_1$. Then $(r_g s_h)^n M_1 \subseteq N_1$ as N_1 is a graded $G2$-absorbing submodule of M_1. So, $(r_g s_h)^n M \subseteq K$ and K is a graded $G2$-absorbing submodule of M. □

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