MATHEMATICA, 65 (88), N° 1, 2023, pp. 3–11

ON h-LOCAL FUNCTIONS IN IDEAL TOPOLOGICAL SPACES

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Abstract. In this research we introduce h-local functions by using h-open sets in an ideal topological space (X, τ, I). Some properties and characterizations of h-local functions are studied. Also, we introduce and research the notions of I_{s^*g}-h-closed and I_g-h-closed sets in an ideal topological space. Additionally, $\mathrm{Cl}^*{}_h$ is defined and its properties are examined.

MSC 2020. 54A05, 54A10.

Key words. Ideal topological space, h-local function, semi h-local function, operation $\mathrm{Cl}^*{}_h$, I_{s^*g}-h-closed set, I_g-h-closed set.

1. INTRODUCTION

In 1966, the concept of local functions was first presented by Kuratowski [7] and Vaidyanathaswamy [11] gave the notion of ideal topological spaces in 1960. Later, Jankovic and Hamlett [4] developed their works on ideal topological spaces in 1990.They gave the notion of I-open sets and studied topologies by ideals.The notion of I_g-closed sets was given by Dontchev et al. [2] in 1999. The concept of I_{s^*g}-closed sets was introduced by Khan and Hamza [5].

In this paper, we define h-local functions by using h-open sets [1] and introduce the operation $\mathrm{Cl}^*{}_h$ and a topology $\tau^*{}_h$. Moreover, I_{s^*g}-h-closed sets and I_g-h-closed sets are introduced and investigated.

2. BASIC CONCEPTS

This section introduces the fundamental principles needed to make this paper self-contained.

Definition 2.1 ([4]). An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies the following conditions:

(1) $A \in I$ and $B \subset A$ implies $B \in I$,
(2) $A \in I$ and $B \in I$ implies $(A \cup B) \in I$.

A topological space (X, τ) with an ideal I is called an ideal topological space and is denoted by (X, τ, I). Throughout this article (X, τ) and (X, τ, I) denote a topological space and an ideal topological space, respectively.

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DOI: 10.24193/mathcluj.2023.1.01

Definition 2.2 ([4], [10]). Let (X, τ, I) be an ideal topological space and $P(X)$ be the power set on X. A set operator $(.)^* : P(X) \to P(X)$, called a local function of A with respect to τ and I, is defined as follows: for any $A \subseteq X$, $A^*(I, \tau) = \{x \in X \mid (U \cap A) \notin I \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau \mid x \in U\}$.We simply write A^* instead of $A^*(I, \tau)$ in case there is no chance for confusion.

Definition 2.3. Let (X, τ) be a topological space. A subset A of X is said to be

(1) semi-open [8] if $A \subset \mathrm{cl}(\mathrm{Int}(A))$,
(2) regular-open [9] if $A = \mathrm{Int}(\mathrm{cl}(A))$.

The family of all semi-open (resp. regular open) sets in X is denoted by SO(X) (resp. RO(X)).

Definition 2.4 ([8]). Let (X, τ) be a topological space and let $A \subseteq X$. Then the union of all semi-open sets contained in A, denoted by sint(A), is called the semi-interior of A.

Definition 2.5 ([3], [6]). Let (X, τ, I) be an ideal topological space and $A \subseteq X$. Then $A^{*s}(I, \tau) = \{x \in X \mid (U \cap A) \notin I \text{ for every } U \in SO(X, x)\}$ is called the semi-local function of A with respect to I and τ, where $SO(X, x) = \{U \in SO(X) \mid x \in U\}$.When there is no ambiguity we write A^{*s} for $A^{*s}(I, \tau)$.

Definition 2.6 ([1]). A subset A of the topological space (X, τ) is said to be h-open if $A \subset \mathrm{Int}(A \cup U)$ for every non-empty open set U in X such that $U \neq X$.The complement of an h-open set is said to be h-closed. We denote the family of all h-open sets of a topological space (X, τ) by τ^h or $hO(X)$.

Remark 2.7 ([1]). Every open set in any topological space (X, τ) is h-open.

Theorem 2.8. *Let (X, τ) be a topological space.Then τ^h is a topology for X.*

Proof. (1) It is obvious that $\emptyset, X \in \tau^h$.

(2) It is shown in Theorem 2.2 of [1] that if $A, B \in \tau^h$ then $A \cap B \in \tau^h$.

(3) We show that τ^h is closed under arbitrary union.Let $A_i \in \tau^h$ for each $i \in I$.Then for each $i \in I$, $A_i \subset \bigcup_{\{i \in I\}} A_i$.For each $i \in I$ and any $U \in \tau$, we have $A_i \subset \mathrm{Int}(A_i \cup U) \subset Int[(\bigcup_{\{i \in I\}} A_i) \cup U]$.Therefore, we have $\bigcup_{\{i \in I\}} A_i \subset Int[(\bigcup_{\{i \in I\}} A_i) \cup U]$.This shows that $\bigcup_{\{i \in I\}} A_i \in \tau^h$. $\square$

Definition 2.9. A subset A of a topological space (X, τ) is said to be S_h-open if $A \subset s\mathrm{Int}(A \cup U)$ for every non-empty open set U in X such that $U \neq X$.The complement of the S_h-open set is said to be S_h-closed.We denote the family of all S_h-open sets of (X, τ) by $S_hO(X)$.

Remark 2.10. Every h-open set is S_h-open. The converse may not always be true as shown by Example 2.11.

EXAMPLE 2.11. $X = \{a, b, c, d\}$ be with the topology $\tau = \{\emptyset, X, \{d\}, \{a, c\}, \{a, c, d\}\}$. From Definition 2.3, we obtain the collection $SO(X) = \{\emptyset, X, \{d\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, c, d\}\}$ and from Definition 2.6, we obtain the collection $\tau^h = \{\emptyset, X, \{d\}, \{a, c\}, \{a, c, d\}\}$. By Definition 2.9, we find the collection $S_hO(X) = \{\emptyset, X, \{b\}, \{d\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, c, d\}\}$. Then $\{b\}, \{b, d\}, \{a, b, c\} \in S_hO(X)$ but $\{b\}, \{b, d\}, \{a, b, c\} \notin \tau^h$.

REMARK 2.12. Every semi-open set is S_h-open. The converse may not always be true.

EXAMPLE 2.13. $X = \{a, b, c, d\}$ be with the topology $\tau = \{\emptyset, X, \{d\}, \{a, c\}, \{a, c, d\}\}$ and $I = \{\emptyset, \{c\}\}$. From Definition 2.4, we obtain the collection $O(X) = \{\emptyset, X, \{d\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, c, d\}\}$. By Definition 2.9, we find the collection $S_hO(X) = \{\emptyset, X, \{b\}, \{d\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, c, d\}\}$ and hence {b} $\in S_hO(X)$ but {b} $\notin SO(X)$.

$$\begin{array}{ccccc} \text{open set} & \rightarrow & h\text{-open set} & \rightarrow & \text{semi } h\text{-open set} \\ & \searrow & & \nearrow & \\ & & \text{semi open set} & & \end{array}$$

DEFINITION 2.14 ([1]). Let (X, τ) be a topological space and let $A \subseteq X$. The h-closure of A is defined as the intersection of all h-closed sets in X containing A and is denoted by hCl(A). It is clear that hCl(A) is h-closed for any subset A of X.

3. *h*-LOCAL FUNCTIONS

In this section, we present definitions of the h-local and semi-h-local functions of a set. We examine the properties of the h-local function. In addition, we give the concept of ${\rm Cl}^*{}_h$ operator and obtain the $\tau^*{}_h$ topology in this way.

DEFINITION 3.1. Let (X, τ) be a topological space and $x \in X$ be given. Every h-open subset containing point x is called an h-open neighborhood of point x.

DEFINITION 3.2. Let (X, τ) be a topological space and x be a point in X. A subset V of X is called an h-neighborhood of a point x if there exists $U \in \tau^h$ such that $x \in U \subseteq V$.

REMARK 3.3. Every h-open neighborhood in a topological space (X, τ) is an h-neighborhood.

REMARK 3.4. The converse of Remark 3.3 may not always be true.

EXAMPLE 3.5. Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ then $\tau^h = hO(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. For $a \in X$, the family of h-open neighborhoods of a is $\{X, \{a\}, \{a, b\}\}$. For $a \in X$, the family of h neighborhoods of a is $\{X, \{a\}, \{a, b\}, \{a, c\}\}$.

Definition 3.6. Let (X, τ, I) be an ideal topological space and a subset A of X. Then $A^*{}_h(I, \tau) = \{x \in X \mid (A \cap U) \notin I$ for every $U \in hO(X, x)\}$ is called the h-local function of A with respect to I and τ, where $hO(X, x) = \{U \in hO(X) \mid x \in U\}$. $A^*{}_h(I, \tau)$ is simply denoted by $A^*{}_h$.

We give an example for the h-local function:

Example 3.7. Let $X = \{k, l, m\}$ be with a topology $\tau = \{\emptyset, X, \{k\}, \{k, l\}\}$ and $I = \{\emptyset, \{m\}\}$. Take $A = \{k, l\}$. Then $\tau^h = \{\emptyset, X, \{k\}, \{k, l\}, \{l\}, \{l, m\}\}$, $A^*{}_h = \{k, l, m\} = X$ and $A^* = \{k, l, m\} = X$.

Definition 3.8. Let (X, τ, I) be an ideal topological space and A be a subset of X. Then $A^{*s}{}_h(I, \tau) = \{x \in X \mid A \cap U \notin I$ for every $U \in S_hO(X, x)\}$ is called the semi-h-local function of A with respect to I and τ, where $S_hO(X, x) = \{U \in S_hO(X) \mid x \in U\}$. When there is no ambiguity we write $A^{*s}{}_h$ for $A^{*s}{}_h(I, \tau)$.

Theorem 3.9. *Let (X, τ, I) be an ideal topological space and $A, B \subseteq X$ any subsets. Then:*

(1) $A \subset B \Rightarrow A^*{}_h \subset B^*{}_h$,
(2) $(A \cup B)^*{}_h = A^*{}_h \cup B^*{}_h$,
(3) $(A \cap B)^*{}_h \subset A^*{}_h \cap B^*{}_h$,
(4) $(A^*{}_h)^*{}_h \subset A^*{}_h$,
(5) $A^*{}_h = \mathrm{hCl}(A^*{}_h) \subset \mathrm{hCl}(A)$ *and* $A^*{}_h$ *is h-closed.*

Proof. (1) Assume $x \notin B^*{}_h$. Then there exists $U \in hO(X, x)$ such that $(U \cap B) \in I$. Since $A \subset B$, $(U \cap A) \in I$ and hence $x \notin A^*{}_h$. Therefore, we have $A^*{}_h \subset B^*{}_h$.

(2) By 1), we have $A^*{}_h \cup B^*{}_h \subset (A \cup B)^*{}_h$. Next, suppose that $x \notin A^*{}_h \cup B^*{}_h$. Then $x \notin A^*{}_h$ and $x \notin B^*{}_h$. Therefore, there exist $U, V \in hO(X, x)$ such that $A \cap U \in I$ and $B \cap V \in I$. Hence, we obtain $(A \cup B) \cap (U \cap V)$=$(A \cap (U \cap V)) \cup (B \cap (U \cap V)) \subset (A \cap U) \cup (B \cap V) \in I$. Therefore, $(A \cup B) \cap (U \cap V) \in I$ and $U \cap V \in hO(X, x)$. This shows that $x \notin (A \cup B)^*{}_h$ and hence $(A \cup B)^*{}_h \subset A^*{}_h \cup B^*{}_h$.

(3) Since $A \cap B \subset A$ and from (1) $(A \cap B)^*{}_h \subset A^*{}_h$. Also $A \cap B \subset B$ hence $(A \cap B)^*{}_h \subset B^*{}_h$. Hence $(A \cap B)^*{}_h \subset A^*{}_h \cap B^*{}_h$.

(4) Suppose that $x \in (A^*{}_h)^*{}_h$. Then $A^*{}_h \cap U \notin I$ for any $U \in hO(X, x)$. Hence $A^*{}_h \cap U \neq \emptyset$ and there exists $y \in A^*{}_h \cap U$ and hence $y \in A^*{}_h$ and $y \in U$. Therefore, $U \cap A \notin I$. This shows that $x \in A^*{}_h$.

(5) From Definition 2.14, $A^*{}_h \subseteq \mathrm{hCl}(A^*{}_h)$. Let $x \in \mathrm{hCl}(A^*{}_h)$. Then $(A^*{}_h \cap U) \neq \emptyset$ for every $U \in hO(X, x)$. Let $y \in (A^*{}_h \cap U)$, then $y \in U$ and $y \in A^*{}_h$, and hence for every $U \in hO(X, x)$, $A \cap U \notin I$ and thus $x \in A^*{}_h$. This shows that $\mathrm{hCl}(A^*{}_h) \subset A^*{}_h$ and consequently $A^*{}_h = \mathrm{hCl}(A^*{}_h)$. Obviously, $A^*{}_h$ is h-closed. Now suppose $x \notin \mathrm{hCl}(A)$. Then, there exists $U \in hO(X, x)$ such that $U \cap A = \emptyset$ and hence $x \notin A^*{}_h$. Therefore, we have $A^*{}_h \subset \mathrm{hCl}(A)$. □

The converses of statements (3) and (4) from Theorem 3.9 may not always be true.

EXAMPLE 3.10. Let $X = \{a, b, c, d\}$ be with the topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, b, d\}\}$ and we consider $I = \{\emptyset, \{d\}\}$. From Definition 2.9, we find the collection of h-open sets $\tau^h = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, d\}, \{b, c, d\}, \{a, b, d\}\}$. If $A = \{a, b\} \subset X$ and $B = \{c\} \subset X$ then $(A \cap B) = \emptyset$ hence $(A \cap B)^*{}_h = (\emptyset)^*{}_h = \emptyset$. $A^*{}_h = \{a, b, c, d\} = X$ and $B^*{}_h = \{c\}$ therefore $A^*{}_h \cap B^*{}_h = \{c\}$. Here the results obvious.

EXAMPLE 3.11. Let $X = \{a, b, c, d\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, b, d\}\}$ and we consider $I = \{\emptyset, \{a\}\}$. By Definition 2.6, we find the collection $\tau^h = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, d\}, \{b, c, d\}, \{a, b, d\}\}$. If $A = \{a, b\}$ then $A^*{}_h = \{a, b, c, d\} = X$, and $(A^*{}_h)^*{}_h = \{b, c, d\}$ then $(A^*{}_h)^*{}_h \subset A^*{}_h$ but $A^*{}_h \nsubseteq (A^*{}_h)^*{}_h$.

REMARK 3.12. Let (X, τ, I) be an ideal topological space and $A \subset X$. Then the following properties hold:

(1) When $I = \{\emptyset\}$, $A^*{}_h = \mathrm{hCl}(A)$,
(2) $\{\emptyset\}^*{}_h = \emptyset$,
(3) It is not essential that $A \subset A^*{}_h$ or $A^*{}_h \subset A$,
(4) $A^*{}_h(I, hO(X)) = A^{*s}(I, \tau)$ if $SO(X) = hO(X)$,
(5) $A^*{}_h(I, hO(X)) = A^*(I, \tau)$ if $hO(X) = \tau(X)$.

Next we give an example of Remark 3.12 (3).

EXAMPLE 3.13. Let $X = \{a, b, c, d\}$ be a set with a topology $\tau = \{\emptyset, X, \{a\}, \{a, b, d\}\}$ and $I = \{\emptyset, \{c\}, \{a, c\}\}$. From Definition 2.6, we find the collection $\tau^h = \{\emptyset, X, \{a\}, \{b, d\}, \{b, c, d\}, \{a, b, d\}\}$. If $A = \{a, b\}$, $B = \{c\}$ then $A^*{}_h = X$ and $B^*{}_h = \emptyset$.

LEMMA 3.14. *Let (X, τ, I) be an ideal topological space. Then the following properties hold:*

(1) $RO(X) \subset \tau \subset \tau^h = hO(X) \subset S_hO(X)$,
(2) $A^{*s}{}_h \subset A^*{}_h \subset A^*$.

Proof. (1) The proof is obvious.

(2) Let $x \in A^{*s}{}_h$. Then $A \cap U \notin I$ for every $U \in S_h(O, x)$. Then $A \cap U \notin I$ for every $U \in hO(X, x)$. Therefore, $x \in A^*{}_h$ and hence $A^{*s}{}_h \subset A^*{}_h$. Let $x \in A^*{}_h$. Then $A \cap U \notin I$ for every $U \in hO(X, x)$. Then $A \cap U \notin I$ for every $U \in \tau(x)$. Therefore, $x \in A^*$ and hence $A^*{}_h \subset A^*$. $\square$

DEFINITION 3.15. Let (X, τ, I) be an ideal topological space. We define $\mathrm{Cl}^*{}_h(A)$ as follows: $\mathrm{Cl}^*{}_h(A) = A \cup A^*{}_h$ for every $A \subset X$.

THEOREM 3.16. *Let (X, τ, I) be an ideal topological space. For any subsets $A, B of X$, the following properties hold:*

(1) $A \subset \mathrm{Cl}^*{}_h(A)$.

(2) $\mathrm{Cl}^*{}_h(\emptyset) = \emptyset$ *and* $\mathrm{Cl}^*{}_h(X) = X$.
(3) *If* $A \subset B$, *then* $\mathrm{Cl}^*{}_h(A) \subset \mathrm{Cl}^*{}_h(B)$.
(4) $\mathrm{Cl}^*{}_h(A) \cup \mathrm{Cl}^*{}_h(B) = \mathrm{Cl}^*{}_h(A \cup B)$.
(5) $(\mathrm{Cl}^*{}_h(A))^*{}_h \subset \mathrm{Cl}^*{}_h(A) = \mathrm{Cl}^*{}_h(\mathrm{Cl}^*{}_h(A))$.

Proof. (1) It is obvious from Definition 3.15.

(2) $\mathrm{Cl}^*{}_h(\emptyset) = \emptyset \cup \{\emptyset\}^*{}_h = \emptyset$ and $\mathrm{Cl}^*{}_h(X) = X \cup X^*{}_h = X$.

(3) Let $A \subset B$. Then, by Theorem 3.9, $A^*{}_h \subset B^*{}_h$ and $\mathrm{Cl}^*{}_h(A) = A \cup A^*{}_h \subset B \cup B^*{}_h = \mathrm{Cl}^*{}_h(B)$.

(4) $\mathrm{Cl}^*{}_h(A \cup B) = (A \cup B) \cup (A \cup B)^*{}_h = (A \cup B) \cup A^*{}_h \cup B^*{}_h = (A \cup A^*{}_h) \cup (B \cup B^*{}_h) = \mathrm{Cl}^*{}_h(A) \cup \mathrm{Cl}^*{}_h(B)$.

(5) By Theorem 3.9, we have $(\mathrm{Cl}^*{}_h(A))^*{}_h = (A \cup A^*{}_h)^*{}_h = A^*{}_h \cup (A^*{}_h)^*{}_h = A^*{}_h \subset \mathrm{Cl}^*{}_h(A)$ and $\mathrm{Cl}^*{}_h(\mathrm{Cl}^*{}_h(A)) = \mathrm{Cl}^*{}_h(A) \cup (\mathrm{Cl}^*{}_h(A))^*{}_h = \mathrm{Cl}^*{}_h(A)$. □

The converse of statement (5) from Theorem 3.16 may not always be true.

Example 3.17. Let's consider the set B in Example 3.13, $B^*{}_h = \emptyset$ is found and so $(\mathrm{Cl}^*{}_h(B))^*{}_h = ((B \cup B^*{}_h)^*{}_h) = \emptyset$ and $\mathrm{Cl}^*{}_h(B) = B$. Therefore, $\mathrm{Cl}^*{}_h(\mathrm{Cl}^*{}_h(B)) = \mathrm{Cl}^*{}_h(B) = B$ and $\emptyset = (\mathrm{Cl}^*{}_h(B))^*{}_h \subset \mathrm{Cl}^*{}_h(B) = \mathrm{Cl}^*{}_h(\mathrm{Cl}^*{}_h(B)) = B$.

Theorem 3.18. *Let (X, τ, I) be an ideal topological space and A, B subsets of X. Then for h-local functions the following properties hold:*

(1) $A^*{}_h - B^*{}_h = (A - B)^*{}_h - B^*{}_h \subset (A - B)^*{}_h$.
(2) *If* $U \in \tau^h$, *then* $U \cap A^*{}_h = U \cap (U \cap A)^*{}_h \subset (U \cap A)^*{}_h$.
(3) *If* $U \in I$, *then* $(A - U)^*{}_h \subset A^*{}_h = (A \cup U)^*{}_h$.

Proof. (1) Since $A - B \subset A$, by Theorem 3.9, $(A - B)^*{}_h \subset (A)^*{}_h$ and so $(A - B)^*{}_h - B^*{}_h \subset A^*{}_h - B^*{}_h$. On the other hand, since $A \subset (A - B) \cup B$, by Theorem 3.9, $A^*{}_h \subset (A - B)^*{}_h \cup B^*{}_h$, and hence $A^*{}_h - B^*{}_h \subset ((A - B)^*{}_h \cup B^*{}_h) - B^*{}_h$. Therefore $A^*{}_h - B^*{}_h \subset (A - B)^*{}_h - B^*{}_h$ and $A^*{}_h - B^*{}_h = (A - B)^*{}_h - B^*{}_h$.

(2) Let $U \in hO(X)$ and $x \in U \cap A^*{}_h$. Then $x \in U$ and $x \in A^*{}_h$. For any $V \in hO(X, x)$, $U \cap V \in hO(X, x)$ by Theorem 2.8. Hence $V \cap (U \cap A) = (U \cap V) \cap A \notin I$. Therefore $x \in (U \cap A)^*{}_h$. So $U \cap A^*{}_h \subset (U \cap A)^*{}_h$. Then $U \cap A^*{}_h \subset U \cap (U \cap A)^*{}_h$, since $(U \cap A) \subset A$. Moreover $U \cap A^*{}_h \subset (U \cap A)^*{}_h \subset A^*{}_h$ and hence $U \cap A^*{}_h \subset U \cap (U \cap A)^*{}_h \subset U \cap A^*{}_h$. Therefore, we obtain $U \cap A^*{}_h = U \cap (U \cap A)^*{}_h$.

(3) By Theorem 3.9 and Definition 3.5, $(A \cup U)^*{}_h = U^*{}_h \cup A^*{}_h = \emptyset \cup A^*{}_h = A^*{}_h$. Since $A - U \subset A$, by Theorem 3.9, $(A - U)^*{}_h \subset A^*{}_h$. □

Theorem 3.19. *Let (X, τ) be a topological space, I_1 and I_2 be ideals on X and let A be a subset of X. Then the following properties hold:*

(1) *If* $I_1 \subset I_2$, *then* $A^*{}_h(I_2) \subset A^*{}_h(I_1)$.
(2) $A^*{}_h(I_1 \cap I_2) = A^*{}_h(I_1) \cup A^*{}_h(I_2)$.

Proof. (1) Suppose that $x \in A^*{}_h(I_2)$. Then for every $U \in hO(X,x)$, $U \cap A \notin I_2$. Hence $U \cap A \notin I_1$ and $x \in A^*{}_h(I_1)$. This shows that $A^*{}_h(I_2) \subset A^*{}_h(I_1)$.

(2) Since $(I_1 \cap I_2) \subset I_1$ and $(I_1 \cap I_2) \subset I_2$, by Theorem 3.19 (1), $A^*{}_h(I_1) \subset A^*{}_h(I_1 \cap I_2)$ and $A^*{}_h(I_2) \subset A^*{}_h(I_1 \cap I_2)$. Hence $A^*{}_h(I_1) \cup A^*{}_h(I_2) \subset A^*{}_h(I_1 \cap I_2)$. On the other hand, let $x \in A^*{}_h(I_1 \cap I_2)$.Then for each $U \in hO(X,x)$, $(U \cap A) \notin (I_1 \cap I_2)$. Hence $(U \cap A) \notin I_1$ or $(U \cap A) \notin I_2$. So, $x \in A^*{}_h(I_1)$ or $x \in A^*{}_h(I_2)$ and $x \in A^*{}_h(I_1) \cup A^*{}_h(I_2)$. Therefore, we obtain $A^*{}_h(I_1 \cap I_2) \subset A^*{}_h(I_1) \cup A^*{}_h(I_2)$ and hence $A^*{}_h(I_1 \cap I_2)$=$A^*{}_h(I1) \cup A^*{}_h(I_2)$. □

By Theorem 3.16., $\mathrm{Cl}^*{}_h(A) = A^*{}_h \cup A$ is a Kuratowski closure operator and we obtain the following theorem:

THEOREM 3.20. *Let (X,τ,I) be an ideal topological space.Put $\tau^*{}_h = \{U \subset X : \mathrm{Cl}^*{}_h(X \setminus U) = (X \setminus U)\}$. Then $\tau^*{}_h$ is a topology for X such that $\tau^* \subset \tau^*{}_h$ and $hO(X) \subset \tau^*{}_h$.*

Proof. $\tau^*{}_h$ is the topology for X generated by $\mathrm{Cl}^*{}_h(A)$. Now we show that $\tau^* \subset \tau^*{}_h$. By Lemma 3.14, $\mathrm{Cl}^*{}_h(A) = (A \cup A^*{}_h) \subset (A \cup A^*) = \mathrm{Cl}^*(A)$. Let A be a τ^*-closed set. Then $\mathrm{Cl}^*(A) = A$ and $\mathrm{Cl}^*{}_h(A) \subset A$. Hence $\mathrm{Cl}^*{}_h(A) = A$ and A is $\tau^*{}_h$-closed. Next, we show that $hO(X) \subset \tau^*{}_h$. Suppose that A is h-closed. If $x \notin A$, then there exists $G \in \tau^h$ containing x such that $A \cap G = \emptyset \in I$. Hence $x \notin A^*{}_h$ and $A^*{}_h \subset A$. Therefore, we have $\mathrm{Cl}^*{}_h(A) = A \cup A^*{}_h = A$ and A is a $\tau^*{}_h$-closed. Hence $hO(X) \subset \tau^*{}_h$. □

THEOREM 3.21. *Let (X,τ,I) be an ideal topological space and $\beta^*(\tau^h, I) = \{U - J : U \in \tau^h, J \in I\}$. Then $\beta^*(\tau^h, I)$ is a basis for $\tau^*{}_h$.*

Proof. Let $U \in \tau^*{}_h$ and $x \in U$.Then $\mathrm{Cl}^*{}_h(X-U) = (X-U)^*{}_h \cup (X-U) = X - U$ and $(X-U)^*{}_h \subset X - U$. Hence $U \subset X - (X-U)^*{}_h$. Since $x \in U$, $x \notin (X-U)^*{}_h$ and there exists $V \in hO(X,x)$ such that $V \cap (X-U) \in I$. Let $V \cap (X-U) = I_0$, then $V - I_0 = V \cap U$ and $x \in V \cap U$. Put $\beta^*{}_0 = V - I_0$, then $x \in \beta^*{}_0 \subset U$ and $\beta^*{}_0 \in \beta^*(\tau^h, I)$. Further, let $\beta^*{}_1, \beta^*{}_2 \in \beta^*$, then we have $\beta^*{}_1 = U_1 - J_1$ and $\beta^*{}_2 = U_2 - J_2$, where $U_1, U_2 \in \tau^h$ and $J_1, J_2 \in I$. Then, we have $\beta^*{}_1 \cap \beta^*{}_2 = (U_1 - J_1) \cap (U_2 - J_2) = (U_1 \cap (X - J_1)) \cap (U_2 \cap (X - J_2)) = (U_1 \cap U_2) - (J_1 \cup J_2) \in \beta^*$, where $(U_1 \cap U_2) \in \tau^h, (J_1 \cup J_2) \in I$. Therefore, $\beta^*{}_1, \beta^*{}_2 \in \beta^*$ and hence $\beta^*(\tau^h, I)$ is a basis for $\tau^*{}_h$. □

4. I_{s^*g}-h-CLOSED SETS

In this section, we introduce the definition of I_{s^*g}-h-closed set. We examine the properties of this set. Also, we compare it with the closed set.

DEFINITION 4.1. Let (X,τ,I) be an ideal topological space. A subset A of X is said to be I_g-h-closed if $A^*{}_h \subset U$ whenever $A \subset U$ and U in $hO(X)$.

THEOREM 4.2. *Let (X,τ,I) be an ideal topological space. For a subset A of X, the following proporties are equivalent:*

(1) *A is I_g-h-closed,*

(2) $\mathrm{Cl}^*{}_h(A) \subset U$ *whenever* $A \subset U$ *and* U *is* h*-open,*
(3) *For each* $x \in \mathrm{Cl}^*{}_h(A)$*,* $\mathrm{hCl}(\{x\}) \cap A \neq \emptyset$*,*
(4) $\mathrm{Cl}^*{}_h(A) - A$ *contains no nonempty* h*-closed set,*
(5) $A^* - A$ *contains no nonempty* h*-closed set.*

Proof. (1) $\Rightarrow$ (2) Suppose that A is I_g-h-closed.By Definition 4.1., $A^*{}_h \subset U$ whenever $A \subset U$ and U in hO(X).Therefore, $A^*{}_h \cup A \subset U$.From here $\mathrm{Cl}^*{}_h(A) \subset U$ whenever $A \subset U$ and U is h-open.

(2) $\Rightarrow$ (3) Suppose that $\mathrm{hCl}(\{x\}) \cap A = \emptyset$ for some $x \in \mathrm{Cl}^*{}_h(A)$. Then $A \subset X - \mathrm{hCl}(\{x\})$, where $X - \mathrm{hCl}(\{x\})$ is h-open and by 2) $\mathrm{Cl}^*{}_h(A) \subset X - \mathrm{hCl}(\{x\})$.Therefore, $\mathrm{Cl}^*{}_h(A) \cap \mathrm{hCl}(\{x\}) = \emptyset$.This is contrary to $x \in \mathrm{Cl}^*{}_h(A)$.

(3) $\Rightarrow$ (4) Suppose that $K \subset \mathrm{Cl}^*{}_h(A) - A$, where K is a nonempty h-closed set and $x \in K$. Then $K \subset X - A$ and $K \cap A = \emptyset$. Therefore, we have $Clh(\{x\}) \cap A \subset K \cap A = \emptyset$. Since $x \in \mathrm{Cl}^*{}_h(A)$, this is contrary to our hypothesis.Therefore, $\mathrm{Cl}^*{}_h(A) - A$ contains no noempty h-closed set.

(4) $\Rightarrow$ (5) The proof is clear from $A^*{}_h \subset \mathrm{Cl}^*{}_h(A)$.

(5) $\Rightarrow$ (1) Let $A \subset U$ and U be any h-open set of X. By Theorem 3.9, $A^*{}_h$ is h-closed and $A^*{}_h \cap (X - U) \subset A^*{}_h - A$, where $A^*{}_h \cap (X - U)$ is h-closed. By 5) $A^*{}_h \cap (X - U) = \emptyset$. Therefore $A^*{}_h \subset U$ and hence A is I_g-h-closed. □

Definition 4.3. Let (X, τ, I) be an ideal topological space.A subset A of X is said to be I_{s^*g}-h-closed (resp. I_{s^*g}-closed [6]) if $A^*{}_h \subset U$ (resp. $A^* \subset U$) whenever $A \subset U$ and U is semi-open.The complement of an I_{s^*g}-h-closed set is said to be I_{s^*g}-h-open.The family of I_{s^*g}-h-closed (resp. I_{s^*g}-closed) sets is denoted by I_{s^*g}-$hC(X)$ (resp. I_{s^*g}-$C(X)$).

Theorem 4.4. *Let* (X, τ, I) *be an ideal topological space and* A *a subset of* X*. If* A *is* I_{s^*g}*-closed, then* A *is* I_{s^*g}*-*h*-closed. However, the converse may not always be true.*

Proof. Let A be an I_{s^*g}-closed set. For each $U \in SO(X)$ containing A, $A^* \subset U$ and by Lemma 3.14 (2), $A^*{}_h \subset A^* \subset U$. Hence A is I_{s^*g}-h-closed. □

Example 4.5. Let $X = \{k, l, m, n\}$ be a set with the topology $\tau = \{\emptyset, X, \{m\}, \{k, l, m\}\}$. Then by Definition 2.5, $hO(X) = \tau^h = \{\emptyset, X, \{m\}, \{k, l\}, \{k, l, m\}, \{k, l, n\}\}$.Considering $I = \{\emptyset, \{n\}\}$ and by Definition 4.3., we find the collection of $I_{s^*g}C(X) = \{\emptyset, X, \{m\}, \{n\}, \{m, n\}, \{k, l, n\}\}$ and $I_{s^*g}C(X) = \{\emptyset, X, \{n\}, \{k, l, n\}\}$. It can be verified at the subsets $\{m\}, \{m, n\}$ of X are I_{s^*g}-$hC(X)$ but not I_{s^*g}-$C(X)$.

Example 4.6. The $\emptyset, X, \{k, l, n\}, \{n\}$ sets given in Example 4.5. are closed and I_{s^*g}-h-closed. But $\{m\}, \{m, n\}$ sets are not closed when I_{s^*g}-h-closed.

Theorem 4.7. *Let* (X, τ, I) *be an ideal topological space and* $A, B \subseteq X$*.*

(1) *If* A *and* B *are* I_{s^*g}*-*h*-closed, then* $(A \cup B)$ *is* I_{s^*g}*-*h*-closed.*
(2) *If* A *is closed in* X*, then* A *is* I_{s^*g}*-*h*-closed.*
(3) *If* U *is open in* X *and* A *is* I_{s^*g}*-*h*-open, then* $(U \cap A)$ *is* I_{s^*g}*-*h*-open.*

Proof. (1) Let $A \cup B \subset U$ and $U \in SO(X)$.Then, $A \subset U$ and $B \subset U$. Since A and B are I_{s^*g}-h-closed, hence $A^*{}_h \subset U$ and $B^*{}_h \subset U$.Hence $(A \cup B)^*{}_h = A^*{}_h \cup B^*{}_h \subset U$. Therefore we have $(A \cup B)$ is I_{s^*g}-h-closed.

(2) Put $A \subset U$ and $U \in SO(X)$. By Lemma 3.14., $A^*{}_h \subset A^* \subset \mathrm{Cl}(A) = A \subset U$. Hence A is I_{s^*g}-h-closed.

(3) The proof follows from the complement of (1) and (2). □

DEFINITION 4.8. Let (X, τ, I) be an ideal topological space and A be a subset of X. Then we define $(\tau^*)^h$ as follows: $(\tau^*)^h = \{A : A \subset \mathrm{Int}^*(A \cup U)$, for every $U \in \tau^*\}$.

QUESTION 4.9. Let (X, τ, I) be an ideal topological space. Find the relationship between families $(\tau^*)^h$ and $\tau^*{}_h$ and give the necessary counter example.

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Received October 21, 2021
Accepted March 26, 2022

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