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COUNTING FORMULAS FOR CERTAIN p-SUBGROUPS OF $GL_n(\mathbb{F}_p)$

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Abstract. Let p be a prime number and $\mathbb{F}_p$ a finite field of order p. Let $GL_n(\mathbb{F}_p)$ denote the general linear group and let U_n denote the unitriangular group of $n\times n$ upper triangular matrices with ones on the diagonal, over the finite field $\mathbb{F}_p$. This is a finite group of order $p^{\frac{n(n-1)}{2}}$ and a Sylow p-subgroup of $GL_n(\mathbb{F}_p)$. In this work, we characterize some p-subgroups of $GL_n(\mathbb{F}_p)$ with respect to a given property. By the Sylow theorems, every p-subgroup of $GL_n(\mathbb{F}_p)$ is contained in some Sylow p-subgroup of $GL_n(\mathbb{F}_p)$ and then it is conjugate to a p-subgroup of U_n, which is why we characterize the p-subgroups of U_n. More precisely, we compute the number of T-invariant p-subgroups of U_n, where T is the diagonal subgroup of $GL_n(\mathbb{F}_p)$. Furthermore, for $n \leq p$, we obtain an interesting formula which computes the number of abelian p-subgroups of order p^t in U_n where $t \leq \left[\frac{n^2}{4}\right]$.

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1. INTRODUCTION

Let p be a prime number, and let $\mathbb{F}_p$ be a finite field of order p. Let $GL_n(\mathbb{F}_p)$ denote the general linear group and let U_n denote the unitriangular group of $n \times n$ upper triangular matrices with ones on the diagonal, over the finite field $\mathbb{F}_p$. In the study of the general linear group $GL_n(\mathbb{F}_p)$, many problems reduce to a characterization of subgroups of its Sylow subgroup U_n. This is a fairly old problem in the theory of finite groups. Notably, Goozeff proves that the maximal order of an abelian p-subgroup of U_n is $p^{\left[\frac{n^2}{4}\right]}$ where $\left[\frac{n^2}{4}\right]$ is the integer part of $\frac{n^2}{4}$ [9]. After two years, Thwaites shows that U_n contains precisely one maximal abelian subgroup of order $p^{\frac{n^2}{4}}$, if n is even, and contains

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precisely two maximal abelian subgroups of order $p^{\frac{n^2-1}{4}}$, if n is odd and $n \geq 5$ [17].

Let $k(U_n)$ denote the number of conjugacy classes of U_n. Bounding $k(U_n)$ is a fundamental problem in group and representation theory. Recently, there are many works about the character theory of U_n and related topics (see e.g. [2, 7, 8, 13]), which are partly motivated by Higman's conjecture that for every n, the number of conjugacy classes of U_n is a polynomial in p with integer coefficients [11]. The primary interest of Higman was not in this conjecture, but rather determining the function that enumerates the number of isomorphism classes of groups of order p^n. Higman originally checked that the conjecture holds for $n \leq 5$ [11]. Gudivok et al. proved later that this conjecture was valid for $n \leq 8$ [10]. Vera-López and Arregi explain a general method to find the conjugacy classes of U_n in [18, 19] and verified Higman's conjecture for $n \leq 13$ in [20]. Pak and Soffer used an indirect enumeration technique to verify Higman's Conjecture for $n \leq 16$ [14, Theorem 1.2]. There are people that believe Higman's conjecture is false based on certain evidence. So, Pak and Soffer conjectured that Higman's conjecture fails for $n \geq 59$ [14, Conjecture 1.6]. But in general, this problem is still open despite of different efforts to solve it.

In this work, we characterize some p-subgroups of U_n with respect to a given property. More precisely, we describe a recursive counting formula for T-invariant p-subgroups of U_n based on the factorization of subgroups of U_n as a semidirect product (see Theorem 2.3). Furthermore, we investigate the number of commuting m-tuples $c_m(G)$ for a finite group G and prove that $c_m(G)$ is divisible by $|G|$ (see Proposition 3.1). The quotient $\frac{c_m(G)}{|G|}$ is then described recursively in terms of numbers of conjugacy classes in iterated centralizers of G (see Corollary 3.3). Finally, we are interested in Higman's Conjecture, and as an application we give a recursive counting formula for abelian subgroups of order p^t in U_n based on the number of commuting t-tuples $c_t(U_n)$ where $n \leq p$ and $t \leq \left[\frac{n^2}{4}\right]$ (see Theorem 4.2).

2. T-INVARIANT p-SUBGROUPS OF U_n

In the matrix ring $M_{n\times n}(\mathbb{F}_p)$, the element $E_{i,j}$ will denote the element which is one in cell (i, j) and zero everywhere else. For each $1 \leq i < j \leq n$, we will let $E(a_{i,j}) = a_{i,j}E_{i,j}$, where $a_{i,j} \in \mathbb{F}_p^{\times}$. Let $D = (d_{ii})_{1\leq i\leq n} \in T$. Recall that a diagonal automorphism φ_D of U_n is an automorphism defined by $\varphi_D(M) = DMD^{-1}$. Thus, if $M = I_n + \sum_{i<j} E(a_{i,j}) \in U_n$, then $\varphi_D(M) = I_n + \sum_{i<j} E(b_{i,j})$ where $b_{i,j} = d_{i,i}a_{i,j}d_{j,j}^{-1}$.

Definition 2.1. A p-subgroup of U_n is called T-invariant if it is invariant under the diagonal automorphisms of U_n.

Let V_{n-1} be the subgroup $\left\{ \begin{pmatrix} I_{n-1} & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{F}_p^{n-1} \right\}$. The group V_{n-1} is elementary abelian of order p^{n-1} and is normal in U_n. Furthermore, we have the following interesting lemma:

Lemma 2.2. *The Sylow p-subgroup U_n is the semidirect product of V_{n-1} by U_{n-1}.*

Proof. Indeed, let π the map described in block matrices as

$$\pi : \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

where $A \in U_{n-1}$ and $b \in \mathbb{F}_p^{n-1}$. Obviously, π is an idempotent endomorphism of U_n and then $U_n = \mathrm{Ker}(\pi) \rtimes \mathrm{Im}(\pi)$. Since

$$\mathrm{Ker}(\pi) = \left\{ \begin{pmatrix} I_{n-1} & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{F}_p^{n-1} \right\}$$

and

$$\mathrm{Im}(\pi) = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \middle| A \in U_{n-1} \right\} \cong U_{n-1},$$

it follows that $U_n \cong V_{n-1} \rtimes U_{n-1}$. □

Suppose U is a T-invariant p-subgroup of U_n, the subgroups $U \cap V_{n-1}$ and $U \cap U_{n-1}$ are T-invariants p-subgroups of U_n. Recall that the number of subgroups of order p^m in an elementary p-group of order p^n is given by the Gaussian coefficients $\left[{n \atop m}\right]_p = \prod_{k=0}^{m-1} \frac{p^{n-k}-1}{p^{m-k}-1}$. Hence, we have the following interesting result:

Theorem 2.3. *Suppose that p is an odd prime number and let n and m be two positive integers, where $m < n$. The number of T-invariant p-subgroups of order p^m in U_n is equal to:*

$$T_m(U_n) = \sum_{k=0}^{m} \left[{n-1 \atop k}\right]_p T_{m-k}(U_{n-1}).$$

Proof. Let U be a T-invariant p-subgroup of U_n. The p-subgroup $U \cap V_{n-1}$ is normal in U and $U \cap V_{n-1} \cap U \cap U_{n-1} = I_{n-1}$, by Lemma 2.2. Obviously, we have $(U \cap V_{n-1}) \rtimes (U \cap U_{n-1}) \leq U$. The only inclusion to prove is $U \leq (U \cap V_{n-1}) \rtimes (U \cap U_{n-1})$. Thus, pick any $M = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \in U$ where $A \in U_{n-1}$ and $b \in \mathbb{F}_p^{n-1}$. The matrix M is written in the form $M = XY$ where $X = \begin{pmatrix} I_{n-1} & b \\ 0 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Let $D = (d_{ii})_{1 \leq i \leq n} \in T$, where $d_{ii} = 1$ if $i = n$ and $d_{ii} = 2$ else. Since U is a T-invariant p-subgroup, it follows that $\varphi_D(M) = XM \in U$, so $X \in U \cap V_{n-1}$ and $Y \in U \cap U_{n-1}$ which deduces the reverse inclusion. Thus, we get $U = V_0 \rtimes U_0$ for some T-invariant p-subgroups $V_0 \leq V_{n-1}$ and $U_0 \leq U_{n-1}$ which concludes the result. □

3. ON THE NUMBER OF COMMUTING m-TUPLES

Let G be a finite group. By [6, Theorem 2.1], the number of commuting pairs of elements in G is equal to the product $k(G)|G|$ where $k(G)$ is the number of conjugacy classes of G. For a positive integer m, let $c_m(G)$ denote the number of commuting m-tuples of elements of G. For $m = 2$, we have $c_2(G) = |G|k(G) = \sum_{x \in G} |C_G(x)|$. For $m = 3$, if we fix the first component x of the triple (x, y, z), the only such pairs fixed by x are the commuting pairs with components which lie in $C_G(x)$. Hence x fixes $k(C_G(x))|C_G(x)|$ commuting pairs and it follows that $c_3(G) = \sum_{x \in G} k(C_G(x))|C_G(x)| = \sum_{x \in G} c_2(C_G(x))$. In general, we have

$$\begin{aligned} & c_{m+1}(G) \\ & = \left|\left\{(x_1, \ldots, x_{m+1}) \in G^{m+1} \mid \forall (i,j) \in \{1, \ldots, m+1\}^2,\ x_i x_j = x_j x_i\right\}\right| \\ & = \sum_{x \in G} \left|\left\{(x_1, \ldots, x_m) \in C_G(x)^m \mid \forall (i,j) \in \{1, \ldots, m\}^2,\ x_i x_j = x_j x_i\right\}\right| \\ & = \sum_{x \in G} c_m(C_G(x)). \end{aligned} \tag{1}$$

Proposition 3.1. *Let G be a finite group and m a positive integer. Then $c_m(G)$ is divisible by $|G|$.*

Proof. If the group G is abelian, then $c_m(G) = |G|^m$, so it is divisible by a very high power of $|G|$. Otherwise, we use induction on m. Assume that the proposition has been proved for $c_{m-1}(G)$. There is nothing to do if $m = 1$. For $m = 2$, we have $c_2(G) = |G|k(G)$. By induction, there exists an integer t_x such that $c_{m-1}(C_G(x)) = t_x|C_G(x)|$. Let $\{x_i : 1 \leq i \leq k(G)\}$ be a system of representatives for the conjugacy classes of G, then by using formula (1), we obtain

$$\begin{aligned} c_m(G) &= \sum_{x \in G} c_{m-1}(C_G(x)) \\ &= \sum_{i=1}^{k(G)} |G : C_G(x_i)| c_{m-1}(C_G(x_i)) \\ &= \sum_{i=1}^{k(G)} t_{x_i} |G : C_G(x_i)||C_G(x_i)| = \sum_{i=1}^{k(G)} t_{x_i} |G| \end{aligned}$$

from which, it follows that $c_m(G) = |G| \sum_{i=1}^{k(G)} t_{x_i}$, which is divisible by $|G|$. □

In view of the preceding proposition, the quotient $\frac{c_{m+1}(G)}{|G|}$ is an integer which we denote by $h_m(G)$. The group $\mathrm{Hom}(\mathbb{Z}^m, G)$ can be identified with the set of ordered m-tuples of commuting elements in G [1]. By using Burnside's lemma, we get that the number of conjugacy classes of homomorphisms $\mathbb{Z}^m \to G$ is equal to $h_m(G)$. Furthermore, we have the following proposition.

PROPOSITION 3.2. *Let $\{x_i : 1 \leq i \leq k(G)\}$ be a system of representatives for the conjugacy classes of G. Then*

$$h_m(G) = \sum_{i=1}^{k(G)} h_{m-1}(C_G(x_i))$$

Proof. Indeed, by using formula (1), we get

$$\begin{aligned} c_{m+1}(G) &= \sum_{x \in G} c_m(C_G(x)) \\ &= \sum_{i=1}^{k(G)} |G : C_G(x_i)| c_m(C_G(x_i)) \\ &= |G| \sum_{i=1}^{k(G)} \frac{c_m(C_G(x_i))}{|C_G(x_i)|}. \end{aligned}$$

Hence, the proposition follows. □

COROLLARY 3.3. *Let G be a finite group and m a positive integer. Set $G_1 = G$ and $G_{s+1} = C_{G_s}(x_{s,i_s})$ for all $1 \leq s \leq m-1$, such that the set $\{x_{s,i_s} : 1 \leq i_s \leq k(G_s)\}$ is a system of representatives for the conjugacy classes of G_s. Then, we have*

$$h_m(G) = \sum_{i_1=1}^{k(G_1)} \sum_{i_2=1}^{k(G_2)} \sum_{i_3=1}^{k(G_3)} \cdots \sum_{i_{m-1}=1}^{k(G_{m-1})} k(G_m)$$

Proof. We proceed by induction on m. If $m = 1$, then $h_1(G) = \frac{c_2(G)}{|G|} = k(G_1)$. Now let $m > 1$ and assume that the corollary has been proved for $h_{m-1}(G)$. By Proposition 3.2, we have

$$h_m(G) = \sum_{i_1=1}^{k(G_1)} h_{m-1}(C_{G_1}(x_{1,i_1}))$$

and then, by induction, we get

$$\begin{aligned} h_m(G) &= \sum_{i_1=1}^{k(G_1)} h_{m-1}(G_2) \\ &= \sum_{i_1=1}^{k(G_1)} \sum_{i_2=1}^{k((G_2)_1)} \sum_{i_3=1}^{k((G_2)_2)} \cdots \sum_{i_{m-1}=1}^{k((G_2)_{m-2})} k((G_2)_{m-1}). \end{aligned}$$

Therefore, the required formula follows since $(G_2)_s = G_{s+1}$ for all $1 \leq s \leq m-1$. □

4. COUNTING THE NUMBER OF ABELIAN p-SUBGROUPS OF U_n FOR $n \leq p$

It can be easily seen that the exponent of U_n is the least power of p greater than or equal to n. However, if $n \leq p$, by [12, Satz 16.5], all non-trivial elements of U_n have order p. In this case, U_n contains $N_p(U_n) = \frac{|U_n|-1}{p-1}$ groups of order p. Furthermore, each abelian subgroup of U_n is elementary abelian. But this is not the case for $n > p$, since U_{p+1} has a cyclic subgroup of order p^2. Let E_{p^t} denote the elementary abelian group of rank t.

As $|Z(U_n)| = p$, $Z(U_n)$ is the only minimal normal subgroup of U_n. Furthermore, we have the following interesting result.

PROPOSITION 4.1. *Suppose that $n \leq p$, and let M be a maximal p-subgroup of U_n. Then $Z(M) = E_{p^t}$ where $t \leq p-1$.*

Proof. If $|Z(M)| = p$, the result holds. Now, assume that $|Z(M)| > p$. Let $X \in U_n - M$ and $U = \langle X, Z(M)\rangle$. Then $Z(U) = Z(U_n)$ and $|Z(U)| = p$. Since $|U : Z(M)| = p$, by [4, Lemma 135.4], U is of maximal class. As in the proof of the second part of [5, Proposition 3.2], we get $|U| \leq p^p$ and the result follows, since $|U| = p\,|Z(M)|$. □

Define the subgroup U_m by the formula: $U_m = \begin{pmatrix} I_m & A \\ 0 & I_{n-m} \end{pmatrix}$ where I_m and I_{n-m} are the identity matrices of size m and $n-m$, respectively, and A ranges over all $m \times (n-m)$ matrices. The subgroup U_m is a maximal abelian normal subgroup of U_n [16, Exercise 3, p. 94]. In fact, the subgroup U_m is an elementary abelian subgroup of U_n of rank $m \times (n-m)$. Let $A_{i,j} = I_n + E_{i,j}$. If there exist $i_1, \ldots, i_m, j_1, \ldots, j_m$ such that the A_{i_k,j_k} pairwise commute, then we have $E_{p^m} \cong \langle A_{i_1,j_1}, \ldots, A_{i_m,j_m}\rangle \subset U_n$. The largest value of m for which such subgroups exist is $m = \left[\frac{n^2}{4}\right]$ [9]. Let $N_{p^t}(U_n)$ denote the number of abelian p-subgroups of order p^t in U_n where $t \leq \left[\frac{n^2}{4}\right]$.

THEOREM 4.2. *Let p be a prime number and n an integer such that $n \leq p$. Then*

$$N_{p^t}(U_n) = \frac{c_t(U_n) - 1 - \sum_{k=1}^{t-1} N_{p^k}(U_n) \prod_{s=0}^{k-1} (p^t - p^s)}{\prod_{k=0}^{t-1} (p^t - p^k)}$$

where $c_t(U_n)$ is the number of commuting t-tuples in U_n.

Proof. Indeed, such t-tuples of U_n must generate a p-subgroup of order p^k, where $0 \leq k \leq t$. As $n \leq p$, every element of U_n has order p and each abelian p-subgroup of order p^k is of rank k and has $\prod_{s=0}^{k-1} (p^t - p^s)$ generating t-tuples. So the number of commuting t-tuples generating abelian p-groups of order p^k

in U_n is $N_{p^k}(U_n)\prod_{s=0}^{k-1}(p^t-p^s)$. Thus, we get

$$c_t(U_n)=1+\sum_{k=1}^{t}N_{p^k}(U_n)\prod_{s=0}^{k-1}(p^t-p^s)$$

and it follows that the number of commuting t-tuples generating abelian p-groups of order p^t in U_n is equal to

$$N_{p^t}(U_n)\prod_{k=0}^{t-1}(p^t-p^k)=c_t(U_n)-1-\sum_{k=1}^{t-1}N_{p^k}(U_n)\prod_{s=0}^{k-1}(p^t-p^s)$$

and then we get the required result. □

Corollary 4.3. *Keep the assumptions of the previous theorem. The number of abelian p-subgroups of order p^2 in U_n is equal to:*

$$N_{p^2}(U_n)=\frac{p+|U_n|\,(k(U_n)-p-1)}{(p^2-1)(p^2-p)}$$

Proof. By [6, Theorem 2.1], the number of commuting pairs of elements in U_n is equal to the product $k(U_n)|U_n|$. Therefore, we conclude the corollary directly from Theorem 4.2, by taking $t=2$. □

Example 4.4. For $n\le 5$, the number of conjugacy classes of U_n has been calculated in [18]. Therefore, by Corollary 4.3, we obtain

$N_{p^2}(U_3)=p+1,$

$N_{p^2}(U_4)=2p^5+3p^4+2p^3+2p^2+p+1,$

$N_{p^2}(U_5)=5p^{10}+5p^9+5p^8+4p^7+4p^6+3p^5+3p^4+2p^3+2p^2+p+1.$

By a similar calculation, we get $N_{p^2}(U_n)$ for $n\ge 6$ whenever $k(U_n)$ is calculated. For $n>3$, we find that $N_{p^2}(U_n)$ is congruent to $1+p+2p^2$ modulo p^3 and this is in agreement with the main result given in [3].

In view of the above, it is natural to ask: How many elementary abelian p-subgroups of rank m are there in $GL_n(\mathbb{F}_p)$? In fact, this question led to the characterization of conjugacy-classes in $GL_n(\mathbb{F}_p)$ of elementary abelian p-subgroups of rank m. However, this is another open problem even for $n\le p$. In the following proposition we consider the case when $m=2$ and $n=3$.

Proposition 4.5. *Suppose that p is an odd prime number. The group $GL_3(\mathbb{F}_p)$ contains $(p^2+p+1)(p^2+1)$ elementary abelian p-subgroup of rank 2.*

Proof. Indeed, an elementary abelian p-subgroup of rank 2 in $GL_3(\mathbb{F}_p)$ is conjugate to exactly one of the p-groups $H_1=\langle I+aE_{12}+aE_{23},I+bE_{13}\rangle$, $H_2=\langle I+aE_{12},I+bE_{13}\rangle$, and $H_3=\langle I+aE_{23},I+bE_{13}\rangle$. By a simple calculation, we get $\left|N_{GL_3(\mathbb{F}_p)}(H_1)\right|=p^3(p-1)^2$ and then, by the Orbit-Stabilizer Theorem, it follows that there are $(p^3-1)(p+1)$ elementary abelian p-subgroup of rank 2 conjugate to H_1. Similarly, the p-groups H_2 and H_3 are both conjugate to (p^2+p+1) elementary abelian p-subgroup of rank 2. In total, we get $(p^2+p+1)(p^2+1)$ elementary abelian p-subgroup of rank 2 in $GL_3(\mathbb{F}_p)$, as required. □

REMARK 4.6. For $n > 3$, it is useful to consider the Quillen complex of elementary abelian subgroups, that is, the complex associated to the poset of elementary abelian subgroups of $GL_n(\mathbb{F}_p)$ ordered by inclusion. The poset of elementary abelian groups of rank at least 2 is homotopy equivalent to the standard Grassmanian complex [15]. However, this way uses hard mathematics to answer easier questions than the one we asked above. So what we can certainly propose now is to use the following GAP function:

```
NumberOfElementaryAbelianpSubgroupsOfRank2InGLnp:=function(n,p)
local G, S, cclS, cclG;
G := Image(IsomorphismPermGroup(GL(n,p)));
S := SylowSubgroup(G,p);
cclS := Filtered(ConjugacyClassesSubgroups(S),
cl->Size(Representative(cl))=p^2
and not IsCyclic(Representative(cl)));
cclG := List(EquivalenceClasses(cclS,
function(cl1,cl2)
return IsConjugate(G,Representative(cl1),
Representative(cl2));
end), Representative);
cclG := List(cclG,cl->Representative(cl)^G);
return Sum(List(cclG,Size));
end;
```

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