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FINITE-TIME BLOWUP AND EXISTENCE OF GLOBAL SOLUTIONS FOR A LOGARITHMIC SEMILINEAR HYPERBOLIC EQUATION

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Abstract. In this paper we consider the semilinear wave equation with the product of logarithmic and polynomial nonlinearities and establish the global existence and finite-time blowup of solutions by using the potential well method.

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Key words. Global existence, finite-time blowup, logarithmic and polynomial nonlinearity, potential well.

1. INTRODUCTION

In this paper, we consider the following initial boundary value problem for a semilinear wave equation with logarithmic nonlinearity

$$\left.\begin{array}{ll} \frac{\partial^2 w}{\partial t^2} - \Delta w = |w|^{\mathtt{p}-2} w \log |w|, & x \in \Omega,\, t > 0, \\ w(x,0) = w_0(x),\, w_t(x,0) = w_1(x), & x \in \Omega, \\ w(x,t) = 0, & x \in \partial\Omega,\, t > 0, \end{array}\right\} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and $w_0(x), w_1(x)$ are given initial data. The parameter $\mathtt{p}$ satisfies

$$2 < \mathtt{p} < 2^* = \begin{cases} +\infty, & n \le 2, \\ \dfrac{2n}{n-2}, & n \ge 3. \end{cases}$$

One of the most important nonlinear evolution equations in the field of mathematical physics and engineering are the semilinear hyperbolic equations and there are various applications in many branches of physics such as nuclear physics, optics and geophysics [1, 4, 10].

Based on the mountain pass theorem and the Nehari manifold, Sattinger [13] firstly studied problem (1) with nonlinear source $|w|^{\mathtt{p}-2}w$ by introducing the potential well method. Using the same method, Payne and Sattinger [11]

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extended the results to the following semilinear hyperbolic equation

$$\frac{\partial^2 w}{\partial t^2} - \Delta w = f(w) \tag{2}$$

with a general source $f(w)$. They studied a series of properties of energy functional and invariant sets, and also proved the finite-time blowup of solutions. Under the same assumptions on $f(w)$ as in [11], Liu and Zhao [9] introduced a family of potential wells and obtained the global existence and blow up of solutions for the initial boundary value problem of (2) with sub-critical initial energy, i.e. $\mathtt{E}(0) < \mathtt{d}$. They also proved the global existence of solutions with critical initial energy $\mathtt{E}(0) = \mathtt{d}$. After that, Liu and Xu [8] extended the results to the initial boundary value problem of (2) with combined nonlinear source terms of different sign, which can not be included by the assumptions of $f(w)$ in [11]. They obtained the global and blow-up solutions with sub-critical initial energy and proved the global existence of solutions with critical initial energy. Subsequently, Xu [14] proved the blow up of solutions for the initial boundary value problem of (2) with critical initial energy and gave the sharp condition for global existence of solutions. Recently, Lian et al. [6] studied the existence of global and blowup of solutions for the following semilinear wave equation

$$\begin{array}{ll} \frac{\partial^2 w}{\partial t^2} - \Delta w = w \log |w|^k, & x \in \Omega,\, t > 0,\, k > 1, \\ w(x,0) = w_0(x),\, w_t(x,0) = w_1(x), & x \in \Omega, \\ w(x,t) = 0, & x \in \partial\Omega,\, t > 0, \end{array}$$

by the potential well method. In [12], Peng and Zhou studied the initial boundary value problem of a semilinear heat equation with logarithmic nonlinearity

$$\begin{array}{ll} \frac{\partial w}{\partial t} - \Delta w = |w|^{\mathtt{p}-2} w \log |w|, & x \in \Omega,\, t > 0, \\ w(x,0) = w_0(x), & x \in \Omega, \\ w(x,t) = 0, & x \in \partial\Omega,\, t > 0. \end{array}$$

By using the potential well method, they obtained the existence of global solutions and finite-time blowup solutions. In [2], Choucha et al. studied a blowup result of solutions to the following nonlinear viscoelastic wave equation with distributed delays,

$$\begin{array}{l} \frac{\partial^2 w}{\partial t^2} - \Delta w - \omega \Delta w_t + \int_0^t \mathtt{g}(t-s) \Delta w(s) \mathrm{d}s + \mu_1 w_t \\ \qquad\qquad + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| w_t(x, t-\varrho) \mathrm{d}\varrho = b|w|^{\mathtt{p}-2},\, x \in \Omega,\, t > 0, \end{array}$$

$$\begin{array}{ll} w(x,0) = w_0(x),\, w_t(x,0) = w_1(x), & x \in \Omega, \\ w(x,t) = 0, & x \in \partial\Omega,\, t > 0, \\ w_t(x,-t) = f_0(x,t), & x \in \Omega,\, 0 < t < \tau_2. \end{array}$$

Motivated by the aforementioned works, in this paper we establish the existence of global and finite-time blowup of solutions for the problem (1), by using the potential well method.

2. PRELIMINARIES

To establish the main results of this paper, we required the following definitions and lemmas. Let $\mathcal{H}^1(\Omega)$ denote the Sobolev space with the usual scalar products and norm. $\mathcal{H}_0^1(\Omega)$ denotes the closure in $\mathcal{H}^1(\Omega)$ of $\mathcal{C}_0^\infty(\Omega)$. For simplicity of notations, hereafter we denote by $\|\cdot\|_{\mathtt{p}}$ the Lebesgue space, $\mathtt{L}^{\mathtt{p}}(\Omega)$, norm and $\|\cdot\|$ denotes $\mathtt{L}^2(\Omega)$ norm. We write the equivalent norm $\|\nabla\cdot\|$ instead of the $\mathcal{H}_0^1(\Omega)$ norm $\|\cdot\|_{\mathcal{H}_0^1(\Omega)}$.

Definition 2.1. The function $w = w(x,t)$ is said to be a weak solution of (1) on $\Omega\times[0,\mathfrak{T})$ if $w \in \mathcal{C}\big([0,\mathfrak{T}],\mathcal{H}_0^1(\Omega)\big)\cap\mathcal{C}^1\big([0,\mathfrak{T}],\mathtt{L}^2(\Omega)\big)$, $w_{tt}\in \mathcal{C}\big([0,\mathfrak{T}],\mathcal{H}^{-1}(\Omega)\big)$ such that $w(x,0)=w_0(x)$ in $\mathcal{H}_0^1(\Omega)$, $w_t(x,0)=w_1(x)$ in $\mathtt{L}^2(\Omega)$ and

$$(w_t,\omega)+\int_0^t(\nabla w,\nabla\omega)\mathrm{d}s=\int_0^t\big(|w|^{\mathtt{p}-2}w\log|w|,\omega\big)\mathrm{d}s+(w_1,\omega)$$

for any $\omega\in\mathcal{H}_0^1(\Omega)$, $t\in[0,\mathfrak{T})$.

2.1. POTENTIAL WELLS

We define energy $\mathtt{E}(t)$ for the problem (1), which satisfies the conservation law $\mathtt{E}(t)=\mathtt{E}(0)$ for every $t\in[0,\mathfrak{T})$, where

$$\mathtt{E}(t)=\frac{1}{2}\|w_t\|^2+\frac{1}{2}\|\nabla w\|^2-\frac{1}{\mathtt{p}}\int_\Omega|w|^{\mathtt{p}}\log|w|\mathrm{d}x+\frac{1}{\mathtt{p}^2}\|w\|_{\mathtt{p}}^{\mathtt{p}}$$

and

$$\mathtt{E}(0)=\frac{1}{2}\|w_1\|^2+\frac{1}{2}\|\nabla w_0\|^2-\frac{1}{\mathtt{p}}\int_\Omega|w_0|^{\mathtt{p}}\log|w_0|\mathrm{d}x+\frac{1}{\mathtt{p}^2}\|w_0\|_{\mathtt{p}}^{\mathtt{p}}.$$

Next, define two real-valued $\mathcal{C}^1$ functionals on $\mathcal{H}_0^1(\Omega)$, known as potential energy and Nehari functionals, which are

$$\mathtt{J}(w)=\frac{1}{2}\|\nabla w\|^2-\frac{1}{\mathtt{p}}\int_\Omega|w|^{\mathtt{p}}\log|w|\mathrm{d}x+\frac{1}{\mathtt{p}^2}\|w\|_{\mathtt{p}}^{\mathtt{p}}$$

and

$$\mathtt{I}(w)=\langle\mathtt{J}'(w),w\rangle=\|\nabla w\|^2-\int_\Omega|w|^{\mathtt{p}}\log|w|\mathrm{d}x,$$

where $\langle\,,\rangle$ denotes the dual pairing between $\mathcal{H}^{-1}(\Omega)$ and $\mathcal{H}_0^1(\Omega)$. Then, it is clear that

$$\mathtt{J}(w)=\frac{1}{\mathtt{p}}\mathtt{I}(w)+\left[\frac{\mathtt{p}-2}{2\mathtt{p}}\right]\|\nabla w\|^2+\frac{1}{\mathtt{p}^2}\|w\|_{\mathtt{p}}^{\mathtt{p}} \tag{3}$$

and

$$\begin{aligned}\mathtt{E}(t) &= \frac{1}{2}\|w_t\|^2 + \mathtt{J}(w)\\ &= \frac{1}{2}\|w_t\|^2 + \frac{1}{\mathtt{p}}\mathtt{I}(w) + \left[\frac{\mathtt{p}-2}{2\mathtt{p}}\right]\|\nabla w\|^2 + \frac{1}{\mathtt{p}^2}\|w\|_{\mathtt{p}}^{\mathtt{p}}.\end{aligned}$$

Also

$$\mathtt{E}(t) = \frac{1}{2}\|w_t\|^2 + \frac{1}{2}\mathtt{I}(w) + \left[\frac{\mathtt{p}-2}{2\mathtt{p}}\right]\int_\Omega |w|^{\mathtt{p}}\log|w|\mathrm{d}x + \frac{1}{\mathtt{p}^2}\|w\|_{\mathtt{p}}^{\mathtt{p}}.$$

Next, define the Nehari manifold as

$$\mathscr{N}(w) = \{w \in \mathcal{H}_0^1(\Omega) \mid \mathtt{I}(w) = 0, \|\nabla w\|^2 \neq 0\}$$

and the depth of the potential well (or mountain pass level) as

$$\mathtt{d} = \inf_{w\in\mathscr{N}} \mathtt{J}(w).$$

Finally, we define the potential well

$$\mathscr{W} = \{w \in \mathcal{H}_0^1(\Omega) \mid \mathtt{I}(w) > 0, \mathtt{J}(w) < \mathtt{d}\} \cup \{0\}$$

and the outer of the potential well

$$\mathscr{V} = \{w \in \mathcal{H}_0^1(\Omega) \mid \mathtt{I}(w) < 0, \mathtt{J}(w) < \mathtt{d}\}.$$

Let ϱ be any fixed positive real number. Now we extend the above single potential wells to the family of potential wells as

$$\mathtt{J}_\varrho(w) = \frac{\varrho}{2}\|\nabla w\|^2 - \frac{1}{\mathtt{p}}\int_\Omega |w|^{\mathtt{p}}\log|w|\mathrm{d}x + \frac{1}{\mathtt{p}^2}\|w\|_{\mathtt{p}}^{\mathtt{p}}$$

and

$$\mathtt{I}_\varrho(w) = \langle \mathtt{J}'_\varrho(w), w\rangle = \varrho\|\nabla w\|^2 - \int_\Omega |w|^{\mathtt{p}}\log|w|\mathrm{d}x.$$

The Nehari manifold is

$$\mathscr{N}_\varrho(w) = \{w \in \mathcal{H}_0^1(\Omega) \mid \mathtt{I}_\varrho(w) = 0, \|\nabla w\|^2 \neq 0\}$$

and the depth of the potential well (or mountain pass level) is

$$\mathtt{d}(\varrho) = \inf_{w\in\mathscr{N}_\varrho} \mathtt{J}(w). \tag{4}$$

Finally, we define the potential well

$$\mathscr{W}_\varrho = \{w \in \mathcal{H}_0^1(\Omega) \mid \mathtt{I}_\varrho(w) > 0, \mathtt{J}_\varrho(w) < \mathtt{d}(\varrho)\} \cup \{0\}$$

and the outer of the potential well

$$\mathscr{V}_\varrho = \{w \in \mathcal{H}_0^1(\Omega) \mid \mathtt{I}_\varrho(w) < 0, \mathtt{J}_\varrho(w) < \mathtt{d}(\varrho)\}.$$

In the next lemma we seek a unique positive critical point of $\mathtt{J}(\lambda w)$.

LEMMA 2.2. *Let* $w \in \mathcal{H}_0^1(\Omega)$ *with* $|w\| \neq 0$ *and let* $\mathtt{f}(\lambda) = \mathtt{J}(\lambda w)$. *Then we have*

(i) $\lim_{\lambda\to 0} \mathtt{f}(\lambda) = 0$ *and* $\lim_{\lambda\to+\infty} \mathtt{f}(\lambda) = -\infty$,

(ii) *there exists a unique* $\lambda^* = \lambda^*(w)$ *in the interval* $(0, +\infty)$ *such that*

$$\mathtt{f}'(\lambda)|_{\lambda=\lambda^*} = 0,$$

(iii) $\mathtt{f}(\lambda)$ *is increasing on* $0 \le \lambda \le \lambda^*$, *decreasing on* $\lambda^* \le \lambda < +\infty$ *and takes maximum at* $\lambda = \lambda^*$, *i.e., there exists a unique* $\lambda^* \in (0, +\infty)$ *such that* $\mathtt{I}(\lambda^* w) = 0$ *and* $\mathtt{I}(\lambda w) = \lambda \mathtt{f}'(\lambda) > 0$ *for* $(0, \lambda^*)$ *and* $\mathtt{I}(\lambda w) < 0$ *for* $(\lambda^*, +\infty)$.

Proof. (i) We have

$$\begin{aligned}\mathtt{f}(\lambda) &= \mathtt{J}(\lambda w)\\ &= \frac{\lambda^2}{2}\|\nabla w\|^2 - \frac{\lambda^{\mathtt{p}}}{\mathtt{p}}\int_\Omega |w|^{\mathtt{p}}\log|\lambda w|\mathrm{d}x + \frac{\lambda^{\mathtt{p}}}{\mathtt{p}^2}\|w\|_{\mathtt{p}}^{\mathtt{p}}\\ &= \frac{\lambda^2}{2}\|\nabla w\|^2 - \frac{\lambda^{\mathtt{p}}}{\mathtt{p}}\int_\Omega |w|^{\mathtt{p}}\log|w|\mathrm{d}x + \frac{\lambda^{\mathtt{p}}}{\mathtt{p}^2}\|w\|_{\mathtt{p}}^{\mathtt{p}} - \frac{\lambda^{\mathtt{p}}}{\mathtt{p}}\log\lambda\|w\|_{\mathtt{p}}^{\mathtt{p}}.\end{aligned}$$

Since $|w\| \ne 0$, it follows that $\lim_{\lambda\to 0} \mathtt{f}(\lambda) = 0$ and $\lim_{\lambda\to+\infty} \mathtt{f}(\lambda) = -\infty$.

(ii) $\mathtt{f}'(\lambda) = 0$, gives

$$\lambda\|\nabla w\|^2 - \lambda^{\mathtt{p}-1}\int_\Omega |w|^{\mathtt{p}}\log|w|\mathrm{d}x - \lambda^{\mathtt{p}-1}\log\lambda\|w\|_{\mathtt{p}}^{\mathtt{p}} = 0,$$

which implies

$$\|\nabla w\|^2 = \lambda^{\mathtt{p}-2}\int_\Omega |w|^{\mathtt{p}}\log|w|\mathrm{d}x + \lambda^{\mathtt{p}-2}\log\lambda\|w\|_{\mathtt{p}}^{\mathtt{p}}. \tag{5}$$

Next, let

$$\mathtt{g}(\lambda) = \lambda^{\mathtt{p}-2}\int_\Omega |w|^{\mathtt{p}}\log|w|\mathrm{d}x + \lambda^{\mathtt{p}-2}\log\lambda\|w\|_{\mathtt{p}}^{\mathtt{p}}.$$

Then it is clear that $\mathtt{g}(\lambda)$ is increasing on $(0, +\infty)$. Also, we have

$$\lim_{\lambda\to 0}\mathtt{g}(\lambda) = 0 \text{ and } \lim_{\lambda\to+\infty}\mathtt{g}(\lambda) = +\infty,$$

that is, there exists a unique λ_0 such that $\mathtt{g}(\lambda_0) = 0$ and $\mathtt{g}(\lambda_0) < 0$ for $0 < \lambda < \lambda_0$ and $\mathtt{g}(\lambda_0) > 0$ for $\lambda_0 < \lambda < \infty$. Hence, for any $\|\nabla w\| > 0$ there exists a unique $\lambda^* > \lambda_0$ such that (5) holds.

(iii) Since

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathtt{J}(\lambda w) = \lambda\big(\|\nabla w\|^2 - \mathtt{g}(\lambda)\big),$$

it follows from the proof of (ii) that

(a) $\mathtt{g}(\lambda) < 0$, $\quad 0 < \lambda \le \lambda_0$,
(b) $\mathtt{g}(\lambda) < \|\nabla w\|^2$, $\quad \lambda_0 < \lambda \le \lambda^*$,
(c) $\mathtt{g}(\lambda) > \|\nabla w\|^2$, $\quad \lambda^* < \lambda \le +\infty$.

That is

(a) $\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathtt{J}(\lambda w) > 0, \quad 0 < \lambda < \lambda^*,$

(b) $\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathtt{J}(\lambda w) < 0, \quad \lambda^* < \lambda < +\infty.$

Also, we have

$$\mathtt{I}(\lambda w) = \lambda^2\|\nabla w\|^2 - \lambda^{\mathtt{p}}\int_\Omega |w|^{\mathtt{p}}\log|\lambda w|\mathrm{d}x = \lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathtt{J}(\lambda w).$$

This completes proof. □

Lemma 2.3. *Let ϱ be any fixed positive real number. Then we have following assertions:*

(i) *if* $0 < \|\nabla w\| \le \delta(\varrho)$, *then* $\mathtt{I}_\varrho(w) > 0$.
(ii) *if* $\mathtt{I}_\varrho(w) < 0$, *then* $\|\nabla w\| > \delta(\varrho)$.
(iii) *if* $\mathtt{I}_\varrho(w) = 0$, *then* $\|\nabla w\| > \delta(\varrho)$ *(or)* $\|\nabla w\| = 0$

where $\delta(\varrho)$ *is the unique real root of the equation* $\alpha^{\mathtt{p}+1}\delta^{\mathtt{p}-1} = \varrho$, *and*

$$\alpha = \sup\left\{\frac{\|w\|_{\mathtt{p}+1}}{\|\nabla w\|} : w \in \mathcal{H}_0^1(\Omega)\right\}. \tag{6}$$

Proof. Let $\Psi(\delta) = \alpha^{\mathtt{p}+1}\delta^{\mathtt{p}-1}$.
(i) Suppose $0 < \|\nabla w\| \le \delta(\varrho)$. Then, we have

$$\int_\Omega |w|^{\mathtt{p}}\log|w|\mathrm{d}x < |w|_{\mathtt{p}+1}^{\mathtt{p}+1} \le \alpha^{\mathtt{p}+1}\|\nabla w\|^{\mathtt{p}+1} = \Psi(\|\nabla w\|)\|\nabla w\|^2 \le \varrho\|\nabla w\|^2.$$

This proves (i).
(ii) Suppose $\mathtt{I}_\varrho(w) < 0$, then

$$\varrho\|\nabla w\|^2 < \int_\Omega |w|^{\mathtt{p}}\log|w|\mathrm{d}x < |w|_{\mathtt{p}+1}^{\mathtt{p}+1} \le \alpha^{\mathtt{p}+1}\|\nabla w\|^{\mathtt{p}+1} = \Psi(\|\nabla w\|)\|\nabla w\|^2,$$

which implies that $\|\nabla w\| > \delta(\varrho)$.
(iii) Suppose $\mathtt{I}_\varrho(w) = 0$, then

$$\varrho\|\nabla w\|^2 = \int_\Omega |w|^{\mathtt{p}}\log|w|\mathrm{d}x < |w|_{\mathtt{p}+1}^{\mathtt{p}+1} \le \alpha^{\mathtt{p}+1}\|\nabla w\|^{\mathtt{p}+1} = \Psi(\|\nabla w\|)\|\nabla w\|^2,$$

so, we have $\|\nabla w\| > \delta(\varrho)$. This completes the proof. □

Lemma 2.4. *Let* $\mathtt{d}(\varrho)$ *be given as in* (4). *We have the following properties:*

(i) $\mathtt{d}(\varrho) = \mathtt{k}(\varrho)\delta^2(\varrho) > 0$ *for* $0 < \delta < \frac{\mathtt{p}}{2}$, $\mathtt{k}(\varrho) = \frac{1}{2} - \frac{\varrho}{\mathtt{p}}$,
(ii) *there exists a unique* $\varrho_0 > \frac{\mathtt{p}}{2}$ *such that* $\mathtt{d}(\varrho_0) = 0$ *and* $\mathtt{d}(\varrho) > 0$ *for* $0 < \varrho < \varrho_0$,
(iii) $\mathtt{d}(\varrho)$ *is strictly increasing on* $0 < \varrho \le 1$, *decreasing on* $1 \le \varrho \le \varrho_0$ *and maximum* $\mathtt{d} = \mathtt{d}(1)$ *at* $\varrho = 1$.

Proof. The proof of the present lemma is similar to the proof of Lemma 2.4 of [6] □

2.2 INVARIANT SETS

THEOREM 2.5. *Let $w_0(x) \in \mathcal{H}_0^1(\Omega)$ and $w_1(x) \in \mathtt{L}^2(\Omega)$ and $0 < e < \mathtt{d}$, then there exits a $\varrho \in (1, \varrho_2)$ such that $D(\varrho) = e$, where ϱ_2 is the constant in Lemma 2.4. Furthermore, if $w = w(x,t)$ is a weak solution of problem* (1) *with $\mathtt{E}(0) = e$, then we have*

(i) *$w \in \mathscr{W}_\varrho$ for any $1 \le \varrho < \varrho_2$, provided $\mathtt{I}(w_0) > 0$.*
(ii) *$w \in \mathscr{V}_\varrho$ for any $1 \le \varrho < \varrho_2$, provided $\mathtt{I}(w_0) < 0$.*

Proof. The proof of the present theorem is similar to the proof of Proposition 2.6 of [6] □

THEOREM 2.6. *Let $w_0(x) \in \mathcal{H}_0^1(\Omega)$ and $w_1(x) \in \mathtt{L}^2(\Omega)$. Suppose that $\mathtt{E}(0) = \mathtt{d}$, $\mathtt{I}(w_0) < 0$ and $(w_0, w_1) \ge 0$, then the set $\mathscr{V}' = \{w \in \mathcal{H}_0^1(\Omega) : \mathtt{I}(w) < 0\}$ is invariant under the flow of* (1).

Proof. The proof of the present theorem is similar to the proof of Lemma 2.7 of [14]. □

THEOREM 2.7. *For some positive constant $a := \alpha^{-\frac{\mathtt{p}+1}{\mathtt{p}-1}}$, where α is given in* (6)*, let*

$$\mathscr{B}_a = \left\{w \in \mathcal{H}_0^1(\Omega) : \|\nabla w\| \ge a\right\}.$$

Then all nontrivial solutions of (1) *with $\mathtt{E}(0) = 0$ belong to $\mathscr{B}_a$.*

Proof. Let $w(t)$ be any solution of (1) with $\mathtt{E}(0) = 0$ and $[0, \mathfrak{T})$ be the time interval of $w(t)$. Now from the energy inequality $\mathtt{E}(t) \le \mathtt{E}(0)$, i.e., $\frac{1}{2}\|w_t\|^2 + \mathtt{J}(w) \le 0$ for $0 \le t < \mathfrak{T}$, which implies $\mathtt{J}(w) \le 0$. From (3), we get

$$\frac{1}{\mathtt{p}}\mathtt{I}(w) + \left[\frac{\mathtt{p}-2}{2\mathtt{p}}\right]\|\nabla w\|^2 + \frac{1}{\mathtt{p}^2}\|w\|_{\mathtt{p}}^{\mathtt{p}} \le 0.$$

Thus,

$$\mathtt{I}(w) \le 0.$$

By the definition of $\mathtt{I}(w)$, we get

$$\|\nabla w\|^2 \le \int_\Omega |w|^{\mathtt{p}} \log|w| \mathrm{d}x \le |w\|_{\mathtt{p}+1}^{\mathtt{p}+1} \le \alpha^{\mathtt{p}+1}\|\nabla w\|^{\mathtt{p}+1}.$$

Hence,

$$\frac{1}{\alpha^{\mathtt{p}+1}} \le \|\nabla w\|^{\mathtt{p}-1}.$$

This completes the proof. □

THEOREM 2.8. *Let $w_0(x) \in \mathcal{H}_0^1(\Omega)$ and $w_1(x) \in \mathtt{L}^2(\Omega)$. Suppose that $\mathtt{E}(0) \le 0$ or $\mathtt{E}(0) = 0$ with $\|\nabla w_0\| \ne 0$, then all solutions of* (1) *belong to $\mathscr{V}_\varrho$ for $0 < \varrho < \mathtt{p}/2$.*

Proof. Let $w(t)$ be any solution of (1) on $[0,\mathfrak{T})$ with $\mathtt{E}(0)=0$. Then by the energy inequality, we have

$$\frac{1}{2}\|w_t\|^2+\frac{1}{\mathtt{p}}\mathtt{I}_\varrho(w)+\mathtt{k}(\varrho)\|\nabla w\|^2\le\frac{1}{2}\|w_t\|^2+\mathtt{J}(w)=\mathtt{E}(0)$$

for $0\le t<\mathfrak{T}$ and $0<\varrho<\mathtt{p}/2$. From the above inequality, by Lemma 2.4 and Theorem 2.7, we have

(i) $\mathtt{I}_\varrho(w)<0$ and $\mathtt{I}(w)<0<\mathtt{d}(\varrho)$ if $\mathtt{E}(0)<0$.

(ii) $\|\nabla w_0\|\ge a$ for $0\le t<\mathfrak{T}$ if $\mathtt{E}(0)=0$ and $\|\nabla w_0\|\neq 0$.

Thus, from (i) and (ii), we have $w(t)\in\mathscr{V}_\varrho$ for $0\le t<\mathfrak{T}$, $0<\varrho<\frac{\mathtt{p}}{2}$. □

3. GLOBAL EXISTENCE AND FINITE-TIME BLOWUP OF SOLUTIONS

In this section by using potential wells, we study the global existence and finite-time blowup property in finite-time for the solution of (1).

THEOREM 3.1. *Let $w_0(x)\in\mathcal{H}_0^1(\Omega)$ and $w_1(x)\in\mathtt{L}^2(\Omega)$. Suppose that $0<\mathtt{E}(0)\le\mathtt{d}$ and $\mathtt{I}(w_0)\ge 0$ or $\|\nabla w_0\|\ge 0$, then the problem (1) has a global weak solution $w(t)\in\mathtt{L}^\infty\big(0,+\infty;\mathcal{H}_0^1(\Omega)\big)$ with $w_t(t)\in\mathtt{L}^\infty\big(0,+\infty;\mathtt{L}^\infty(\Omega)\big)$ and $w(t)\in\mathscr{W}$ for $0\le t<\infty$.*

Proof. We divide the proof into two steps.

Step 1. The case when $\mathtt{E}(0)<\mathtt{d}$ and $\mathtt{I}(w_0)>0$ or $\|\nabla w_0\|=0$.

With the similar arguments employed in Theorem 3.1 in [6] and in Theorem 3.2 in [7], we can construct approximate solutions $w_\ell(x,t)$ of (1) such that

$$\frac{1}{2}\|w_{\ell t}\|^2+\mathtt{J}(w_\ell)=\mathtt{E}_\ell(0)<\mathtt{d},\ 0\le t<+\infty \tag{7}$$

and $w_\ell(t)\in\mathscr{W}$ for sufficiently large ℓ and $0\le t<+\infty$. Now by the potential energy functional, we get

$$\mathtt{J}(w_\ell)=\frac{1}{\mathtt{p}}\mathtt{I}(w_\ell)+\left[\frac{\mathtt{p}-2}{2\mathtt{p}}\right]\|\nabla w_\ell\|^2+\frac{1}{\mathtt{p}^2}|w_\ell|_{\mathtt{p}}^{\mathtt{p}}\ge\left[\frac{\mathtt{p}-2}{2\mathtt{p}}\right]\|\nabla w_\ell\|^2. \tag{8}$$

From (7) and (8), we have

$$\left[\frac{\mathtt{p}-2}{2\mathtt{p}}\right]\|\nabla w_\ell\|^2\le\frac{1}{2}\|w_{\ell t}\|^2+\left[\frac{\mathtt{p}-2}{2\mathtt{p}}\right]\|\nabla w_\ell\|^2<\mathtt{d},\ 0\le t<+\infty,$$

Thus, for $0\le t<+\infty$, we have

$$\|\nabla w_\ell\|^2<\frac{2\mathtt{pd}}{\mathtt{p}-2}, \tag{9}$$

$$\|w_\ell\|_{\mathtt{p}}^2\le\alpha^2\|\nabla w_\ell\|^2<\alpha^2\frac{2\mathtt{pd}}{\mathtt{p}-2}, \tag{10}$$

$$\int_\Omega|w_\ell|^{\mathtt{p}}\log|w_\ell|dx<\|w_\ell\|_{\mathtt{p}+1}^{\mathtt{p}+1}\le\alpha^{\mathtt{p}+1}\|\nabla w_\ell\|^{\mathtt{p}+1}<\alpha^{\mathtt{p}+1}\left[\frac{2\mathtt{pd}}{\mathtt{p}-2}\right]^{\frac{\mathtt{p}+1}{2}} \tag{11}$$

and

$$\|w_{\ell t}\|^2 < 2\mathtt{d}. \tag{12}$$

Hence, by the compactness method (see, for example [3,6]) and from (9)-(12), the problem (1) has a global weak solution $w(t) \in \mathtt{L}^\infty\big([0,+\infty),\mathcal{H}_0^1(\Omega)\big)$ with $w_t(t) \in \mathtt{L}^\infty\big([0,+\infty),\mathtt{L}^2(\Omega)\big)$ and $w(t) \in \mathscr{W}$ for $0 \le t < +\infty$.

Step 2. The case when $\mathtt{E}(0) = \mathtt{d}$ and $\mathtt{I}(w_0) \ge 0$ or $\|\nabla w_0\| \ge 0$.

We further divide the proof into two cases.

Case (i). $\|\nabla w_0\| > 0$.

Let $\lambda_{0\ell} = 1 - \frac{1}{\ell}$ and $w_{0\ell} = \lambda_\ell w_0$, $\ell = 2, 3, \ldots$. Consider the initial conditions

$$w(x,0) = w_{0\ell}(x),\ w_t(x,0) = w_1(x)$$

and the corresponding problem

$$\left.\begin{array}{ll} \frac{\partial^2 w}{\partial t^2} - \Delta w = |w|^{\mathtt{p}-2} w \log|w|, & x \in \Omega,\ t > 0,\\ w(x,0) = w_{0\ell}(x),\ w_t(x,0) = w_1(x), & x \in \Omega,\\ w(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{array}\right\} \tag{13}$$

By Lemma 2.2 and $\mathtt{I}(w_0) \ge 0$, we get $\lambda^* = \lambda^*(w_0) \ge 1$. Therefore $\mathtt{I}(w_{0\ell}) > 0$ and $\mathtt{J}(w_{0\ell}) = \mathtt{J}(\lambda_\ell w_0) < \mathtt{J}(w_0)$. Further,

$$0 < \mathtt{E}_\ell(0) \equiv \frac{1}{2}\|w_1\|^2 + \mathtt{J}(w_{0\ell}) < \frac{1}{2}\|w_1\|^2 + \mathtt{J}(w_0) = \mathtt{E}(0) = \mathtt{d}.$$

It follows from Step 1 that for each ℓ, the problem (13) has a global solution $w_\ell(t) \in \mathtt{L}^\infty\big([0,+\infty),\mathcal{H}_0^1(\Omega)\big)$ and $w_\ell(t) \in \mathscr{W}$ for $0 \le t < +\infty$ satisfying

$$(w_{\ell t},\omega) + \int_0^t (\nabla w_\ell, \nabla\omega)\mathrm{d}s = \int_0^t \big(|w_\ell|^{\mathtt{p}-2} w_\ell \log|w_\ell|, \omega\big)\mathrm{d}s + (w_1,\omega) \tag{14}$$

for every $\omega \in \mathcal{H}_0^1(\Omega)$, $0 \le t < +\infty$ and

$$\frac{1}{2}\|w_{\ell t}\|^2 + \mathtt{J}(w_\ell) = \mathtt{E}_\ell(0) < \mathtt{d},\ 0 \le t < +\infty. \tag{15}$$

The remaining proof is similar to Step 1.

Case (ii). $\|\nabla w_0\| = 0$.

Then $\mathtt{J}(w_0) = 0$ and $\frac{1}{2}\|\nabla w_1\|^2 = \mathtt{E}(0) = \mathtt{d}$. Let $\lambda_\ell = 1 - \frac{1}{\ell}$ and $w_{1\ell} = \lambda_\ell w_1$, $\ell = 2, 3, \ldots$. Consider the initial conditions

$$w(x,0) = w_0(x),\ w_t(x,0) = w_{1\ell}(x)$$

and the corresponding problem

$$\left.\begin{array}{ll} \frac{\partial^2 w}{\partial t^2} - \Delta w = |w|^{\mathtt{p}-2} w \log|w|, & x \in \Omega,\ t > 0,\\ w(x,0) = w_0(x),\ w_t(x,0) = w_{1\ell}(x), & x \in \Omega,\\ w(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{array}\right\} \tag{16}$$

Also,

$$0 < \mathtt{E}_\ell(0) \equiv \frac{1}{2}\|w_{1\ell}\|^2 + \mathtt{J}(w_0) = \frac{1}{2}\|\lambda_\ell w_1\|^2 < \mathtt{E}(0) = \mathtt{d}.$$

It follows from Step 1 that for each ℓ, problem (16) has a global solution $w_\ell(t) \in \mathtt{L}^\infty\big([0,+\infty), \mathcal{H}_0^1(\Omega)\big)$ with $w_t(t) \in \mathtt{L}^\infty\big([0,+\infty), \mathtt{L}^2(\Omega)\big)$ and $w_\ell(t) \in \mathscr{W}$ for $0 \le t < +\infty$ satisfying (14) and (15). The rest of the proof is similar to Case (i). □

THEOREM 3.2. *Let* $w_0(x) \in \mathcal{H}_0^1(\Omega)$ *and* $w_1(x) \in \mathtt{L}^2(\Omega)$. *Suppose that* $0 < \mathtt{E}(0) \le \mathtt{d}$, $\mathtt{I}(w_0) < 0$ *and* $(w_0, w_1) \ge 0$, *then the weak solution of* (1) *blows up in finite-time such that* $\lim_{t\to\mathfrak{T}^-} \|w(\cdot,t)\| = +\infty$.

Proof. We divide the proof into two steps.

Step 1. The case when $\mathtt{E}(0) < \mathtt{d}$ **and** $\mathtt{I}(w_0) < 0$.

Suppose $w(x,t)$ be any solution of (1) and define a function $\mathscr{F} : [0,+\infty) \to (0,+\infty)$ by $\mathscr{F}(t) = \|w\|^2$. Then $\mathscr{F}'(t) = 2(w, w_t)$ and

$$\begin{aligned} \mathscr{F}''(t) &= 2\|w_t\|^2 + 2(w, w_{tt}) \\ &= 2\|w_t\|^2 - 2\left[\|\nabla w\|^2 - \int_\Omega |w|^{\mathtt{p}} \log|w| \mathrm{d}x\right] \qquad (17) \\ &= 2\|w_t\|^2 - 2\mathtt{I}(w). \end{aligned}$$

Now from the energy inequality, we have

$$\frac{1}{2}\|w_t\|^2 + \frac{1}{2}\|\nabla w\|^2 - \frac{1}{\mathtt{p}}\int_\Omega |w|^{\mathtt{p}} \log|w| \mathrm{d}x + \frac{1}{\mathtt{p}^2}\|w\|_{\mathtt{p}}^{\mathtt{p}} \le \mathtt{E}(0).$$

That is

$$\begin{aligned} 2\int_\Omega |w|^{\mathtt{p}} \log|w| \mathrm{d}x &\ge \mathtt{p}\|w_t\|^2 + \mathtt{p}\|\nabla w\|^2 + \frac{2}{\mathtt{p}}\|w\|_{\mathtt{p}}^{\mathtt{p}} - 2\mathtt{p}\mathtt{E}(0) \qquad (18) \\ &\ge \mathtt{p}\|w_t\|^2 + \mathtt{p}\|\nabla w\|^2 - 2\mathtt{p}\mathtt{E}(0) \end{aligned}$$

Substituting (18) into (17), we get

$$\begin{aligned} \mathscr{F}''(t) &\ge (\mathtt{p}+2)\|w_t\|^2 + (\mathtt{p}-2)\|\nabla w\|^2 - 2\mathtt{p}\mathtt{E}(0) \qquad (19) \\ &\ge (\mathtt{p}+2)\|w_t\|^2 + (\mathtt{p}-2)\lambda_1\mathscr{F}(t) - 2\mathtt{p}\mathtt{E}(0), \end{aligned}$$

where $\lambda_1 > 0$ is the first eigenvalue of $\Delta\omega + \lambda\omega = 0$ with $\omega = 0$ on $\partial\Omega = 0$. If $\mathtt{E}(0) \le 0$, then (19) gives

$$\mathscr{F}''(t) \ge (\mathtt{p}+2)\|w_t\|^2. \qquad (20)$$

Suppose $0 < \mathtt{E}(0) \le \mathtt{d}$, then from Theorem 2.5, we have $wt \in \mathscr{V}_\varrho$ for $1 < \varrho < \varrho_2$ and $t > 0$. So, $\mathtt{I}_\varrho(w) < 0$ and by Lemma 2.3, we get $\|\nabla w\| \ge \delta(\varrho)$ for

$1 < \varrho < \varrho_2$ and $t > 0$. That is, $\mathtt{I}_{\varrho_2}(w) < 0$ and $\|\nabla w\| \geq \delta(\varrho_2)$ for $t > 0$. Since $\mathscr{F}'(0) = 2(w_0, w_1) \geq 0$ and from (17), it follows that

$$\mathscr{F}''(t) \geq 2(\varrho_2 - 1)\|\nabla w\|^2 - 2\mathtt{I}_{\varrho_2}(w) \geq 2(\varrho_2 - 1)\delta^2(\varrho_2) > 0,$$
$$\mathscr{F}'(t) \geq 2(\varrho_2 - 1)\delta^2(\varrho_2)t + \mathscr{F}'(0) \geq 2(\varrho_2 - 1)\delta^2(\varrho_2)t,$$

and

$$\mathscr{F}(t) \geq (\varrho_2 - 1)\delta^2(\varrho_2)t^2 + \mathscr{F}(0) \geq (\varrho_2 - 1)\delta^2(\varrho_2)t^2.$$

So, there exists a $\hat{t} > 0$ such that for $t \geq \hat{t}$, we get $(\mathtt{p} - 2)\lambda_1\mathscr{F}(t) > 2\mathtt{p}\mathtt{E}(0)$. Therefore, from (19) we obtain (20) and from (20), for all $t \geq \hat{t}$, we have

$$\mathscr{F}(t)\mathscr{F}''(t) - \left[\frac{\mathtt{p}+2}{4}\right]\left(\mathscr{F}'(t)\right)^2 \geq (\mathtt{p}+2)\left[\|w\|^2\|w_t\|^2 - (w, w_t)^2\right] > 0. \tag{21}$$

Step 2. The case when $\mathtt{E}(0) = \mathtt{d}$ and $\mathtt{I}(w_0) < 0$.

From (19), we have

$$\begin{aligned}\mathscr{F}''(t) &\geq (\mathtt{p}+2)\|w_t\|^2 + (\mathtt{p}-2)\lambda_1\mathscr{F}(t) - 2\mathtt{p}\mathtt{E}(0)\\ &= (\mathtt{p}+2)\|w_t\|^2 + (\mathtt{p}-2)\lambda_1\mathscr{F}(t) - 2\mathtt{p}\mathtt{d}.\end{aligned} \tag{22}$$

From (17) and by Theorem 2.6, it follows that $\mathscr{F}''(t) > 0$ for $0 \leq t < +\infty$, i.e., $\mathscr{F}'(t)$ is strictly increasing for $0 \leq t < +\infty$. Since $\mathscr{F}'(0) = 2(w_0, w_1) \geq 0$, for any $t^* > 0$ we get

$$\mathscr{F}'(t) \geq \mathscr{F}'(t^*) > 0,\ t \geq t^*$$

and

$$\mathscr{F}(t) \geq \mathscr{F}'(t^*)(t - t^*) + \mathscr{F}(t^*) > \mathscr{F}'(t^*)(t - t^*),\ t \geq t^*.$$

So, there exists a $\hat{t} > 0$ such that for $t \geq \hat{t}$, we obtain

$$(\mathtt{p} - 2)\lambda_1\mathtt{F}(t) \geq 2(\mathtt{P} + 1)\mathtt{d}.$$

From (22), we get $\mathscr{F}''(t) \geq (\mathtt{p} + 2)\|w_t\|^2$. Hence,

$$\mathscr{F}(t)\mathscr{F}''(t) - \left[\frac{\mathtt{p}+2}{4}\right]\left(\mathscr{F}'(t)\right)^2 \geq (\mathtt{p}+2)\left[\|w\|^2\|w_t\|^2 - (w, w_t)^2\right] > 0 \text{ for all } t \geq \hat{t}.$$

Finally, by Levine's concavity argument [5] that $\mathscr{F}(t)$ can not remain finite for all $t \geq \hat{t}$, we reach a contradiction. This completes proof. □

REFERENCES

[1] H. Buljan, A. Šiber, M. Soljačić, T. Schwartz, M. Segev and D. N. Christodoulides, *Incoherent white light solitons in logarithmically saturable noninstantaneous nonlinear media*, Phys. Rev. E, **68** (2003), 1–5.

[2] A. Choucha, D. Ouchenane and S. Boulaaras, *Blowup of a nonlinear viscoelastic wave equation with distributed delay combined with strong damping and source terms*, Journal of Nonlinear Functional Analysis, **2020** (2020), 1–10.

[3] P. Górka, *Logarithmic Klein-Gordan equation*, Acta Phys. Polon. B, **40** (2009), 59–66.

[4] W. Królikowski, D. Edmundson and O. Bang, *Unified model for partially coherent solitons in logarithmically nonlinear media*, Phys. Rev. E, **61** (2000), 3122–3126.

[5] H. A. Levine, *Some nonexistence and instability theorems for solutions of formally parabolic equation of the form $Pu_t = -Au + \mathcal{F}u$*, Arch. Ration. Mech. Anal., **51** (1973), 371–386.
[6] W. Lian, M. S. Ahmed and R. Xu, *Global existence and blow up solution for semilinear hyperbolic equation with logarithmic nonlinearity*, Nonlinear Anal., **184** (2019), 239–257.
[7] Y. Liu, *On potential wells and vacuum isolating of solutions for semilinear wave equations*, J. Differential Equations, **192** (2003), 155–169.
[8] Y. Liu and R. Xu, *Wave equations and reaction-diffusion equations with several nonlinear source terms of different sign*, Discrete Contin. Dyn. Syst. Ser. B, **7** (2007), 171–189.
[9] Y. Liu and J. Zhao, *On potential wells and applications to semilinear hyperbolic equations and parabolic equations*, Nonlinear Anal., **64** (2006), 2665–2687.
[10] S. De Martino, M. Falanga, C. Godon and G. Lauro, *Logarithmic Schrödinger-like equation as a model for magma transport*, Europhys. Lett., **63** (2003), 472–475.
[11] L. E. Payne and D. H. Sattinger, *Saddle points and instability of nonlinear hyperbolic equations*, Israel J. Math., **22** (1975), 273–303.
[12] J. Peng and J. Zhou, *Global existence and blow-up of solutions to a semilinear heat equation with logarithmic nonlinearity*, Appl. Anal., **100** (2021), 2804–2824.
[13] D. H. Sattinger, *On global solution of nonlinear hyperbolic equations*, Arch. Ration. Mech. Anal., **30** (1968), 148–172.
[14] R. Xu, *Initial boundary value problem for semilinear hyperbolic equations and parabolic equations with critical initial data*, Quart. Appl. Math., **68** (2010), 459–468.

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