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ON A GENERALIZED CAYLEY GRAPH OF COLUMN MATRICES OF ELEMENTS OF A FINITE GROUP

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Abstract. Let G be a finite group and let X_m be an $m \times 1$ column matrix of elements of G. Let S be a nonempty subset of G such that $e \notin S$ and $S^{-1} \subseteq S$. If $\mathrm{Cay}(G, S)$ is the usual Cayley graph, whose vertices are all elements of G and two vertices x and y are adjacent if and only if $xy^{-1} \in S$, then the generalized Cayley graph $\mathrm{Cay}_m(G, S)$ is a graph with vertex set consisting of all column matrices X_m, and two vertices X_m and Y_m are adjacent if and only if $X_m[(Y_m)^{-1}]^t \in M(S)$, where $Y_m{}^{-1}$ is a column matrix that each entry is the inverse of the corresponding entry of Y_m, $M(S)$ is an $m \times m$ matrix with all entries in S, $[Y^{-1}]^t$ is the transpose of Y^{-1}, and $m \geq 1$. It is obvious that if $m = 1$, then $\mathrm{Cay}_m(G, S)$ and $\mathrm{Cay}(G, S)$ coincide. In this article, we establish some basic properties of the new graph and determine the structure of $\mathrm{Cay}_m(G, S)$ when $\mathrm{Cay}(G, S)$ is a cycle, C_n, for every $n \geq 3$.

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Key words. Cayley graph, cycle graph, generalized Cayley graph, column matrix.

1. INTRODUCTION AND BASIC RESULTS

The notion of Cayley graphs was introduced by Arthur Cayley in 1878 [2]. He gave a geometrical representation of groups by means of a set of generators. This translates groups into geometrical objects that can be studied from a geometrical view. In particular, it provides a rich source of highly symmetric graphs, known as transitive graphs, which play an important role in many graph theoretical problems as well as group theoretical problems, like expanders, width of groups, the representation of interconnection networks, Hamiltonian path and cycles that naturally arise in computer science, and so on.

In this paper, we introduce a new generalization of Cayley graphs. Previously, some kinds of generalization of the Cayley graphs have been introduced and studied by several authors. For instance, Marušič, Scapellato, and Salvi [5] gave a generalization of Cayley graphs in terms of an automorphism of a group. Later, Zho [7] introduced the Cayley graph of a semigroup. Recently,

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the second author introduced a generalized Cayley graph for all $m \times 1$ matrices, namely $\mathrm{Cay}_m(G,S)$, which is a new generalization of usual $\mathrm{Cay}(G,S)$.
We recall that for any group G and any nonempty set S of G such that $e \notin S$ and $S^{-1} \subseteq S$, the Cayley graph $\mathrm{Cay}(G,S)$ is an undirected simple graph whose vertices are all elements of G and two vertices x and y are adjacent if and only if $xy^{-1} \in S$ [3]. It is known that $\mathrm{Cay}(G,S)$ is connected whenever S is a generating set of G and that it is always regular and vertex transitive (see [3] for more details). Now, we are going to define $\mathrm{Cay}_m(G,S)$ as follows.

Definition 1.1. For each $m \geq 1$, the generalized Cayley graph of $\mathrm{Cay}(G,S)$ denoted by $\mathrm{Cay}_m(G,S)$ is an undirected simple graph with vertex set consisting all $m \times 1$ matrices $\begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}^t$, where $x_i \in G,\ 1 \leq i \leq m$, and two vertices $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}^t$ and $Y = \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}^t$ are adjacent if and only if

$$X(Y^{-1})^t = \begin{bmatrix} x_1{y_1}^{-1} & x_1{y_2}^{-1} & \cdots & x_1{y_m}^{-1} \\ x_2{y_1}^{-1} & x_2{y_2}^{-1} & \cdots & x_2{y_m}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_m{y_1}^{-1} & x_m{y_2}^{-1} & \cdots & x_m{y_m}^{-1} \end{bmatrix} \in M_{m\times m}(S), \text{ where}$$

$$M_{m\times m}(S) = \left\{ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mm} \end{bmatrix} \mid x_{ij} \in S\ ,\ 1 \leq i,j \leq m \right\}.$$

Note that, in this paper, we always assume that $e \notin S$, that $S^{-1} \subseteq S$, and that S is a generating set. Hence, $\mathrm{Cay}(G,S)$ is connected.
In the following lemma from [6], we can find a necessary and sufficient condition for two arbitrary vertices in $\mathrm{Cay}_m(G,S)$ to be adjacent.

Lemma 1.2 ([6]). *Let* $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}^t$ *and* $Y = \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}^t$ *be two vertices of* $\mathrm{Cay}_m(G,S)$, *where* $x_i, y_j \in G$ *for* $1 \leq i,j \leq m$. *Then* X *and* Y *are adjacent in* $\mathrm{Cay}_m(G,S)$ *if and only if* x_i *is adjacent to* y_j *in* $\mathrm{Cay}(G,S)$ *for all* $1 \leq i,j \leq m$.

The following lemma gives a formula for the degree of any vertex, $\deg(\cdot)$, in $\mathrm{Cay}_m(G,S)$ in terms of some right cosets of S.

Lemma 1.3 ([6]). *Let* $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}^t$ *be a vertex of* $\mathrm{Cay}_m(G,S)$. *Then* $\deg(X) = |\bigcap_{i=1}^{m} Sx_i|$.

As mentioned earlier, $\mathrm{Cay}(G,S)$ is connected (by assuming S to be a generating set of G), so there is no isolated vertex. Indeed, $\mathrm{Cay}_m(G,S)$ is not necessary connected, even when S is a generating set and we have some isolated vertices. The following lemma states that under some conditions, we may have an isolated vertex in $\mathrm{Cay}_m(G,S)$.

Lemma 1.4 ([6]). *Let* $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}^t$ *be a vertex of* $\mathrm{Cay}_m(G,S)$. *If* $d(x_i,x_j) \neq 2$ *in* $\mathrm{Cay}(G,S)$ *for some* $1 \leq i \neq j \leq m$ *and* $\mathrm{Cay}(G,S)$ *is triangle free, then* X *is an isolated vertex in* $\mathrm{Cay}_m(G,S)$, *where* $d(x_i,x_j)$ *is the distance between* x_i *and* x_j *in* $\mathrm{Cay}(G,S)$.

One more property of the usual Cayley graph $\mathrm{Cay}(G,S)$ that fails for $\mathrm{Cay}_m(G,S)$, is regularity. We know that $\mathrm{Cay}(G,S)$ is $|S|$-regular. Indeed, $\mathrm{Cay}_m(G,S)$ is not necessary regular. For instance, if $G =< x \mid x^2 = e >= \{e,x\}$ is a cyclic group of order two, then $\mathrm{Cay}(G,S) = P_2$ [1]. Now, we can easily check that $\mathrm{Cay}_2(G,S) = K_2 \cup \overline{K_2}$, which has two isolated vertices, so it is not regular (see Figure 1.1).

In this paper, we are going to investigate the structure of $\mathrm{Cay}_m(G,S)$ whenever $\mathrm{Cay}(G,S)$ is a cycle of length n for all $n \geq 3$ and $m \geq 2$. In fact, the following theorem is the main result of the paper.

Fig. 1.1 – $\mathrm{Cay}(\mathbb{Z}_2,\{x\})$ and $\mathrm{Cay}_2(\mathbb{Z}_2,\{x\})$

Theorem 1.5. *Let* G *be a group and let* S *be a subset of* G *such that* $e \notin S$, $S^{-1} \subseteq S$, *and* $\mathrm{Cay}(G,S) = C_n$ *for* $n \geq 3$. *Then:*

(i) $\mathrm{Cay}_m(G,S) = K_{2^m,2^m} \cup \overline{K}_{2^{m+1}(2^{m-1}-1)}$, *if* $n = 4$.

(ii) $\mathrm{Cay}_m(G,S) = (C_n \circ K_{2^m-2}) \cup \overline{K}_{n^m - n(2^m-1)}$, *if* $n \neq 4$, *where* $C_n \circ \overline{K}_{2^m-2}$ *is the corona product of graphs* C_n *and* K_{2^m-2}.

In the next section, we prove some necessary lemmas and results, which will be used in the proof of Theorem 1.5. Moreover, we will give an application of this theorem in the case when G is a dihedral group of order $2n$.

2. PROOF OF THEOREM 1.5

First, let us recall the definition of the corona product, which we need to use in the proof of Theorem 1.5.

Definition 2.1 ([4]). Suppose that G and H are two graphs. Then the corona product of G and H denoted by $G \circ H$ is obtained by taking one copy of G and $|V(G)|$ copies of H and by joining each vertex of ith copy of H to the ith vertex of G, where $1 \leq i \leq |V(G)|$.

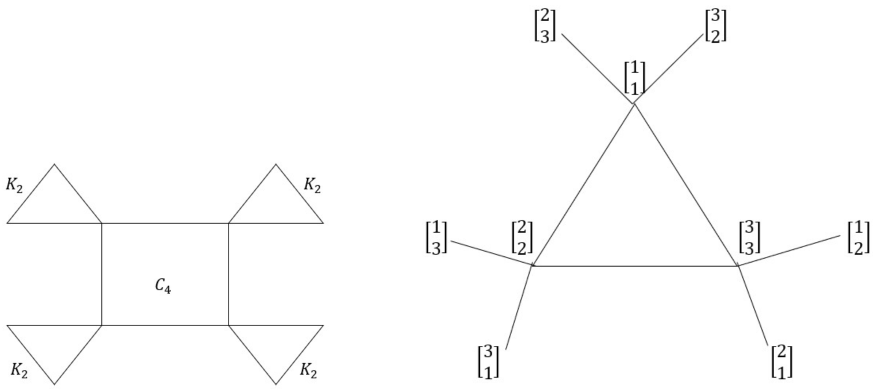
For example, the corona product of graphs C_4 and K_2 is shown in Figure 2.2 (a).

In the following three lemmas, we determine some special cases of Theorem 1.5 for $n = 3, 4$ and $m = 2, 3$.

LEMMA 2.2. *Let* $\mathrm{Cay}(G, S) = C_3$. *Then we have* $\mathrm{Cay}_2(G, S) = C_3 \circ \overline{K_2}$ *and* $\mathrm{Cay}_3(G, S) = (C_3 \circ \overline{K_6}) \cup \overline{K_6}$.

Proof. Assume that $\mathrm{Cay}(G, S) = C_3$. Then we have $|G| = 3$ and $|S| = 2$. Hence, G is a cyclic group of order three and $S = G - \{e\}$.

If $G =< x \mid x^3 = e >= \{e, x, x^2\}$, then $S = \{x, x^2\}$. Thus $\mathrm{Cay}_2(G, S)$ has $3^2 = 9$ vertices, and we can see that $\mathrm{Cay}_2(G, S) = C_3 \circ \overline{K_2}$. The graph of $\mathrm{Cay}_2(G, S)$ is shown in Figure 2.2 (b). In the case of $\mathrm{Cay}_3(G, S)$, we have $3^3 = 27$ vertices and the graph is shown in Figure 2.3. □



(a) Graph $C_4 \circ K_2$ (b) Graph $\mathrm{Cay}_2(G, S)$

Fig. 2.2 – Graph $C_4 \circ K_2$ and Graph $\mathrm{Cay}_2(G, S)$

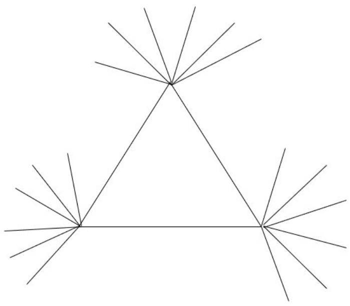


Fig. 2.3 – A component of the graph $\mathrm{Cay}_3(G, S)$ of C_3

Lemma 2.3. *Let* $\mathrm{Cay}(G,S)=C_4$. *Then* $\mathrm{Cay}_2(G,S)=K_{4,4}\cup\overline{K_8}$.

Proof. Assume that $G=\{x_1,x_2,x_3,x_4\}$ and that $\mathrm{Cay}(G,S)$ is the cycle $x_1-x_2-x_3-x_4-x_1$. Then we have $4^2=16$ vertices in $\mathrm{Cay}_2(G,S)$ and the graph is a complete bipartite graph $K_{4,4}$ with eight isolated vertices as shown in Figure 2.4. □

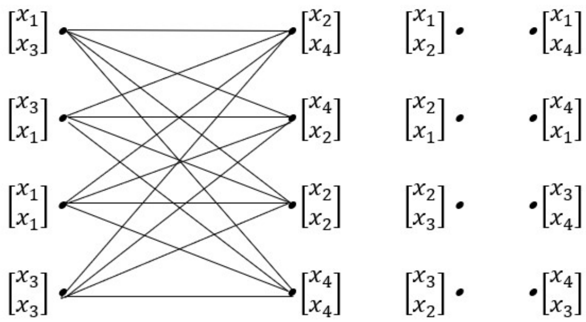


Fig. 2.4 – Graph $\mathrm{Cay}_2(G,S)$ of C_4

Lemma 2.4. *Let* $\mathrm{Cay}(G,S)=C_4$. *Then* $\mathrm{Cay}_3(G,S)=K_{8,8}\cup\overline{K}_{48}$.

Proof. Assume that $G=\{x_1,x_2,x_3,x_4\}$ and that $\mathrm{Cay}(G,S)$ is a cycle $x_1-x_2-x_3-x_4-x_1$. We know that $\mathrm{Cay}_3(G,S)$ has $4^3=64$ vertices. Put $A=\{[a,b,c]^t : a,b,c\in\{x_1,x_3\}\}$ and $B=\{[a,b,c]^t : a,b,c\in\{x_2,x_4\}\}$. It is clear that both sets A and B are independent sets and that each vertex in A is adjacent to each vertex in B and vice versa. Moreover, A and B are disjoint. Hence, the subgraph induced by $A\dot{\cup}B$ is complete and bipartite. The rest of vertices outside $A\dot{\cup}B$ are all isolated vertices. Hence, we have $\mathrm{Cay}_3(G,S)=K_{|A|,|B|}\cup\overline{K}_{4^3-(|A|+|B|)}=K_{8,8}\cup\overline{K}_{48}$, as required. □

In the following theorem, we extend Lemmas 2.3 and 2.4.

Theorem 2.5. *Let* $\mathrm{Cay}(G,S)=C_4$. *Then*

$$\mathrm{Cay}_m(G,S)=K_{2^m,2^m}\cup\overline{K}_{2^{m+1}(2^{m-1}-1)}$$

for all $m\geq 2$.

Proof. Assume that $G=\{x_1,x_2,x_3,x_4\}$ and that $\mathrm{Cay}(G,S)$ is the cycle $x_1-x_2-x_3-x_4-x_1$ of length four. We have $V=V(\mathrm{Cay}_m(G,S))=\left\{\begin{bmatrix}a_1 & a_2 & \cdots & a_m\end{bmatrix}^t : a_i\in G, 1\leq i\leq m\right\}$, so $|V|=4^m$. Consider the subsets A and B of V as follows:

$$A=\left\{\begin{bmatrix}a_1 & a_2 & \cdots & a_m\end{bmatrix}^t : a_i\in\{x_1,x_3\}, 1\leq i\leq m\right\},$$
$$B=\left\{\begin{bmatrix}a_1 & a_2 & \cdots & a_m\end{bmatrix}^t : a_i\in\{x_2,x_4\}, 1\leq i\leq m\right\}.$$

By the same method as in the proof of Lemma 2.4, we can see that A and B are independent sets and that every vertex from one is adjacent to another set. Hence, the union of disjoint sets, $A\dot{\cup}B$, induces a complete bipartite graph and the rest of vertices are all isolated vertices. Therefore, we have $\mathrm{Cay}_m(G,S) = K_{|A|,|B|} \cup \overline{K}_{|V|-(|A|+|B|)} = K_{2^m,2^m} \cup \overline{K}_{4^m-(2^m+2)} = K_{2^m,2^m} \cup \overline{K}_{2^{m+1}(2^{m-1}-1)}$ for all $m \geq 2$. □

Now, we are in position to prove Theorem 1.5.

Proof of Theorem 1.5. (i) It follows directly from Theorem 2.5.

(ii) Let $V(\mathrm{Cay}(G,S)) = \{x_1, x_2, \ldots, x_n\}$ and let $\mathrm{Cay}(G,S)$ be a cycle $x_1 - x_2 - \cdots - x_n - x_1$ of length n. We know that $\mathrm{Cay}_m(G,S)$ has n^m vertices. Put

$$A_1 = \left\{ \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix}^t \mid a_i \in \{x_2, x_n\}, 1 \leq i \leq m \right\},$$
$$A_j = \left\{ \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix}^t \mid a_i \in \{x_{j-1}, x_{j+1}\}, 1 \leq i \leq m \right\}$$

for $j = 2, 3, \ldots, n-1$, and

$$A_n = \left\{ \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix}^t \mid a_i \in \{x_1, x_{n-1}\}, 1 \leq i \leq m \right\}.$$

Then we can see that the vertex $[x_j x_j \ldots x_j]^t$ is adjacent to all vertices in A_j and that $|A_j| = 2^m$ for every $j = 1, 2, \ldots, n$. Consider the sets

$$B_1 = A_1 - \left\{ [x_2 x_2 \ldots x_2]^t, [x_n x_n \ldots x_n]^t \right\},$$
$$B_j = A_j - \left\{ [x_{j-1} x_{j-1} \ldots x_{j-1}]^t, [x_{j+1} x_{j+1} \ldots x_{j+1}]^t \right\}$$

for $1 \leq j-1 \leq n-1$, and

$$B_n = A_n - \left\{ [x_1 x_1 \ldots x_1]^t, [x_{n-1} x_{n-1} \ldots x_{n-1}]^t \right\}.$$

We have $|B_j| = |A_j| - 2 = 2^m - 2$ for all $1 \leq j \leq n$. All sets $B_1, B_2, \ldots, B_n$ are disjoint and independent sets and the subgraph induced by $\bigcup_{j=1}^{n} A_j$ is the Corona product of C_n and $|B_j| = 2^m - 2$ isolated vertices. Hence, $\mathrm{Cay}_m(G,S)$ has a component consisting of $C_n \circ \overline{K}_{2^m-2}$, and the rest of the components are all isolated vertices. The number of isolated components is $|V(\mathrm{Cay}(G,S))| - n(2^m-2) - n = n^m - n(2^m-2) - 1$. Therefore, $\mathrm{Cay}_m(G,S) = C_n \circ \overline{K}_{2^m-2} \cup \overline{K}_{n^m-n(2^m-1)}$, as required. □

Finally, we give an application of the main result: we determine the generalized Cayley graph $\mathrm{Cay}_m(G,S)$ for the case when $G = D_{2n}$ and $|S| = 1$ *or* 2. We restate that the dihedral group of order $2n$ is defined as $D_{2n} =< a, b \mid a^n = b^2 = e, b^{-1}ab = a^{-1} >$. For instance, if $n = 4$, then $D_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$. The next corollary gives us the structure of the generalized Cayley graph of D_{2n} when the order of S is one.

Corollary 2.6. *Let* D_{2n} *be a dihedral group of order* $2n$ *and let* $|S| = 1$. *Then for every* $m \geq 1$, *it follows that* $\mathrm{Cay}_m(D_{2n}, S) = nK_2 \cup \overline{K}_{(2n)^m - 2n}$, *where* $n \geq 3$.

Proof. It is obvious that when $|S| = 1$, then $\mathrm{Cay}(G, S) = nK_2$. Assume that $D_{2n} = \{x_1, x_2, \ldots, x_{2n}\}$ and that $\mathrm{Cay}(D_{2n}, S)$ is the union of n edges

$$x_1 - x_2, \ \ x_3 - x_4, \ \ \ldots, \ \ x_{2n-1} - x_{2n}.$$

Then $\mathrm{Cay}_m(D_{2n}, S)$ has a subgraph consisting of n edges

$$\begin{aligned}
&[x_1\, x_1\, \ldots\, x_1]^t - [x_2\, x_2\, \ldots\, x_2]^t,\\
&[x_3\, x_3\, \ldots\, x_3]^t - [x_4\, x_4\, \ldots\, x_4]^t,\\
&\ldots\\
&[x_{2n-1} x_{2n-1} \ldots x_{2n-1}]^t - [x_{2n} x_{2n} \ldots x_{2n}]^t.
\end{aligned}$$

All other vertices are isolated. □

Likewise, in the next corollary, we find the construction of the generalized Cayley graph of the dihedral group D_{2n}, when $|S| = 2$ and $S = \{x, x^{-1}\}$ such that $x \neq x^{-1}$.

Corollary 2.7. *Let* D_{2n} *be the dihedral group of order* $2n$. *If* $S \subseteq D_{2n}$ *such that* $S = \{x, x^{-1}\}$, *where* $x \neq x^{-1}$ *and* $o(x) = t$, *then the following hold:*

(i) *If* $t = 4$, *then* $\mathrm{Cay}_m(D_{2n}, S) = \frac{n}{2}\left[K_{2^m, 2^m} \cup \overline{K}_{2^{m+1}(2^{m-1}-1)}\right]$.
(ii) *If* $t \neq 4$, *then* $\mathrm{Cay}_m(G, S) = (C_t \circ K_{2^m - 2}) \cup \overline{K}_{t^m - t(2^m - 1)}$.

Proof. We know from [1] that $\mathrm{Cay}(D_{2n}, S) = \frac{2n}{t} C_t$. Applying Theorem 1.5 completes the proof. □

Corollary 2.8. *Let* D_{2n} *be the dihedral group of order* $2n$. *Let* $S \subseteq D_{2n}$ *such that* $S = \{x, y\}$, *where* $x^2 = y^2 = e$ *and* $o(xy) = t$. *If* $t = 2$, *then*

$$\mathrm{Cay}_m(D_{2n}, S) = \frac{n}{2}\left[K_{2^m, 2^m} \cup \overline{K}_{2^{m+1}(2^{m-1}-1)}\right],$$

and if $t \neq 2$, *then*

$$\mathrm{Cay}_m(G, S) = (C_{2t} \circ K_{2^m - 2}) \cup \overline{K}_{2t^m - 2^m + 2}.$$

Proof. By applying Theorem 1.5, it follows from the fact that

$$\mathrm{Cay}(D_{2n}, S) = \frac{n}{t} C_{2t},$$

where $o(xy) = t$. □

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