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NEW QUANTUM INEQUALITIES OF HERMITE-HADAMARD TYPE VIA GREEN FUNCTION

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Abstract. In this study, the Hermite-Hadamard inequality for q^{κ_2}-integrals is demonstrated by a new method called the Green Function Technique. For this purpose, we first obtain certain identities. Then, by using these identities, we establish many new inequalities for functions whose second derivative is convex, monotone and concave in absolute value.

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1. INTRODUCTION

Quantum Calculus is a limitless calculation study. In the eighteenth century, the famous mathematician, Newton, laid the foundation for q-calculus by initializing the q parameter in his work about infinite series. Later, in the twentieth century, Jackson [8] began a study of q-calculus in a symmetrical way and introduced q-integrals. Many researchers have studied various integral inequalities through the use of classical convexity in the context of q-derivatives and q-integrals. The most famous of these is the Hermite-Hadamard inequality. The Hermite-Hadamard inequality discovered by C. Hermite and J. Hadamard (see, for example, [5], [15, p.137]) is one of the most well-established inequalities in the theory of convex functions with geometric interpretation and many applications. This inequality states that if $F : I_o \to \mathbb{R}$ is a convex function at the interval I_o of real numbers and $\kappa_1, \kappa_2 \in I_o$ with $\kappa_1 < \kappa_2$, then

$$F\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\chi)\mathrm{d}\chi \leq \frac{F(\kappa_1)+F(\kappa_2)}{2}.$$

Both inequalities remain in the reversed direction if F is concave. We note that the Hermite-Hadamard inequality can be seen as a refining of the concept of convexity and can easily be traced back to Jensen's inequality. Many researchers have worked extensively on the development and refining of the Hermit-Hadamard inequality. Noor et al. [12], Sudsutad et al. [16] and Zhang

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et al. [19] have contributed a great deal to ongoing research and have developed quantum estimates for the right part of the quantum analog of the Hermite-Hadamard inequality through q-differentiable convex functions and q-differentiable quasi-convex functions. The main idea of this paper is to reshape the inequality presented in [4] with a novel approach using Green Functions, while also establishing some new useful results and identities. The anatomy of this paper includes the introduction of a quantum calculus with preliminary results, and the main results are demonstrated and discussed in the next section with concluding remarks at the end. We note that the opinion and technique of this work may stimulate new research in this field.

2. PRELIMINARIES OF q-CALCULUS AND SOME INEQUALITIES

Many integral inequalities are well known in the classical analysis of this kind, such as Hólder's inequality, the Hermit-Hadamard inequality and Ostrowski's inequality. The Cauchy-Bunyakovsky-Schwarz inequality, Gruss's inequality, the Gruss-Cebysev inequality, and other integral inequalities have been established and applied for q-calculus using classical convexity. For other results for q-calculus, please refer to [6, 7, 10, 12, 14, 16, 20].

In this section, we present some of the definitions required and related to the q-calculus inequalities. Here, too, we use the following notation (see [9]):

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + ... + q^{n-1}, \quad q \in (0,1).$$

In [8], Jackson gave the q-Jackson integral from 0 to $\kappa 2$ for $0 < q < 1$ as follows:

$$\int_0^{\kappa_2} F(\chi) \ \mathrm{d}_q\chi \ = (1-q)\kappa_2 \sum_{n=0}^{\infty} q^n F(\kappa_2 q^n)$$

provided the sum converges absolutely.
Jackson in [8] gave the q-Jackson integral in a generic interval $[\kappa_1, \kappa_2]$ as:

$$\int_{\kappa_1}^{\kappa_2} F(\chi) \ \mathrm{d}_q\chi \ = \int_0^{\kappa_2} F(\chi) \ \mathrm{d}_q\chi \ - \int_0^{\kappa_1} F(\chi) \ \mathrm{d}_q\chi \ .$$

Definition 2.1 ([17]). For a continuous function $\psi : [\kappa_1, \kappa_2] \to \mathbb{R}$, the q_{κ_1}-derivative of F at $\chi \in [\kappa_1, \kappa_2]$ is characterized by the expression

$$(1) \qquad {}_{\kappa_1}D_q F(\chi) \ = \frac{F(\chi) - F(q\chi + (1-q)\kappa_1)}{(1-q)(\chi - \kappa_1)}, \ \chi \neq \kappa_1.$$

Since $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a continuous function, we have ${}_{\kappa_1}D_q F(\kappa_1) = \lim_{\chi \to \kappa_1} {}_{\kappa_1}D_q F(\chi)$. The function F is said to be q-differentiable on $[\kappa_1, \kappa_2]$ if ${}_{\kappa_1}D_q F(\tau)$ exists for all $\chi \in [\kappa_1, \kappa_2]$. If $\kappa_1 = 0$ in (1), then ${}_0D_q F(\chi) =$

$D_q F(\chi)$, where $D_q F(\chi)$ is the familiar q-derivative of F at $\chi \in [\kappa_1, \kappa_2]$ defined by the expression (see [9])

$$D_q F(\chi) = \frac{F(\chi) - F(q\chi)}{(1-q)\chi}, \ \chi \neq 0.$$

DEFINITION 2.2 ([4]). For a continuous function $F : [\kappa_1, \kappa_2] \to \mathbb{R}$, the q^{κ_2}-derivative of F at $\chi \in [\kappa_1, \kappa_2]$ is characterized by the expression

$$^{\kappa_2}D_q F(\chi) = \frac{F(q\chi + (1-q)\kappa_2) - F(\chi)}{(1-q)(\kappa_2 - \chi)}, \ \chi \neq \kappa_2.$$

DEFINITION 2.3 ([17]). Let $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a continuous function. The q_{κ_1}-definite integral on $[\kappa_1, \kappa_2]$ is defined as

$$\begin{aligned}\int_{\kappa_1}^{\kappa_2} F(\chi)_{\kappa_1} \mathrm{d}_q\chi &= (1-q)(\kappa_2 - \kappa_1)\sum_{n=0}^{\infty} q^n F(q^n\kappa_2 + (1-q^n)\kappa_1) \\ &= (\kappa_2 - \kappa_1)\int_0^1 F((1-\tau)\kappa_1 + \tau\kappa_2)\,\mathrm{d}_q\tau\end{aligned}$$

In [3], Alp et al. proved the following q_{κ_1}-Hermite-Hadamard inequalities for convex functions in the setting of quantum calculus:

THEOREM 2.4. *If $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a convex differentiable function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then q-Hermite-Hadamard inequalities are as follows:*

$$F\left(\frac{q\kappa_1 + \kappa_2}{1+q}\right) \leq \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} F(\chi) \ {}_{\kappa_1}\mathrm{d}_q\chi \leq \frac{qF(\kappa_1) + F(\kappa_2)}{1+q}. \tag{2}$$

In [3, 13], the authors established some bounds for the left and right hand sides of the inequality (2).

On the other hand, Bermudo et al. gave the following new definition and related Hermite-Hadamard type inequalities:

DEFINITION 2.5 ([4]). Let $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a continuous function. Then, the q^{κ_2}-definite integral on $[\kappa_1, \kappa_2]$ is defined as

$$\begin{aligned}\int_{\kappa_1}^{\kappa_2} F(\chi)^{\kappa_2} \mathrm{d}_q\chi &= (1-q)(\kappa_2 - \kappa_1)\sum_{n=0}^{\infty} q^n F(q^n\kappa_1 + (1-q^n)\kappa_2) \\ &= (\kappa_2 - \kappa_1)\int_0^1 F(\tau\kappa_1 + (1-\tau)\kappa_2)\,\mathrm{d}_q\tau.\end{aligned}$$

THEOREM 2.6 ([4]). *Let $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a convex function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then, q-Hermite-Hadamard inequalities are as follows:*

$$F\left(\frac{\kappa_1 + q\kappa_2}{1+q}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\chi) \ {}^{\kappa_2}\mathrm{d}_q\chi \ \leq \frac{F(\kappa_1) + qF(\kappa_2)}{1+q}.$$

From Theorem 2.4 and Theorem 2.6, one can the following inequalities:

COROLLARY 2.7 ([4]). *For any convex function $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ and $0 < q < 1$, we have*

$$\begin{aligned} & F\left(\frac{q\kappa_1 + \kappa_2}{1+q}\right) + F\left(\frac{\kappa_1 + q\kappa_2}{1+q}\right) \\ & \leq \frac{1}{\kappa_2 - \kappa_1} \left\{ \int_{\kappa_1}^{\kappa_2} F(\chi) \ {}_{\kappa_1}\mathrm{d}_q\chi \ + \int_{\kappa_1}^{\kappa_2} F(\chi) \ {}^{\kappa_2}\mathrm{d}_q\chi \right\} \\ & \leq F(\kappa_1) + F(\kappa_2) \end{aligned}$$

and

$$\begin{aligned} & F\left(\frac{\kappa_1 + \kappa_2}{2}\right) \\ & \leq \frac{1}{2(\kappa_2 - \kappa_1)} \left\{ \int_{\kappa_1}^{\kappa_2} F(\chi) \ {}_{\kappa_1}\mathrm{d}_q\chi \ + \int_{\kappa_1}^{\kappa_2} F(\chi) \ {}^{\kappa_2}\mathrm{d}_q\chi \right\} \\ & \leq \frac{F(\kappa_1) + F(\kappa_2)}{2}. \end{aligned}$$

LEMMA 2.8. *Let $n \in \mathbb{R} \setminus \{-1\}$, then we have the formula*

$$\int_{\chi}^{\kappa_2} (\kappa_2 - \tau)^n \ {}^{\kappa_2}\mathrm{d_q}\tau = \frac{1}{[n+1]_q} (\kappa_2 - \chi)^{n+1}.$$

3. MAIN RESULTS

We will use the following lemma to prove our main results.

LEMMA 3.1 ([2,11]). *Let ϕ be the Green function defined on $[\kappa_1, \kappa_2] \times [\kappa_1, \kappa_2]$ by*

$$\phi(\chi, \tau) = \begin{cases} \kappa_1 - \tau, & \kappa_1 \leq \tau \leq \chi \\ \kappa_1 - \chi, & \chi \leq \tau \leq \kappa_2. \end{cases}$$

We can express any $F \in C^2([\kappa_1, \kappa_2])$ as

$$F(\chi) = F(\kappa_1) + \frac{q}{q+1}(\chi - \kappa_1)F'(\kappa_2) + \int_{\kappa_1}^{\kappa_2} \phi(\chi, \tau) F''(\tau) \mathrm{d}\tau. \tag{3}$$

Our main results are as follows.

THEOREM 3.2. *Let $F : [\kappa_1, \kappa_2] \to R$ be a convex function twice differentiable on (κ_1, κ_2). If $0 < q < 1$, then*

$$F\left(\frac{\kappa_1 + q\kappa_2}{q+1}\right) \leq \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} F(\chi)^{\kappa_2}\mathrm{d}_q\chi \leq \frac{F(\kappa_1) + qF(\kappa_2)}{q+1}.$$

Proof. If we set $\chi = \frac{\kappa_1 + q\kappa_2}{q+1}$ in (3), then we obtain

$$\begin{aligned}
&F\left(\frac{\kappa_1 + q}{\kappa_2}q + 1\right)\\
&= F(\kappa_1) + \left(\frac{\kappa_1 + q\kappa_2}{q+1} - \kappa_1\right)F'(\kappa_2) + \int_{\kappa_1} \kappa_2\phi\left(\frac{\kappa_1 + q\kappa_2}{q+1}, \tau\right)F''(\tau)\mathrm{d}\tau\\
&= F(\kappa_1) + \frac{q}{q+1}(\kappa_2 - \kappa_1)F'(\kappa_2) + \int_{\kappa_1}^{\kappa_2} \phi\left(\frac{\kappa_1 + q\kappa_2}{q+1}, \tau\right)F''(\tau)\mathrm{d}\tau.
\end{aligned} \tag{4}$$

By evaluating, we obtain that

$$\begin{aligned}
&\frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} F(\chi)^{\kappa_2}\mathrm{d}_q\chi\\
&= \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2}\left\{F(\kappa_1) + (\chi - \kappa_1)F'(\kappa_2) + \int_{\kappa_1}^{\kappa_2} \phi(\chi, \tau)F''(\tau)\mathrm{d}\tau\right\}^{\kappa_2}\mathrm{d}_q\chi\\
&= F(\kappa_1) + \frac{q}{q+1}(\kappa_2 - \kappa_1)F'(\kappa_2) + \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_1}^{\kappa_2} \phi(\chi, \tau)F''(\tau)\mathrm{d}\tau^{\kappa_2}\mathrm{d}_q\chi.
\end{aligned} \tag{5}$$

Subtracting (5) from (4), we get

$$\begin{aligned}
&F\left(\frac{\kappa_1 + q\kappa_2}{q+1}\right) - \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} F(\chi)^{\kappa_2}\mathrm{d}_q\chi\\
&= F(\kappa_1) + \frac{q}{q+1}(\kappa_2 - q)F'(\kappa_2) + \int_{\kappa_1}^{\kappa_2} \phi\left(\frac{\kappa_1 + q\kappa_2}{q+1}, \tau\right)F''(\tau)\mathrm{d}\tau\\
&- F(\kappa_1) - \frac{q}{q+1}(\kappa_2 - q)F'(\kappa_2) - \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_1}^{\kappa_2} \phi(\chi, \tau)F''(\tau)\mathrm{d}\tau\ {}^{\kappa_2}\mathrm{d}_q\chi
\end{aligned}$$

$$\begin{aligned}
&= \int_{\kappa_1}^{\kappa_2} \phi\left(\frac{\kappa_1 + q\kappa_2}{q+1}, \tau\right)F''(\tau)\mathrm{d}\tau\\
&- \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_1}^{\kappa_2} \phi(\chi, \tau)F''(\tau)\mathrm{d}\tau\ {}^{\kappa_2}\mathrm{d}_q\chi\\
&+ \int_{\kappa_1}^{\kappa_2}\left\{\phi\left(\frac{\kappa_1 + q\kappa_2}{q+1}, \tau\right) + \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} \phi(\chi, \tau)^{\kappa_2}\mathrm{d}_q\chi\right\}F''(\tau)\mathrm{d}\tau
\end{aligned} \tag{6}$$

$$\begin{aligned}
&= \int_{\kappa_1}^{\kappa_2}\left[\phi\left(\frac{\kappa_1 + q\kappa_2}{q+1}, \tau\right)\right.\\
&\left. + \frac{q}{(q+1)(\kappa_2 - \kappa_1)}\left\{(\kappa_2 - \tau)^2 - (\kappa_2 - \kappa_1)^2\right\}\right]F''(\tau)\mathrm{d}\tau.
\end{aligned}$$

Now we contemplate the function

$$g(\tau)=\phi\left(\frac{\kappa_1+q\kappa_2}{q+1},\tau\right)+\frac{q}{(q+1)(\kappa_2-\kappa_1)}\{(\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2\}.$$

We have the following possible cases.

Case I. If $\kappa_1\leq\tau\leq\frac{\kappa_1+q\kappa_2}{q+1}$, then

$$\begin{aligned}g(\tau)&=\kappa_1-\tau-\frac{q}{(q+1)(\kappa_2-\kappa_1)}[(\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2]\\ g'(\tau)&=-1+\frac{2q(\kappa_2-\tau)}{(q+1)(\kappa_2-\kappa_1)}\\ g''(\tau)&=\frac{-2q}{(q+1)(\kappa_2-\kappa_1)}<0.\end{aligned}$$

This implies that g' is decreasing and $g'(\kappa_1)=0$, which shows that $g'(\tau)\leq 0$. Thus g is also decreasing, and $g(\kappa_1)=0$, that is, $g(\tau)\leq 0$ for all $\tau\in[\kappa_1,\frac{\kappa_1+q\kappa_2}{q+1}]$.

Case II. If $\frac{\kappa_1+q\beta}{q+1}\leq\tau\leq\kappa_2$, then

$$\begin{aligned}g(\tau)&=\frac{-q(\kappa_2-\kappa_1)}{q+1}-\frac{q}{(q+1)(\kappa_2-\kappa_1)}[(\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2]\\ g'(\tau)&=\frac{2q(\kappa_2-\tau)}{(q+1)(\kappa_2-\kappa_1)}\geq 0\end{aligned}$$

Hence g is increasing and $g(\kappa_2)=0$. So, $g(\tau)\leq 0$ for all $\tau\in[\frac{\kappa_1+q\kappa_2}{q+1},\kappa_2]$. Now, using (6) and the fact that $F''(\tau)\geq 0$ for all $\tau\in[\kappa_1,\kappa_2]$, since g is convex, we obtain the first inequality:

$$F\left(\frac{\kappa_1+q\kappa_2}{q+1}\right)\leq\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(\chi)_{\kappa_2}\mathrm{d}_q\chi.$$

For the right hand side inequality, from (3), we have

$$F(\kappa_2)=F(\kappa_1)+(\kappa_2-\kappa_1)F'(\kappa_2)+\int_{\kappa_1}^{\kappa_2}\phi(\kappa_2,\tau)F''(\tau)\mathrm{d}\tau$$

and thus

$$\begin{aligned}&\frac{F(\kappa_1)+qF(\kappa_2)}{q+1}\\ &=F(\kappa_1)+\frac{q(\kappa_2-\kappa_1)}{q+1}F'(\kappa_2)+\frac{q}{q+1}\int_{\kappa_1}^{\kappa_2}\phi(\kappa_2,\tau)F''(\tau)\mathrm{d}\tau.\end{aligned}\tag{7}$$

Subtracting (5) from (7), we get

$$\begin{aligned}&\frac{F(\kappa_1)+qF(\kappa_2)}{q+1}-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(\chi)_{\kappa_2}\mathrm{d}_q\chi\\ &=\int_{\kappa_1}^{\kappa_2}\left[\frac{q\phi(\kappa_2,\tau)}{q+1}-\frac{q}{(q+1)(\kappa_2-\kappa_1)}\{(\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2\}\right]F''(\tau)\mathrm{d}\tau.\end{aligned}\tag{8}$$

Let

$$G(\tau) = \frac{q\phi(\kappa_2,\tau)}{q+1} - \frac{q((\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2)}{(q+1)(\kappa_2-\kappa_1)}.$$

Then

$$\begin{aligned} G'(\tau) &= \frac{-q}{q+1} - \frac{2q(\kappa_2-\tau)}{(q+1)(\kappa_2-\kappa_1)} \\ G''(\tau) &= -\frac{2q}{(q+1)(\kappa_2-\kappa_1)}. \end{aligned}$$

Case III. If $\kappa_1 \le \tau \le \frac{\kappa_1+\kappa_2}{2}$, then $G''(\tau) < 0$. This implies that G' is decreasing, and also $G'\left(\frac{\kappa_1+\kappa_2}{2}\right) = 0$, which shows that $G'(\tau) \ge 0$. Moreover, G is increasing, and $G(\kappa_1) = 0$. Hence $G(\tau) \ge 0$, for all $\tau \in [\kappa_1, \frac{\kappa_1+\kappa_2}{2}]$.

Case IV. Also, if $\frac{\kappa_1+\kappa_2}{2} \le \tau \le \kappa_2$, then $G''(\tau) < 0$. This implies that G' is decreasing, and $G'\left(\frac{\kappa_1+\kappa_2}{2}\right) = 0$, which implies that $G'(\tau) \le 0$. Hence G is decreasing, and $G(\kappa_2) = 0$, and then $G(\tau) \ge 0$, for all $\tau \in [\frac{\kappa_1+\kappa_2}{2}, \kappa_2]$.

Combining the above two cases, we conclude that $G(\tau) \ge 0$, for all $\tau \in [\kappa_1, \kappa_2]$. Applying (8) and the convexity of F, we establish the right-hand side of the desired inequality. The proof is completed. □

Now for the class of monotone and convex functions, we prove new quantum Hermite-Hadamard type inequalities.

THEOREM 3.3. *Let $F \in C^2([\kappa_1, \kappa_2])$ and $0 < q < 1$. Then:*

(i) *If $|F''|$ is an increasing function, then*

$$\left| \frac{F(\kappa_1) + qF(\kappa_2)}{q+1} - \frac{1}{\kappa_2-\kappa_1} \int_{\kappa_1}^{\kappa_2} F(\chi)^{\kappa_2} \mathrm{d}_q\chi \right| \le |F''(\kappa_2)| \left[\frac{q(\kappa_2-\kappa_1)^2}{6(q+1)} \right].$$

(ii) *If $|F''|$ is a decreasing function, then*

$$\left| \frac{F(\kappa_1) + qF(\kappa_2)}{q+1} - \frac{1}{\kappa_2-\kappa_1} \int_{\kappa_1}^{\kappa_2} F(\chi)^{\kappa_2} \mathrm{d}_q\chi \right| \le |F''(\kappa_1)| \left[\frac{q(\kappa_2-\kappa_1)^2}{6(q+1)} \right].$$

(iii) *If $|F''|$ is a convex function, then*

$$\begin{aligned} &\left| \frac{F(\kappa_1) + qF(\kappa_2)}{q+1} - \frac{1}{\kappa_2-\kappa_1} \int_{\kappa_1}^{\kappa_2} F(\chi)^{\kappa_2} \mathrm{d}_q\chi \right| \\ &\le \max\{|F''(\kappa_1)|, |F''(\kappa_2)|\} \left[\frac{q(\kappa_2-\kappa_1)^2}{6(q+1)} \right] \end{aligned}$$

Proof. To prove (i), by (8) we get:

$$\left|\frac{F(\kappa_1)+qF(\kappa_2)}{q+1}-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(\chi)\,{}^{\kappa_2}\mathrm{d}_q\chi\right|$$
$$=\left|\int_{\kappa_1}^{\kappa_2}\left[\frac{q\phi(\kappa_1,\tau)}{q+1}-\frac{q}{(q+1)(\kappa_2-\kappa_1)}\left\{(\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2\right\}\right]F''(\tau)\mathrm{d}\tau\right|$$
$$\leq|F''(\kappa_2)|\left|\int_{\kappa_1}^{\kappa_2}\left[\frac{q(\kappa_1-\tau)}{q+1}-\frac{q}{(q+1)(\kappa_2-\kappa_1)}\left\{(\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2\right\}\right]\mathrm{d}\tau\right|$$
$$=|F''(\kappa_2)|\left[\frac{q(\kappa_2-\kappa_1)^2}{6(q+1)}\right].$$

which proves the inequality from part (i). The second part can be proved in a similar manner. For part (iii), using (8) and the fact that $|F''|$ is bounded above, on the interval $[\kappa_1,\kappa_2]$, by $\max\{|F''(\kappa_1)|,|F''(\kappa_2)|\}$ as a convex function, we obtain

$$\left|\frac{F(\kappa_1)+qF(\kappa_2)}{q+1}-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(\chi)\,{}^{\kappa_2}\mathrm{d}_q\chi\right|$$
$$\leq\max\{|F''(\kappa_1)|,|F''(\kappa_2)|\}\left[\frac{q(\kappa_2-\kappa_1)^2}{6(q+1)}\right].$$

□

THEOREM 3.4. *Let $F\in C^2([\kappa_1,\kappa_2])$, and let $|F''|$ be a concave function. Then for $0<q<1$*

$$\left|\frac{F(\kappa_1)+qF(\kappa_2)}{q+1}-\frac{1}{\kappa_2-\kappa_1}\int_{\alpha}^{\kappa_2}F(\chi)\,{}^{\kappa_2}\mathrm{d}_q\chi\right|$$
$$\leq\frac{3q(\kappa_2-\kappa_1)^2}{2(q+1)}\left|F''\left(\frac{\kappa_1+2\kappa_2}{3}\right)\right|+\frac{(\kappa_2-\kappa_1)^2q}{3(q+1)}\left|F''\left(\frac{3\kappa_1+\kappa_2}{4}\right)\right|.$$

Proof. By using the identity (8), we have

$$\left|\frac{F(\kappa_1)+qF(\kappa_2)}{q+1}-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(\chi)\,{}^{\kappa_2}\mathrm{d}_q\chi\right|$$
$$=\left|\int_{\kappa_1}^{\kappa_2}\left[\frac{q\phi(\kappa_1,\tau)}{q+1}-\frac{q}{(q+1)(\kappa_2-\kappa_1)}\left\{(\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2\right\}\right]F''(\tau)\mathrm{d}\tau\right|$$
$$=\left|\int_{\kappa_1}^{\kappa_2}\left[\frac{q(\tau-\kappa_1)}{q+1}-\frac{q}{(q+1)(\kappa_2-\kappa_1)}\left\{(\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2\right\}\right]F''(\tau)\mathrm{d}\tau\right|.$$

Suppose $\tau=(1-\chi)\kappa_1+\chi\kappa_2$ with $\chi\in[0,1]$. Then

$$\left|\frac{F(\kappa_1)+qF(\kappa_2)}{q+1}-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(\chi)\,{}^{\kappa_2}\mathrm{d}_q\chi\right|$$

$$=\left|\int_{\kappa_1}^{\kappa_2}\left[\frac{q(\kappa_2-\kappa_1)^2\chi}{q+1}-\frac{q(\kappa_2-\kappa_1)^2}{(q+1)}\left\{(1-\chi)^2-1\right\}\right]F''((1-\chi)\kappa_1+\chi\kappa_2))\mathrm{d}\chi\right|$$

$$\leq \frac{-q(\kappa_2-\kappa_1)^2}{q+1}\left|\int_0^1 \chi F''((1-\chi)\kappa_1+\chi\kappa_2)\mathrm{d}\chi\right|$$
$$+\frac{(\kappa_2-\kappa_1)^2 q}{q+1}\left|\int_0^1 (\chi^2-2\chi)F''((1-\chi)\kappa_1+\chi\kappa_2)\mathrm{d}\chi\right|.$$

Now by using Jensen's integral inequality, we obtain the following estimates:

$$\begin{aligned}
I_1 &= \left|\int_0^1 \chi F''((1-\chi)\kappa_1+\chi\kappa_2)\mathrm{d}\chi\right| \\
&\leq \int_0^1 \chi \mathrm{d}\chi \left|F''\left(\frac{\int_0^1 \chi((1-\chi)\kappa_1+\chi\kappa_2)\mathrm{d}\chi}{\int_0^1 \chi \mathrm{d}\chi}\right)\right| \\
&= \frac{1}{2}\left|F''\left(\frac{\kappa_1\int_0^1(\chi-\chi^2)\mathrm{d}\chi+\kappa_2\int_0^1\chi^2\mathrm{d}\chi}{\frac{1}{2}}\right)\right| \\
&= \frac{1}{2}\left|F''\left(\frac{\kappa_1+2\kappa_2}{3}\right)\right|
\end{aligned}$$

$$\begin{aligned}
I_2 &= \left|\int_0^1 (\chi^2-2\chi)F''((1-\chi)\kappa_1+\chi\kappa_2)\mathrm{d}\chi\right| \\
&= \int_0^1 \chi^2 \mathrm{d}\chi \left|F''\left(\frac{\int_0^1 \chi^2((1-\chi)\kappa_1+\chi\kappa_2)\mathrm{d}\chi}{\int_0^1 \chi^2 \mathrm{d}\chi}\right)\right| \\
&+2\int_0^1 \chi \mathrm{d}\chi \left|F''\left(\frac{\int_0^1 \chi((1-\chi)\kappa_1+\chi\kappa_2)\mathrm{d}\chi}{\int_0^1 \chi \mathrm{d}\chi}\right)\right| \\
&= \frac{1}{3}\left|F''\left(\frac{3\kappa_1+\kappa_2}{4}\right)\right|+\left|F''\left(\frac{\kappa_1+2\kappa_2}{3}\right)\right|.
\end{aligned}$$

By substituting the values of I_1 and I_2, we get the required result. □

THEOREM 3.5. *Let $F \in C^2([\kappa_1,\kappa_2])$ and $q \in (0,1)$. Then:*

(i) *If $|F''|$ is an increasing function, then*

$$\left|F\left(\frac{\kappa_1+q\kappa_2}{q+1}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\chi)^{\kappa_2}\mathrm{d}_q\chi\right|$$
$$\leq |F''(\kappa_2)|\left(\frac{(\kappa_2-\kappa_1)^2 q(7q^3+24q^2+27q+103)}{6(q+1)(q^3+3q^2+3q+1)}\right).$$

(ii) *If $|F''|$ is a decreasing function, then*

$$\left|F\left(\frac{\kappa_1+q\kappa_2}{q+1}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} F(\chi)^{\kappa_2}\mathrm{d}_q\chi\right|$$
$$\leq |F''(\kappa_1)|\left(\frac{(\kappa_2-\kappa_1)^2 q(7q^3+24q^2+27q+103)}{6(q+1)(q^3+3q^2+3q+1)}\right).$$

(iii) *If $|F''|$ is a convex function, then*

$$\begin{aligned}
&\left|F\left(\frac{\kappa_1+q\kappa_2}{q+1}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(\chi)^{\kappa_2}\mathrm{d}_q\chi\right| \\
&= \max\{|F''(\kappa_2)|,|F''(\kappa_1)|\} \\
&\cdot|F''(\kappa_2)|\left(\frac{(\kappa_2-\kappa_1)^2q(7q^3+24q^2+27q+10)}{6(q+1)(q^3+3q^2+3q+1)}\right).
\end{aligned} \tag{9}$$

Proof. To prove (i), we use (6) and the fact that $|F''|$ is an increasing function to get

$$\begin{aligned}
&\left|F\left(\frac{\kappa_1+q\kappa_2}{q+1}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(\chi)^{\kappa_2}\mathrm{d}_q\chi\right| \\
&=\left|\int_{\kappa_1}^{\kappa_2}\left[\phi\left(\frac{\kappa_1+q\kappa_2}{q+1},\tau\right)+\frac{q}{(q+1)(\kappa_2-\kappa_1)}\left\{(\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2\right\}\right]F''(\tau)\mathrm{d}\tau\right| \\
&\leq|F''(\kappa_2)|\int_{\kappa_1}^{\kappa_2}\left|\left[\phi\left(\frac{\kappa_1+q\kappa_2}{q+1},\tau\right)+\frac{q}{(q+1)(\kappa_2-\kappa_1)}\left\{(\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2\right\}\right]\right|\mathrm{d}\tau \\
&\leq|F''(\kappa_2)|\cdot\left[\int_{\kappa_1}^{\frac{\kappa_1+q\kappa_2}{q+1}}\left|(\kappa_1-\tau)+\frac{q}{(q+1)(\kappa_2-\kappa_1)}\left\{(\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2\right\}\right|\mathrm{d}\tau\right. \\
&\left.+\int_{\frac{\kappa_1+q\kappa_2}{q+1}}^{\kappa_2}\left|\frac{q(\kappa_1-\kappa_2)}{q+1}+\frac{q}{(q+1)(\kappa_2-\kappa_1)}\left\{(\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2\right\}\right|\mathrm{d}\tau\right] \\
&=|F''(\kappa_2)|\cdot\left[\int_{\kappa_1}^{\frac{\kappa_1+q\kappa_2}{q+1}}\left|(\tau-\kappa_1)-\frac{q}{(q+1)(\kappa_2-\kappa_1)}\left\{(\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2\right\}\right|\mathrm{d}\tau\right. \\
&\left.+\int_{\frac{\kappa_1+q\kappa_2}{q+1}}^{\kappa_2}\left|\frac{q(\kappa_2-\kappa_1)}{q+1}-\frac{q}{(q+1)(\kappa_2-\kappa_1)}\left\{(\kappa 2-\tau)^2-(\kappa_2-\kappa_1)^2\right\}\right|\mathrm{d}\tau\right] \\
&=|F''(\kappa_2)|\left[\frac{\begin{array}{c}2q\tau^2+((3\kappa_1-9\kappa_2)q-3\kappa_2+3\kappa_1)\\+(18\kappa_1\kappa_2-12\kappa_1^2)q+6\kappa_1\kappa_2-6\kappa_1^2\end{array}}{6(\kappa_2-\kappa_1)(q+1)}\right]_{\kappa_1}^{\frac{\kappa_1+q\kappa_2}{q+1}} \\
&+|F''(\kappa_2)|\left[\frac{-q\tau(\tau^2-3\kappa_2\tau-3\kappa_2^2+12\kappa_1\kappa_2-6\kappa_1^2)}{3(\kappa_2-\kappa_1)(q+1)}\right]_{\frac{\kappa_1+q\kappa_2}{q+1}}^{\kappa_2} \\
&=|F''(\kappa_2)|\left[\left(\frac{q^2(\kappa_2-\kappa_1)^2(7q^2+12q+3)}{6(q+1)(q^3+3q^2+3q+1)}\right)\right. \\
&\left.+|F''(\kappa_2)|\left(\frac{q(\kappa_2-\kappa_1)^2(6q^2+12q+5)}{3(q+1)(q^3+3q^2+3q+1)}\right)\right].
\end{aligned}$$

Second part can be proved in a similar manner. For the third part, using (6) and the fact that $|F''|$ is bounded above, on the interval $[\kappa_1, \kappa_2]$, by $\max\{|\kappa_1|, |\kappa_2|\}$ as a convex function, we obtain the inequality (9).

This completes the proof. □

THEOREM 3.6. *Let $F \in C^2([\kappa_1, \kappa_2])$, and let $|F''|$ be a convex function. Then for $q \in (0, 1)$, the following inequality holds:*

$$\left|\frac{F(\kappa_1) + qF(\kappa_2)}{q+1} - \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} F(\chi)^{\kappa_2} \mathrm{d}_q\chi\right|$$
$$\leq |F''(\kappa_2)|\left[\frac{-9q(\kappa_2-\kappa_1)^2}{12(q+1)}\right] + |F''(\kappa_1)|\left[\frac{-5q(\kappa_2-\kappa_1)^2}{12(q+1)}\right].$$

Proof. By using (8), we get

$$\left|\frac{F(\kappa_1) + qF(\kappa_2)}{q+1} - \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} F(\chi)^{\kappa_2} \mathrm{d}_q\chi\right|$$
$$= \left|\int_{\kappa_1}^{\kappa_2}\left[\frac{q\phi(\kappa_2,\tau)}{q+1} - \frac{q}{(q+1)(\kappa_2-\kappa_1)}\left\{(\kappa_2-\tau)^2 - (\kappa_2-\kappa_1)^2\right\}\right]F''(\tau)\mathrm{d}\tau\right|$$
$$\leq \int_{\kappa_1}^{\kappa_2}\left|\frac{q(\kappa_1-\tau)}{q+1} - \frac{q}{(q+1)(\kappa_2-\kappa_1)}\left\{(\kappa_2-\tau)^2 - (\kappa_2-\kappa_1)^2\right\}\right||F''(\tau)|\mathrm{d}\tau$$
$$= \int_{\kappa_1}^{\kappa_2}\left[\frac{q(\kappa_1-\tau)}{q+1} - \frac{q}{(q+1)(\kappa_2-\kappa_1)}\left\{(\kappa_2-\tau)^2 - (\kappa_2-\kappa_1)^2\right\}\right]|F''(\tau)|\mathrm{d}\tau.$$

Suppose $\tau = (1-\chi)\kappa_1 + \chi\kappa_2$ with $\chi \in [0, 1]$, we have

$$\left|\frac{F(\kappa_1) + qF(\kappa_2)}{q+1} - \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2} F(\chi)^{\kappa_2} \mathrm{d}_q\chi\right|$$

$$\leq (\kappa_2-\kappa_1)\cdot\int_0^1\left[\frac{-q(\kappa_2-\kappa_1)\chi}{q+1} - \frac{q(\kappa_2-\kappa_1)}{(q+1)}\left\{\chi^2 - 2\chi\right\}\right]|F''((1-\chi)\kappa_1 + \chi\kappa_2)|\mathrm{d}\chi$$

$$\leq (\kappa_2-\kappa_1)^2\cdot\int_0^1\left[\frac{-q\chi}{q+1} - \frac{q}{(q+1)}\left\{\chi^2 - 2\chi\right\}\right][(1-\chi)|F''(\kappa_1)| + \chi|F''(\kappa_2)|]\mathrm{d}\chi$$

$$= |F''(\kappa_1)|\int_0^1(1-\chi)\left[\frac{-q\chi}{q+1} - \frac{q}{(q+1)}\left\{\chi^2 - 2\chi\right\}\right]\mathrm{d}\chi$$
$$+ |F''(\kappa_2)|\int_0^1\chi\left[\frac{-q\chi}{q+1} - \frac{q}{(q+1)}\left\{\chi^2 - 2\chi\right\}\right]\mathrm{d}\chi$$

$$= |F''(\kappa_2)|\left[\frac{q(\kappa_2-\kappa_1)^2}{12(q+1)}\right] + |F''(\kappa 1)|\left[\frac{q(\kappa_2-\kappa_1)^2}{12(q+1)}\right].$$

which is the required result. □

Theorem 3.7. *Let $F \in [\kappa_1, \kappa_2]$ be such that $|F''|$ is a convex function. Then for any $q \in (0,1)$, the following inequality holds:*

$$\begin{aligned}&\left|F\left(\frac{\kappa_1+q\kappa_2}{q+1}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(\chi)^{\kappa_2}\mathrm{d}_q\chi\right|\\ &\le |F''(\kappa_1)|\left[\left(\frac{q(\kappa_2-\kappa_1)^2(5q^2+12q+9)}{12(q+1)^3}\right)\right]\\ &+|F''(\kappa_2)|\left[\left(\frac{q(\kappa_2-\kappa_1)^2(9q^2+22q+11)}{12(q+1)^3}\right)\right].\end{aligned}$$

Proof. By using (6), we get

$$\begin{aligned}&\left|F\left(\frac{\kappa_1+q\kappa_2}{q+1}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}F(\chi)^{\kappa_2}\mathrm{d}_q\chi\right|\\
&=\left|\int_{\kappa_1}^{\kappa_2}\left[\phi\left(\frac{\kappa_1+q\kappa_2}{q+1},\tau\right)+\frac{q}{(q+1)(\kappa_2-\kappa_1)}\left\{(\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2\right\}\right]F''(\tau)\mathrm{d}\tau\right|\\
&\le\int_{\kappa_1}^{\kappa_2}\left|\left[\phi\left(\frac{\kappa_1+q\kappa_2}{q+1},\tau\right)+\frac{q}{(q+1)(\kappa_2-\kappa_1)}\left\{(\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2\right\}\right]\right||F''(\tau)|\mathrm{d}\tau\\
&=\left[\int_{\alpha}^{\frac{\kappa_1+q\kappa_2}{q+1}}\left[(\tau-\kappa_1)-\frac{q}{(q+1)(\kappa_2-\kappa_1)}\left\{(\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2\right\}\right]|F''(\tau)|\mathrm{d}\tau\right.\\
&\left.+\int_{\frac{\kappa_1+q\kappa_2}{q+1}}^{\kappa_2}\left[\frac{q(\kappa_2-\kappa_1)}{q+1}-\frac{q}{(q+1)(\kappa_2-\kappa_1)}\left\{(\kappa_2-\tau)^2-(\kappa_2-\kappa_1)^2\right\}\right]|F''(\tau)|\mathrm{d}\tau\right]\\
&=(\kappa_2-\kappa_1)\cdot\left[\int_0^{\frac{q}{q+1}}\left|(\kappa_2-\kappa_1)\chi-\frac{q(\kappa_2-\kappa_1)}{(q+1)}\left\{\chi^2-2\chi\right\}\right||F''((1-\chi)\kappa_1+\chi\kappa_2)|\mathrm{d}\chi\right.\\
&\left.+(\kappa_2-\kappa_1)\cdot\int_{\frac{q}{q+1}}^{1}\left|\frac{q(\kappa_2-\kappa_1)}{q+1}-\frac{q(\kappa_2-\kappa_1)}{(q+1)}\left\{\chi^2-2\chi\right\}\right||F''((1-\chi)\kappa_1+\chi\kappa_2)|\mathrm{d}\chi\right]\\
&=(\kappa_2-\kappa_1)^2\cdot\left[\int_0^{\frac{q}{q+1}}\left|\chi-\frac{q}{(q+1)}\left\{\chi^2-2\chi\right\}\right|[(1-\chi)|F''(\kappa_1)|+\chi|F''(\kappa_2)|]\mathrm{d}\chi\right.\\
&\left.+(\kappa_2-\kappa_1)^2\cdot\int_{\frac{q}{q+1}}^{1}\left|\frac{q}{q+1}-\frac{q}{(q+1)}\left\{\chi^2-2\chi\right\}\right|[(1-\chi)|F''(\kappa_1)|+\chi|F''(\kappa_2)|]\mathrm{d}\chi\right]\\
&=|F''(\kappa_1)|\left[\left(\frac{q(\kappa_2-\kappa_1)^2(5q^2+12q+9)}{12(q+1)^3}\right)\right]\\
&+|F''(\kappa_2)|\left[\left(\frac{q(\kappa_2-\kappa_1)^2(9q^2+22q+11)}{12(q+1)^3}\right)\right]\end{aligned}$$

which is the required result. □

4. CONCLUSION

We used the inspiring concept of quantum calculus to study the Hermite-Hadamard inequality in a different way. We have made new estimates by employing newly developed identities. Using the method presented in this paper, we anticipate a number of other inequalities that will stimulate further research in this area.

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