MATHEMATICA, 64 (87), N° 2, 2022, pp. 211–221

CLOSED CO-HOPFIAN MODULES

THAAR YOUNIS GHAWI

Abstract. In this paper, we properly generalize the notion of co-Hopficity for modules to the concept of closed co-Hopficity. A module M is said to be closed co-Hopfian if any injective endomorphism of M has a closed submodule image. The aim of this paper is to study and investigate this class of modules. In addition, some relations for this class with other types of modules are provided.

MSC 2010. Primary 16P20, 16P40; Secondary 16D10, 16D70.

Key words. Closed submodule, closed co-Hopfian module, weakly co-Hopfian module.

1. INTRODUCTION

The notion of co-Hopfian modules have been studied previously. Co-Hopfian modules were given and have been investigated by Varadarajan K. in [9]. Recall that an R-module M is called co-Hopfian if every injective R-endomorphism of M is automatically an R-isomorphism. A submodule N of a module M is called closed if it has no proper essential extensions inside M, i.e. if N essential in K of M implies $N = K$.

In this paper, we introduce and study the concept of closed co-Hopfian modules. An R-module M is called closed co-Hopfian if any injective R-endomorphism of M has a closed submodule image. Clearly, all co-Hopfian modules are closed co-Hopfian. We dealt with some properties of closed co-Hopfian modules and rings. Several relations between this concept and other classes of modules are given. An R-module M is called weakly co-Hopfian if every injective R-endomorphism of M is essential [5]. Recall [11], that an R-module M is called a generalized co-Hopfian module, briefly a GCH module, if any essential injective R-endomorphism of M is an isomorphism. It is shown that every closed co-Hopfian module is GCH. We proved that the concepts co-Hopfian, closed co-Hopfian and GCH modules are equivalent under weakly co-Hopfian modules. Many examples of closed co-Hopfian modules and their relations with other concepts have been obtained.

Throughout this paper, all rings are associative with identity and all modules are unital left modules unless otherwise specified. For any R-module M,

The author would like to thank the referee for valuable suggestions and comments, which improved the present article.

DOI: 10.24193/mathcluj.2022.2.06

if $m \in M$ then $l_R(m)$ will denote the left annihilator of m over R. We will denote the R-endomorphisms ring of M by $\mathrm{End}_R(M)$. The notations $N \subseteq M$, $N \leq M$, $N \trianglelefteq M$, $N \leq^c M$ and $N \leq^{\oplus} M$ mean that N is a subset, a submodule, an essential submodule, a closed submodule, or a direct summand of M, respectively. For definitions and notations that are unexplained in this work, we refer the reader to [3, 6].

2. CLOSED CO-HOPFIAN MODULES

An R-module M is called co-Hopfian if any injective R-endomorphism of M is an isomorphism [9]. Recall that an R-module M is called semi co-Hopfian if any injective R-endomorphism of M has a direct summand image, i.e. every injective R-endomorphism of M splits [2]. This section is devoted to the study of one of the generalizations of co-Hopfian modules, namely, closed co-Hopfian modules, as follows:

Definition 2.1. A left R-module M is called closed co-Hopfian if the image of every injective R-endomorphism of M is a closed submodule. A ring R is left closed co-Hopfian if R is left closed co-Hopfian as a R-module. Also, a module M_1 is called closed co-Hopfian related to a module M_2 if for every injective R-homomorphism $\varphi : M_1 \to M_2$, $\mathrm{Im}\varphi$ is a closed submodule of M_2. As a matter of fact, a module M is closed co-Hopfian if and only if M is closed co-Hopfian related to itself.

Clearly, every co-Hopfian module is semi co-Hopfian, hence a closed co-Hopfian module. Also, every semisimple (simple) module is closed co-Hopfian, but not conversely, for example: the $\mathbb{Z}$-module $\mathbb{Q}$ is closed co-Hopfian but it is neither semisimple nor simple. However, the $\mathbb{Z}$-module $\mathbb{Z}$ is not closed co-Hopfian. Notice that $\mathbb{Z}$ is closed co-Hopfian as a ring, because the only ring monomorphism of $\mathbb{Z}$ into itself is the identity map.

Remarks 2.2. (1) A closed co-Hopfian property for a module is not inherited by its submodules, such as in the following example: it is well known that $\mathbb{Q}$ as $\mathbb{Z}$-module is closed co-Hopfian, while $\mathbb{Z}$ is not a closed co-Hopfian as $\mathbb{Z}$-submodule in $\mathbb{Q}$.

(2) If the factor module of an arbitrary module M is closed co-Hopfian, then M may not be closed co-Hopfian, for example: let $M = \mathbb{Z}$ as $\mathbb{Z}$-module and $N = p\mathbb{Z} \leq M$ where p is a prime number, then $M/N \cong \mathbb{Z}_p$ is simple and hence closed co-Hopfian as $\mathbb{Z}$-module, while $M = \mathbb{Z}$ is not a closed co-Hopfian $\mathbb{Z}$-module. Later, we will give the necessary condition for this property to be achieved, in general (see Proposition 2.25).

The following gives a characterization for a closed co-Hopfian module.

Theorem 2.3. *Let M be an R-module. Then M is closed co-Hopfian if and only if for any submodule N of M with $N \cong M$, N is a closed submodule of M.*

Proof. Let N be any submodule of a closed co-Hopfian R-module M such that $N \cong M$. So there exists an R-isomorphism $\varphi : M \to N$. Then, we have that $\psi = i \circ \varphi$ is an injective R-endomorphism of M, where i is the inclusion map. By closed co-Hopficity for M, we get $\mathrm{Im}\psi \leq^c M$, notice that $\mathrm{Im}\psi = N$, hence N is closed in M. Conversely, let α be any injective R-endomorphism of M, i.e. $\mathrm{Ker}\alpha = 0$. We have $\mathrm{Im}\alpha \cong M/\mathrm{Ker}\alpha \cong M$, hence by the condition, $\mathrm{Im}\alpha$ is a closed submodule of M, and this completes the proof. □

COROLLARY 2.4. *Let M be a module. If for any non-closed submodule of M is closed co-Hopfian, then M is closed co-Hopfian.*

Proof. If false, then there exists a non-closed submodule N of M such that $N \cong M$ by Theorem 2.3, so N is not closed co-Hopfian, which contradicts the assumption. □

In the following, we will consider the descending chain condition (DCC) on non-closed submodules.

PROPOSITION 2.5. *If M is a module which has the DCC on non-closed submodules, then M is closed co-Hopfian.*

Proof. Suppose that M is not a closed co-Hopfian module, so by Theorem 2.3, there exists a non-closed submodule K_1 of M such that $K_1 \cong M$. Also K_1 is not closed co-Hopfian, hence there exists a non-closed submodule K_2 of K_1 such that $K_2 \cong K_1$. If K_2 is a closed submodule of M, then K_2 is closed in K_1, contradiction, so K_2 must be a non-closed submodule of M. By repeating this argument, we get $M \supset K_1 \supset K_2 \supset ...$, which is a strictly descending chain of non-closed submodules of M, that contradicts the assumption. Hence M is a closed co-Hopfian module. □

From Varadarajan K. [10], recall that a nonzero module M is said to be anti co-Hopfian if M is non-simple and any nonzero submodule of M is isomorphic to M. Note that, anti co-Hopfian and closed co-Hopfian modules are different notions, as in the following example: the $\mathbb{Z}$-module $\mathbb{Z}$ is clearly anti co-Hopfian but it is not closed co-Hopfian, while $\mathbb{Q}$ as a $\mathbb{Z}$-module is closed co-Hopfian but it is not anti co-Hopfian. However, in the following theorem we give a condition such that the notions coincide. Before that, we will prove the following lemma in detail which appeared in [6, Exercise 6(c), p.139].

LEMMA 2.6. *Let M be a module. Then M is semisimple if and only if every submodule of M is closed.*

Proof. Suppose M is a semisimple module, then every submodule of M is a direct summand, and so all its submodules are closed. Conversely, assume M is a module for which all its submodules are closed. Let $N \leq M$, so by [3, Proposition 1.3], there exists a submodule K of M such that $N \oplus K \trianglelefteq M$, so by assumption, $N \oplus K = M$. Thus, $N \leq^{\oplus} M$ and M is semisimple. □

Theorem 2.7. *An anti co-Hopfian module M is closed co-Hopfian if and only if it is semisimple.*

Proof. The sufficiency is clear. Conversely, assume that N is a non trivial submodule of a closed co-Hopfian module M, then $N \cong M$, by the anti co-Hopficity of M, and so by Theorem 2.3, N is a closed submodule of M, so the result follows by Lemma 2.6. □

Proposition 2.8. *Consider the following for a ring R:*
(1) *R is left semi co-Hopfian.*
(2) *R is left closed co-Hopfian.*
(3) *For $x \in R$, if $l_R(x) = 0$ then xR is a closed submodule of R.*
(4) *For $x \in R$, if $l_R(x) = 0$ then $xR = R$.*
(5) *Every R-isomorphism $xR \to R$, $x \in R$, can be extended to R.*
Then (1) $\Rightarrow$ (2), (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2). If R extends, then all five statements are equivalent.

Proof. (1) $\Rightarrow$ (2) Obvious.

(5) $\Rightarrow$ (4) Assume $l_R(x) = 0$, for $x \in R$. Define $\varphi : xR \to R$ by $\varphi(xr) = r$, for all $r \in R$. It is easy to see that φ is well-defined and a R-epimorphism. If $xr \in \text{Ker}\varphi$, then $r = 0$, and so $xr = 0$ implies $\text{Ker}\varphi = 0$. Thus φ is an R-isomorphism. So by (5), φ can be extended to R by ψ. Hence, we have $1 = \varphi(x) = \psi(x) = x.\psi(1) \in xR$, thus $xR = R$.

(4) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (2) Suppose $\alpha : R \to R$ is a left R-monomorphism. If we assume $r \in l_R(\alpha(1))$, then $\alpha(r) = 0$, which implies $r \in \text{Ker}\alpha = 0$, so $l_R(\alpha(1)) = 0$. Thus by (3), $\alpha(1).R = \alpha(R)$ is a closed submodule of R; that is $\text{Im}\alpha \leq^c R$. Hence R is closed co-Hopfian.

(2) $\Rightarrow$ (5) Suppose $xR \to R$ is an R-isomorphism, $x \in R$, i.e. $xR \cong R$. Thus, by Theorem 2.3, $xR \leq^c R$ implies $xR \leq^{\oplus} R$ by extending for R, therefore (5) is achieved. By extending R, (2) $\Rightarrow$ (1). □

Corollary 2.9. *Consider the following for a uniform ring R:*
(1) *R is field.*
(2) *R is left co-Hopfian.*
(3) *R is left semi co-Hopfian.*
(4) *R is left closed co-Hopfian.*
(5) *For $x \in R$, if $l_R(x) = 0$ then xR is a closed submodule of R.*
(6) *For $x \in R$, if $l_R(x) = 0$ then $xR = R$.*
(7) *Every R-isomorphism $xR \to R$, $x \in R$, can be extended to R.*
Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7). If R is an integral domain (so it is uniform), then (1) through (7) are equivalent.

Proof. Since R is a uniform ring, it is extending. Thus, we have (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) by Proposition 2.8.

(2) $\Rightarrow$ (3) Clear. (3) $\Rightarrow$ (2) Let $\varphi : R \to R$ be a left R-monomorphism, so $\mathrm{Im}\varphi$ is a direct summand of R, because R is semi co-Hopfian. Since R is a uniform ring and $\mathrm{Im}\varphi \neq 0$, it implies $\mathrm{Im}\varphi \trianglelefteq R$, and so $\mathrm{Im}\varphi = R$. Hence R is co-Hopfian.

(1) $\Rightarrow$ (2) Let $\varphi : R \to R$ be a left R-monomorphism. Since R is a field and $\mathrm{Im}\varphi \neq 0$, then $\mathrm{Im}\varphi = R$, this means that φ is surjective and so R is co-Hopfian.

(2) $\Rightarrow$ (1) Let $(0 \neq)t \in R$, then t is a non-zero divisor (since R is an integral domain), so $l_R(t) = 0$. By (6), we have $tR = R$; that is $1 = tr$ for some $r \in R$, hence t is an invertible element in R, so R is field. □

Proposition 2.10. *Let M be a module. If any proper submodule of M is closed co-Hopfian, then M is closed co-Hopfian.*

Proof. It is easy to check. □

Proposition 2.11. *Let M be a module such that N is closed co-Hopfian whenever N is a non-closed submodule of M, then M is closed co-Hopfian.*

Proof. It is easy to check. □

A module M is said to have the closed sum property, briefly the CSP, if the sum of any two closed submodules of M is again closed [4]. Hadi I. M-A. and Ghawi T.Y. in [4, Corollary 1.9] present the following result.

Corollary 2.12. *Let M be a module which has the CSP. For every decomposition $M = M_1 \oplus M_2$ and any homomorphism $\varphi : M_1 \to M_2$,* $\mathrm{Im}\varphi$ *is a closed submodule of M.*

The following is a immediate consequence of Corollary 2.12.

Corollary 2.13. *Let M be a module. If $M \oplus M$ has the CSP, then M is closed co-Hopfian.*

The reverse of Corollary 2.13, is not true, in general, for example: let $M = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ as a $\mathbb{Z}$-module. Thus M is co-Hopfian (in fact, M is Artinian), and hence M is closed co-Hopfian. Define $\varphi : M \to M$ as a $\mathbb{Z}$-homomorphism by $\varphi(\bar{x}, \bar{y}) = (2\bar{x}, 2\bar{y})$ for all $\bar{x}, \bar{y} \in \mathbb{Z}_4$. Thus, $\mathrm{Im}\varphi = 2\mathbb{Z}_4 \oplus 2\mathbb{Z}_4$ is not a closed submodule of M (in fact, $2\mathbb{Z}_4 \oplus 2\mathbb{Z}_4 \trianglelefteq \mathbb{Z}_4 \oplus \mathbb{Z}_4$). So, by Corollary 2.12, $M \oplus M$ does not have the CSP.

Following [5], a module M is called weakly co-Hopfian if any injective endomorphism of M has an essential image. Note that, in general, if M is a weakly co-Hopfian (not simple) module then M is not closed co-Hopfian. Since $\mathbb{Z}$ as a $\mathbb{Z}$-module is uniform, it is easy to see that $\mathbb{Z}$ is weakly co-Hopfian as a $\mathbb{Z}$-module, and that is another reason for $\mathbb{Z}$ not to be closed co-Hopfian as a $\mathbb{Z}$-module.

The following proposition gives a connection between co-Hopfian, closed co-Hopfian and weakly co-Hopfian modules.

Proposition 2.14. *Let M be an R-module. Then M is co-Hopfian if and only if M is closed co-Hopfian and weakly co-Hopfian.*

Proof. The necessity is clear. Conversely, assume that M is a closed co-Hopfian and a weakly co-Hopfian R-module. Let $\varphi \in \mathrm{End}_R(M)$ be an injection, so $\mathrm{Im}\varphi \trianglelefteq M$ and $\mathrm{Im}\varphi \leq^c M$, hence it follows that $\mathrm{Im}\varphi = M$, i.e. φ is surjective, and hence M is co-Hopfian. □

We recall from Wang Y. [11], that a module M is called a generalized co-Hopfian module, briefly a GCH module, if any essential injective endomorphism of M is an isomorphism.

It is easy to prove that any co-Hopfian module is a GCH module. Now, we weaken this condition as follows.

Proposition 2.15. *Every closed co-Hopfian module is a GCH module.*

Proof. Suppose $\varphi : M \to M$ is a left essential R-monomorphism. Since M is a closed co-Hopfian module, then $\mathrm{Im}\varphi \leq^c M$, but $\mathrm{Im}\varphi \trianglelefteq M$, so $\mathrm{Im}\varphi = M$. This means φ is a surjective. Hence M is a GCH module. □

The $\mathbb{Z}$-module $\mathbb{Z}$ is a GCH module while it is not closed co-Hopfian; this indicates that the GCH modules are considered a proper generalization of closed co-Hopfian modules.

Proposition 2.16. *Let M be an R-module with the property that for any R-endomorphism φ of M, there is a positive integer n such that $\mathrm{Ker}\varphi^n + \mathrm{Im}\varphi^n \trianglelefteq M$. If M is a GCH module, then M is co-Hopfian, and hence it is closed co-Hopfian.*

Proof. Suppose $\varphi : M \to M$ is a left R-monomorphism. By the condition, there is an integer $n \geq 1$ such that

$$\mathrm{Ker}\varphi^n + \mathrm{Im}\varphi^n \trianglelefteq M. \tag{1}$$

Since φ is injective, then it is clear that so is φ^n, i.e. $\mathrm{Ker}\varphi^n = 0$, then by (1), we get $\mathrm{Im}\varphi^n \trianglelefteq M$. But, we have that $\mathrm{Im}\varphi^n \subseteq \mathrm{Im}\varphi$, which implies $\mathrm{Im}\varphi \trianglelefteq M$, hence M is also a GCH module, so φ is a left R-epimorphism and therefore M is a co-Hopfian module. □

Corollary 2.17. *The following are equivalent for an R-module M.*
(1) *M is co-Hopfian.*
(2) *M is closed co-Hopfian and weakly co-Hopfian.*
(3) *M is GCH and weakly co-Hopfian.*

Proof. (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) By Propositions 2.14 and 2.15.

(3) $\Rightarrow$ (1) Suppose $\varphi : M \to M$ is an injective R-endomorphism. Since M is weakly co-Hopfian, then $\mathrm{Im}\varphi \trianglelefteq M$, which implies that φ is surjective, as M is a GCH module, and hence M is co-Hopfian. □

As an analogous concept of the notion of indecomposable modules we recall from Hadi I. M-A. and Ghawi T.Y. [4], that a module M is called closed simple if the trivial submodules are the only closed submodules of M. It is clear that every closed simple module extends. Also, we know that the concepts of closed co-Hopfian and semi co-Hopfian modules coincide when extended. However, we present the following.

PROPOSITION 2.18. *The following are equivalent for a closed simple R-module M.*
(1) *M is co-Hopfian.*
(2) *M is semi co-Hopfian.*
(3) *M is closed co-Hopfian.*

Proof. It is easy to check. □

Recall from Roman C.S. [8], that an R-module M is called to be mono-endo if all nonzero R-endomorphisms are monomorphisms, or, equivalently, for any R-endomorphism f of M, Ker$f$ is either M or 0. A left R-module M is called dual Rickart (closed dual Rickart), briefly d-Rickart (c-d-Rickart), if for any $f \in \mathrm{End}_R(M)$, Imf is a direct summand (resp. closed) submodule of M, (see [4, 7]). So, we have:

PROPOSITION 2.19. *Every d-Rickart (c-d-Rickart) R-module is semi co-Hopfian (resp. closed co-Hopfian). The converse holds, under an mono-endo R-module.*

Proof. The necessity is clear. Conversely, assume that M is a closed co-Hopfian R-module. Let $f \in \mathrm{End}_R(M)$. If $f = 0$ then Imf is trivially closed in M. Let $f \neq 0$ and as M is mono-endo, then f is an R-monomorphism, hence Im$f \leq^c M$, by the closed co-Hopficity for M. Thus M is a c-d-Rickart R-module. □

The following is immediate according to Propositions 2.18 and 2.19.

COROLLARY 2.20. *If M is a closed simple and a mono-endo module, then all of the five following statements are equivalent.*
(1) *M is co-Hopfian.*
(2) *M is semi co-Hopfian.*
(3) *M is closed co-Hopfian.*
(4) *M is d-Rickart.*
(5) *M is c-d-Rickart.*

The property of closed co-Hopfian is inherited by the direct summands of closed co-Hopfian modules, as follows.

PROPOSITION 2.21. *Every direct summand of a closed co-Hopfian module is also closed co-Hopfian.*

Proof. Suppose M is a closed co-Hopfian module and N a direct summand of M, then $M = N \oplus K$ for some $K \leq M$. Let $f : N \to N$ be a monomorphism. We can define an endomorphism $g : M \to M$ by $g(n+k) = f(n) + k$ where $n \in N$ and $k \in K$. It is easy to prove that g is injective. Since M is closed co-Hopfian, $\mathrm{Im}g \leq^c M$. We have $\mathrm{Im}g = \mathrm{Im}f \oplus K$, so $\mathrm{Im}f \leq^{\oplus} \mathrm{Im}g$ which implies $\mathrm{Im}f \leq^c \mathrm{Im}g$. By the transitive property for closed submodules, we get that $\mathrm{Im}f$ is closed in M, and therefore also in N. □

A natural question is whether or not a direct sum of closed co-Hopfian modules is again closed co-Hopfian. We do not have a counterexample to this question. Moreover, we present the following Proposition. Before that, we will prove the following lemma in detail which appeared in [3, Exercise 15, p.20].

LEMMA 2.22. *Let $\{A_i\}$ and $\{B_i\}$ be collections of modules such that A_i is a closed submodule of B_i for all $i \in I$. Then $\bigoplus_{i\in I} A_i$ is a closed submodule of $\bigoplus_{i\in I} B_i$.*

Proof. It is enough to prove when the index set consists exactly of two elements, say $I = \{1, 2\}$. Let $A_1 \leq^c B_1$ and $A_2 \leq^c B_2$. Then, A_1 is a relative complement of $X_1 \leq B_1$, i.e. A_1 is a maximal submodule of B_1 with the property $A_1 \cap X_1 = 0$. Similarly, A_2 is a relative complement of $X_2 \leq B_2$, i.e. A_2 is a maximal submodule of B_2 with the property $A_2 \cap X_2 = 0$. It follows that $X_1 \oplus X_2 \leq B_1 \oplus B_2$ and $(A_1 \oplus A_2) \cap (X_1 \oplus X_2) = 0$. Suppose $L \supset A_1 \oplus A_2$ in $B_1 \oplus B_2$ with $L \cap (X_1 \oplus X_2) = 0$. Hence $L \supset A_1$ and $L \supset A_2$ such that $L \cap X_1 \subseteq L \cap (X_1 \oplus X_2) = 0$ and $L \cap X_2 \subseteq L \cap (X_1 \oplus X_2) = 0$, which contradicts the maximality for A_1 and A_2, respectively. Hence $A_1 \oplus A_2$ is a relative complement for $X_1 \oplus X_2 \leq B_1 \oplus B_2$, and hence $A_1 \oplus A_2 \leq^c B_1 \oplus B_2$. □

PROPOSITION 2.23. *Let $M = M_1 \oplus M_2$ be an R-module such that $l_R(M_1) \oplus l_R(M_2) = R$. Then M is closed co-Hopfian if and only if M_i is closed co-Hopfian for all $i = 1, 2$.*

Proof. The necessity follows directly by Proposition 2.21. Conversely, assume $f : M \to M$ is a left R-monomorphism. Since $l_R(M_1) \oplus l_R(M_2) = R$ and $\mathrm{Im}f \leq M_1 \oplus M_2$, by [1, Proposition 1.4.2] $\mathrm{Im}f = A \oplus B$ where $A \leq M_1$ and $B \leq M_2$, that is $\mathrm{Im}(f|_{M_1}) \oplus \mathrm{Im}(f|_{M_2}) = A \oplus B$ implies that $\mathrm{Im}(f|_{M_1})$ and $\mathrm{Im}(f|_{M_2})$ are submodules of M_1 and M_2, respectively. As M_1 and M_2 are closed co-Hopfian modules and f is a left monomorphism (so are $f|_{M_1}$ and $f|_{M_2}$), then $\mathrm{Im}(f|_{M_1}) \leq^c M_1$ and $\mathrm{Im}(f|_{M_2}) \leq^c M_2$, so by Lemma 2.22 $\mathrm{Im}(f|_{M_1}) \oplus \mathrm{Im}(f|_{M_2}) \leq^c M_1 \oplus M_2$, thus $\mathrm{Im}f \leq^c M$. Hence M is closed co-Hopfian. □

A submodule N of an R-module M is called fully invariant if $f(N) \subseteq N$ for each $f \in \mathrm{End}_R(M)$.

In the following, we gave another case under which an infinite direct sum of closed co-Hopfian modules is closed co-Hopfian.

Proposition 2.24. *Let $M = \bigoplus_{i\in I} M_i$ such that M_i is fully invariant under any injection of M for all $i \in I$. Then M is closed co-Hopfian if and only if M_i is closed co-Hopfian for all $i \in I$.*

Proof. Suppose that M_i is a closed co-Hopfian module for all $i \in I$. Let $\varphi : M \to M$ be injective. Let φ_i be the restriction of φ to M_i , i.e. $\varphi_i =: \varphi|_{M_i}$ for all $i \in I$. Since φ is a monomorphism, so is φ_i for all $i \in I$. As M_i is closed co-Hopfian and fully invariant, then $\mathrm{Im}\varphi_i \leq^c M_i$ for all $i \in I$. Lemma 2.22 implies $\bigoplus_{i\in I} \varphi_i \leq^c \bigoplus_{i\in I} M_i$, thus $\mathrm{Im}\varphi \leq^c M$, and hence M is a closed co-Hopfian module. The converse follows by Proposition 2.21. □

Proposition 2.25. *Let M be a module and N a fully invariant closed submodule of M. If N is co-Hopfian and M/N is closed co-Hopfian, then M is closed co-Hopfian.*

Proof. Let $\varphi : M \to M$ be a monomorphism. Put $\psi = \varphi|_N$, then ψ is a monomorphism. As N is a fully invariant co-Hopfian submodule, hence $\psi(N) = N$ and so $\varphi(N) = N$. Define the endomorphism $h : M/N \to M/N$ by $h(m + N) = \varphi(m) + N$ for all $m \in M$. It is easy to show that h is injective. Since M/N is closed co-Hopfian, then $\mathrm{Im}h \leq^c M/N$, so that $\mathrm{Im}\varphi/N \leq^c M/N$ implies that $\mathrm{Im}\varphi \leq^c M$, by $N \leq^c M$. Therefore M is a closed co-Hopfian module. □

Proposition 2.26. *Let N be a proper closed submodule of a module M such that for any injective endomorphism φ of M, $N \subseteq \mathrm{Im}\varphi$ and $M/\varphi^{-1}(N)$ is closed co-Hopfian related to each other proper factor of M, then M is closed co-Hopfian.*

Proof. Let $\varphi : M \to M$ be a monomorphism. Since N is a proper closed submodule of M, then M/N is proper factor, so by assumption we have $N \subseteq \mathrm{Im}\varphi$ and $M/\varphi^{-1}(N)$ is closed co-Hopfian related to M/N. If $N = 0$, then there is nothing to prove. Suppose $N \neq 0$, so we can define $\psi : M/\varphi^{-1}(N) \to M/N$ by $\psi(m + \varphi^{-1}(N)) = \varphi(m) + N$ for all $m \in M$. It is easy to show that ψ is well defined and a monomorphism. Since $M/\varphi^{-1}(N)$ is closed co-Hopfian related to M/N, then $\mathrm{Im}\psi = \mathrm{Im}\varphi/N$ is closed in M/N, which implies that $\mathrm{Im}\varphi \leq^c M$, by $N \leq^c M$. Therefore M is a closed co-Hopfian module. □

Theorem 2.27. *A nonzero R-module M is closed co-Hopfian if and only if M is closed co-Hopfian related to each of its nonzero submodules.*

Proof. Suppose M is a closed co-Hopfian R-module. Let N be any nonzero submodule of M such that $\varphi : M \to N$ is a left R-monomorphism. Thus, we have that $h = i \circ \varphi \in \mathrm{End}_R(M)$ and h is injective, where $i : N \to M$ is the inclusion map. Since M is closed co-Hopfian, $\mathrm{Im}h \leq^c M$. As $\mathrm{Im}h = \mathrm{Im}\varphi$, it follows that $\mathrm{Im}\varphi \leq^c N$, and hence M is closed co-Hopfian related to N. The converse is clear. □

Let M be a module over an integral domain R. Then M is called torsion free if the submodule $T(M)$=$\{m \in M \mid rm = 0$ for some $(0 \neq)r \in R\}$ is equal to zero. Notice that the concepts of torsion free and closed co-Hopfian modules are different, as in the following example: the left $\mathbb{Z}$-module $\mathbb{Z}$ is torsion free but it is not closed co-Hopfian, while $\mathbb{Z}_p$ as $\mathbb{Z}$-module is closed co-Hopfian but it is not torsion free, where p is a prime number. However, we have the following consequence.

Proposition 2.28. *Let M be a torsion free module over an integral domain R. If M is a closed co-Hopfian R-module, then M is injective.*

Proof. If $(0 \neq)r \in R$. Define an R-endomorphism $\varphi : M \to M$ by $\varphi(m) = rm$ for all $m \in M$. Now, if $m \in \mathrm{Ker}\varphi$, $rm = 0$ implies $m \in T(M) = 0$, thus φ is injective. Since M is closed co-Hopfian, then $\mathrm{Im}\varphi = rM$ is a closed submodule of M. But, we have $rM \trianglelefteq M$, to see this: if $(0 \neq)m \in M$, then $m \notin T(M)$, so $sm \neq 0$ for all nonzero element $s \in R$, hence $rm \neq 0$ and $rm \in rM$. So we get $rM = M$ for all $(0 \neq)r \in R$, this means that M is a torsion free divisible R-module, therefore M is an injective R-module. □

Notice that the condition (torsion free) in Proposition 2.28, is necessary, for example; the $\mathbb{Z}$-module $\mathbb{Z}_6$ is closed co-Hopfian but it is not injective, in fact $\mathbb{Z}_6$ as $\mathbb{Z}$-module is not torsion free.

Let M be an R-module. The set $\{\sum m_i x^i \mid m_i \in M, i \in I\}$ is denoted by $M[x]$. Then $M[x]$ can be considered as a $R[x]$-module. This module is called a polynomial module.

Theorem 2.29. *Let M be an R-module. If $M[x]$ is a closed co-Hopfian $R[x]$-module, then M is a closed co-Hopfian R-module.*

Proof. Let $\varphi : M \to M$ be an R-monomorphism, so $\varphi[x] : M[x] \to M[x]$ defined by $\varphi[x](\sum m_i x^i) = \sum \varphi(m_i)x^i$ is an $R[x]$-endomorphism. To prove that $\varphi[x]$ is injective, if $\sum m_i x^i \in \mathrm{Ker}(\varphi[x])$, then $\sum \varphi(m_i)x^i = 0$ this implies $\varphi(m_i) = 0$, so $m_i \in \mathrm{Ker}\varphi$ for i, hence $\sum m_i x^i = 0$. As $M[x]$ is a closed co-Hopfian $R[x]$-module, then $\mathrm{Im}(\varphi[x]) = \mathrm{Im}\varphi[x]$ is a closed submodule of $M[x]$. We claim that $\mathrm{Im}\varphi \leq^c M$. If $\mathrm{Im}\varphi \trianglelefteq N \subseteq M$, then $\mathrm{Im}\varphi \cap L \neq 0$ for every nonzero submodule L of N, thus $\mathrm{Im}\varphi[x] \cap L[x] = (\mathrm{Im}\varphi \cap L)[x] \neq 0$ for any nonzero submodule $L[x]$ of $N[x]$ in $M[x]$, we deduce that $\mathrm{Im}\varphi[x] \trianglelefteq N[x] \subseteq M[x]$, so that $\mathrm{Im}\varphi[x] = N[x]$ (because $\mathrm{Im}\varphi[x] \leq^c M[x]$), thus $\mathrm{Im}\varphi = N$. Therefore $\mathrm{Im}\varphi \leq^c M$, and M is a closed co-Hopfian R-module. □

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Received April 10, 2021
Accepted May 30, 2021

University of AL-Qadisiyah
College of Education
Department of Mathematics
Diwaniyah, Iraq
E-mail: thar.younis@qu.edu.iq
https://orcid.org/0000-0002-0637-1908