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POSITIVE SOLUTIONS FOR A $(p,2)$-LAPLACIAN STEKLOV PROBLEM

ABDELMAJID BOUKHSAS, ABDELLAH ZEROUALI, OMAR CHAKRONE, and BELHADJ KARIM

Abstract. In this work, we study positive solutions of a Steklov problem driven by the $(p,2)$-Laplacian operator by using the variational method. A sufficient condition for the existence of positive solutions is characterized by the eigenvalues of a linear eigenvalue problem and another nonlinear eigenvalue problem.

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1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear Steklov eigenvalue problem:

$$(S_{p,2}) \begin{cases} -\Delta_p u - \Delta u + |u|^{p-2}u + u = 0 & \text{in } \Omega, \\ \langle |\nabla u|^{p-2}\nabla u + \nabla u, \nu \rangle = f(x,u) & \text{on } \partial\Omega. \end{cases}$$

Here for any $p > 2$ by Δ_p we denote the p-Laplacian differential operator defined by

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) \text{ for all } u \in W^{1,p}(\Omega).$$

When $p = 2$, we write $\Delta_2 = \Delta$ (the standard Laplace differential operator). ν is the outward unit normal vector on $\partial\Omega$, $\langle .,. \rangle$ is the scalar product of $\mathbb{R}^N$, while the reaction term $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

In problem $(S_{p,2})$, the differential operator $u \mapsto -\Delta_p u - \Delta u$ is nonhomogeneous. We mention that equations involving the sum of a p-Laplacian and a Laplacian (also known as $(p,2)$-equations) arise in mathematical physics, see, for example the works of Benci et al. [2] (quantum physics), Cherfils and Il'yasov [6] (plasma physics) and Zhirkov [16] (homogenization of composites consisting of two different materials with distinct hardening exponents, double phase problems). Recently, in a series of papers, problem $(S_{p,2})$ has been investigated for $p > 2$, under the boundary condition $u = 0$. In [12], the authors

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studied the following Dirichlet problem

$$(D_{p,2})\begin{cases} -\Delta_p u - \Delta u &= f(x,u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

They impose certain conditions on the reaction term $f(x,u)$ to make equation resonant at $\pm\infty$ and zero. Using variational methods and critical groups, they obtain existence and multiplicity results. In [8], the authors consider the case with a reaction term $f(x,u)$ which is superlinear in the positive direction (without satisfying the Ambrosetti-Rabinowitz condition) and sublinear resonant in the negative direction. They apply Morse theory and variational methods to establish the existence of at least three non-trivial smooth solutions. Using critical point theory, truncation and comparison techniques, and Morse theory, Papageorgiou and Rădulescu [11] proved multiplicity results for $(D_{p,2})$ for both $p > 2$ and $p < 2$.

A more general problem with a (p,q)-Laplacian equation under a Steklov boundary condition ($1 < q < p < \infty$), was studied in [3–5, 14, 15]. Elliptic equations involving differential operators of the form

$$Au := \operatorname{div}(D(u)\nabla u) = \Delta_p u + \Delta_q u,$$

where $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{q-2})$, usually called (p,q)-Laplacian, occurs in many important concrete situations. For instance, this happens when one seeks stationary solutions to the reaction-diffusion system

$$u_t = Au + c(x,u), \tag{1}$$

where $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{q-2})$. This system has a wide range of applications in physics and related sciences like chemical reaction design [1], biophysics [7] and plasma physics [13]. In such applications, the function u describes a concentration, the first term on the right-hand side of (1) corresponds to the diffusion with a diffusion coefficient $D(u)$; whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x;u)$ has a polynomial form with respect to the concentration.

Now, we give our hypothesis on the reaction term $f(x,u)$:

$H(f)_1$ $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $f(x,t) \geq 0$ for any $x \in \partial\Omega$, $t > 0$.

$H(f)_2$ For $f_0, f_\infty < \infty$, the limits

$$\lim_{t\to 0^+} \frac{f(x,t)}{t} = f_0, \ \lim_{t\to\infty} \frac{f(x,t)}{t^{p-1}} = f_\infty, \tag{2}$$

exist uniformly for $x \in \partial\Omega$.

Remark 1.1. Since we are looking for positive solutions and the above hypotheses concerns the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality we assume that

$$f(x,t) = 0 \text{ for a.e. } x \in \partial\Omega, \text{for all } t \leq 0.$$

The asymptotic behaviors of f near zero and infinity lead us to define

$$
(3)\qquad
\begin{aligned}
\mu_1 &= \inf\left\{ \int_\Omega (|\nabla u|^2 + u^2)\mathrm{d}x : u \in H^1(\Omega), \int_{\partial\Omega} |u|^2 \mathrm{d}\sigma = 1 \right\},\\
\lambda_1 &= \inf\left\{ \int_\Omega (|\nabla u|^p + |u|^p)\mathrm{d}x : u \in W^{1,p}(\Omega), \int_{\partial\Omega} |u|^p \mathrm{d}\sigma = 1 \right\}.
\end{aligned}
$$

Throughout this paper, $\|u\|_{1,r} := \left(\int_\Omega (|\nabla u|^r + |u|^r)\mathrm{d}x \right)^{1/r}$ is the $W^{1,r}(\Omega)$-norm, where $W^{1,r}(\Omega)^*$ denotes the dual space, and by $\langle .,. \rangle$ we denote the duality pairing between $W^{1,r}(\Omega)$ and $W^{1,r}(\Omega)^*$. The letters $C_1, C_2, ...$ will denote various positive constants whose exact values are not essential to the analysis of the problem. Let $P = \{u \in W^{1,p}(\Omega) : u(x) \geq 0,\ \text{a.e } \overline{\Omega}\}$ and $p^* = Np/(N-p)$ if $p < N$ or $p^* = \infty$ if $p \geq N$. We always assume $H(f)_1$ and $H(f)_2$ hold with $f_0 < \mu_1$ and $f_\infty > \lambda_1$. Hence, for any given $\varepsilon > 0$, there exist $C_\varepsilon > 0$ such that

$$|f(x,t) - f_\infty t^{p-1}| \leq \varepsilon t^{p-1} + C_\varepsilon,\ x \in \partial\Omega, t \geq 0,$$

which implies that

$$(4)\qquad F(x,t) \geq \frac{1}{p}(\lambda_1 - \varepsilon)t^p - C_\varepsilon,\ x \in \partial\Omega, t \geq 0.$$

And there exists $q \in (p, p^*)$ such that

$$|f(x,t) - f_0(1-\varepsilon)t| \leq C_\varepsilon t^{q-1},\ x \in \partial\Omega, t \in \mathbb{R}.$$

Subsequently, we have that

$$(5)\qquad F(x,t) \leq \frac{1}{2}(1-\varepsilon)\mu_1 t^2 - C_\varepsilon t^q,\ x \in \partial\Omega, t \in \mathbb{R},$$

where $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$.

The following function illustrates our main result

Example 1.2.

$$f(x,t) = \begin{cases} (\frac{\mu_1}{2} + t\varepsilon(x))t + 2\lambda_1 t^{p-1} & \text{if } t \geq 0,\\ 0 & \text{if } t < 0\ , \end{cases}$$

where $\varepsilon : \partial\Omega \to \mathbb{R}$ is a measurable function.

2. PRELIMINARIES

In this section, we state some preliminary results which will be used to prove our main theorem in this paper. First, recall a theorem from [9].

Theorem 2.1. *Let $(E, \|.\|)$ be a Banach space and $U \subset \mathbb{R}_+$ an interval. Consider the family of C^1 functionals on E,*

$$(6)\qquad J_\kappa(u) = S(u) - \kappa T(u),\ \kappa \in U,$$

with $J_\kappa(0) = 0$, $\kappa \in U, T$ *nonnegative and either* $S(u) \to \infty$ *or* $T(u) \to \infty$ *as* $\|u\| \to \infty$. *for any* $\kappa \in U$, *we set*

$$\Gamma_\kappa = \{\gamma \in C([0,1], E) : \gamma(0) = 0, J_\kappa(\gamma(1)) < 0\}. \tag{7}$$

If for every $\kappa \in U$ *the set* Γ_κ *is nonempty and*

$$c_\kappa = \inf_{\gamma \in \Gamma_\kappa} \max_{t \in [0,1]} J_\kappa(\gamma(1)) > 0, \tag{8}$$

then for almost every $\kappa \in U$ *there exists a sequence* $\{u_n^\kappa\} \subset E$ *such that*

(i) $\{u_n^\kappa\}$ *is bounded;*
(ii) $J_\kappa(\{u_n^\kappa\}) \to c_\kappa$ *as* $n \to \infty$;
(iii) $J'_\kappa(\{u_n^\kappa\}) \to 0$ *in the dual* E^* *as* $n \to \infty$.

Next, we state the following inequality that will be used later.

Lemma 2.2 ([10, Lemma 4.2]). *If* $p \geq 2$, *then*

$$|w|^p - |v|^p - p|v|^{p-2}v.(w-v) \geq \frac{|w-v|^p}{2^{p-1}-1}$$

for all points v *and* w *in* $\mathbb{R}^n$.

In the setting of Theorem 2.1 we have $E = W^{1,p}(\Omega)$, $U = [\delta, 1]$

$$\begin{aligned} S(u) &= \frac{1}{2}\|u\|_{1,2}^2 + \frac{1}{p}\|u\|_{1,p}^p, \ T(u) = \int_{\partial\Omega} F(x,u)\mathrm{d}\sigma, \\ J_\kappa(u) &= \frac{1}{2}\|u\|_{1,2}^2 + \frac{1}{p}\|u\|_{1,p}^p - \kappa \int_{\partial\Omega} F(x,u)\mathrm{d}\sigma, \ u \in W^{1,p}(\Omega), \kappa \in U. \end{aligned} \tag{9}$$

It is easy to verify that

$$\begin{aligned} \langle J'_\kappa(u), v\rangle = &\int_\Omega (\nabla u \cdot \nabla v + uv)\mathrm{d}x + \int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla v + |u|^{p-2}uv)\mathrm{d}x \\ &- \kappa \int_{\partial\Omega} f(x,u)v\mathrm{d}\sigma, \quad u \in W^{1,p}(\Omega), \kappa \in U. \end{aligned} \tag{10}$$

Firstly, we show that J_κ satisfies the conditions of Theorem 2.1 by proving several lemmas.

Lemma 2.3. $\Gamma_\kappa \neq \emptyset$ *for any* $\kappa \in U$.

Proof. Let $\phi_1 > 0$ be a λ_1-eigenfunction. For $t > 0$, we have by (4) that

$$\begin{aligned} J_\kappa(t\phi_1) &= \frac{1}{2}t^2\|\phi_1\|_{1,2}^2 + \frac{1}{p}t^p\|\phi_1\|_{1,p}^p - \kappa \int_{\partial\Omega} F(x, t\phi_1)\mathrm{d}\sigma \\ &\leq \frac{1}{2}t^2\|\phi_1\|_{1,2}^2 + \frac{1}{p}t^p\lambda_1\|\phi_1\|_{L^p(\partial\Omega)}^p - \frac{1}{p}(\lambda_1 - \varepsilon)\delta t^p\|\phi_1\|_{L^p(\partial\Omega)}^p + C_1 \\ &= \frac{1}{2}t^2\|\phi_1\|_{1,2}^2 + \frac{1}{p}t^pC_2\|\phi_1\|_{L^p(\partial\Omega)}^p + C_1, \end{aligned}$$

where $C_2 = \lambda_1(1-\delta) + \varepsilon\delta > 0$. We can choose $t_0 > 0$ large enough so that $J_\kappa(t_0\phi_1) < 0$, where t_0 is independent of $\kappa \in U$. The proof is completed. $\square$

LEMMA 2.4. *There exists a constant $c > 0$ such that $c_\kappa \geq c$ for any $\kappa \in U$.*

Proof. For any $u \in W^{1,p}(\Omega)$, it follows from (5) that

$$\begin{aligned} J_\kappa(u) &= \frac{1}{2}\|u\|_{1,2}^2 + \frac{1}{p}\|u\|_{1,p}^p - \kappa \int_{\partial\Omega} F(x,u)\mathrm{d}\sigma \\ &\geq \frac{1}{2}\|u\|_{1,2}^2 + \frac{1}{p}\|u\|_{1,p}^p - \frac{1}{2}(1-\varepsilon)\mu_1 \int_{\partial\Omega} |u|^2 \mathrm{d}\sigma - C_\varepsilon \int_{\partial\Omega} |u|^q \mathrm{d}\sigma \\ &\geq \frac{\mu_1}{2}\|u\|_{L^2(\partial\Omega)}^2 + \frac{1}{p}\|u\|_{1,p}^p - \frac{1}{2}(1-\varepsilon)\mu_1\|u\|_{L^2(\partial\Omega)}^2 - C_\varepsilon\|u\|_{L^q(\partial\Omega)}^q \\ &\geq \frac{1}{p}\|u\|_{1,p}^p - C_\varepsilon\|u\|_{L^q(\partial\Omega)}^q. \end{aligned}$$

By trace embedding $W^{1,p}(\Omega) \to L^p(\partial\Omega)$, we conclude that there exists $\rho > 0$ and $c > 0$ such that $J_\kappa(u) > 0$ for $\|u\| \in (0,\rho)$ and $J_\kappa(u) \geq c$, $\|u\|_{1,p} = \rho$. Fix $\kappa \in U$ and $\gamma \in \Gamma_\kappa$. By definition of Γ_κ, we have that $\|\gamma(1)\| > \rho$. Hence, there exists $t_\gamma \in (0,1)$ such that $\|\gamma(t_\gamma)\| = \rho$. So

$$c_\kappa = \inf_{\gamma\in\Gamma_\kappa} \max_{t\in[0,1]} J_\kappa(\gamma(t)) \geq \inf_{\gamma\in\Gamma_\kappa} J_\kappa(\gamma(t_\gamma)) \geq c. \tag{11}$$

The proof is completed. □

LEMMA 2.5. *For any $\kappa \in U$, if $\{u_n\}$ is bounded and $J'_\kappa(u_n) \to 0$ in $W^{1,p}(\Omega)^*$ as $n \to \infty$, then $\{u_n\}$ admits a convergent subsequence.*

Proof. Given $\kappa \in U$, assume that $\{u_n\}$ is bounded and $J'_\kappa(u_n) \to 0$ in $W^{1,p}(\Omega)^*$ as $n \to \infty$. By extracting a subsequence, we may suppose that there exists $u \in W^{1,p}(\Omega)$ such that as $n \to \infty$

$$u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega),\ u_n \to u \text{ in } L^p(\Omega),\ u_n \to u \text{ in } L^s(\partial\Omega),\ s \in [1,p^*). \tag{12}$$

Noting that

$$\begin{aligned} &\langle J'_\kappa(u_n) - J'_\kappa(u), u_n - u\rangle \\ &= \langle J'_\kappa(u_n), u_n - u\rangle - \langle J'_\kappa(u), u_n - u\rangle \\ &= \int_\Omega \nabla u_n \cdot \nabla(u_n - u)\mathrm{d}x + \int_\Omega u_n(u_n - u)\mathrm{d}x \\ &+ \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - u)\mathrm{d}x \\ &+ \int_\Omega |u_n|^{p-2}u_n(u_n - u)\mathrm{d}x - \kappa\int_{\partial\Omega} f(x,u_n)(u_n - u)\mathrm{d}\sigma \\ &- \int_\Omega \nabla u \cdot \nabla(u_n - u)\mathrm{d}x + \int_\Omega u(u_n - u)\mathrm{d}x \\ &+ \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(u_n - u)\mathrm{d}x \\ &- \int_\Omega |u|^{p-2}u(u_n - u)\mathrm{d}x + \kappa\int_{\partial\Omega} f(x,u)(u_n - u)\mathrm{d}\sigma \end{aligned} \tag{13}$$

$$
\begin{aligned}
&= \int_\Omega (|\nabla(u_n - u)|^2 + |u_n - u|^2 \mathrm{d}x \\
&+ \int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot \nabla(u_n - u)\mathrm{d}x \\
&+ \int_\Omega (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\mathrm{d}x - \kappa \int_{\partial\Omega} f(x, u_n)(u_n - u)\mathrm{d}\sigma \\
&+ \kappa \int_{\partial\Omega} f(x, u)(u_n - u)\mathrm{d}\sigma
\end{aligned}
$$

and using the inequality in Lemma 2.2 we deduce the following inequality

$$
\begin{aligned}
&\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot \nabla(u_n - u)\mathrm{d}x \\
&\geq \frac{2}{p(2^{p-1} - 1)} \int_\Omega |\nabla(u_n - u)|^p \mathrm{d}x.
\end{aligned} \tag{14}
$$

It follows from (13) and (14) that

$$
\begin{aligned}
&\frac{2}{p(2^{p-1} - 1)} \int_\Omega |\nabla(u_n - u)|^p \mathrm{d}x \leq \langle J'_\kappa(u_n) - J'_\kappa(u), u_n - u\rangle \\
&+ \kappa \int_{\partial\Omega} f(x, u_n)(u_n - u)\mathrm{d}\sigma - \kappa \int_{\partial\Omega} f(x, u)(u_n - u)\mathrm{d}\sigma.
\end{aligned} \tag{15}
$$

Note that

$$
\langle J'_\kappa(u_n) - J'_\kappa(u), u_n - u\rangle \to 0, \text{ as } n \to \infty. \tag{16}
$$

It follows from $H(f)_1$ and $H(f)_2$ that there exists $C_1, C_2 > 0$ such that

$$
f(x, t) \leq C_1|t| + C_2|t|^{p-1}, \; x \in \partial\Omega, t \in \mathbb{R}. \tag{17}
$$

Hence, by Holder's inequality and trace embedding $W^{1,p}(\Omega) \to L^p(\partial\Omega)$, we have

$$
\begin{aligned}
&\left| \int_{\partial\Omega} f(x, u_n)(u_n - u)\mathrm{d}\sigma \right| \leq C_1 \int_{\partial\Omega} |u_n||u_n - u|\mathrm{d}\sigma \\
&+ C_2 \int_{\partial\Omega} |u_n|^{p-1}|u_n - u|\mathrm{d}\sigma \\
&\leq C_1 \left(\int_{\partial\Omega} |u_n|^2 \mathrm{d}\sigma \right)^{1/2} \left(\int_{\partial\Omega} |u_n - u|^2 \mathrm{d}\sigma \right)^{1/2} \\
&+ C_2 \left(\int_{\partial\Omega} |u_n|^p \mathrm{d}\sigma \right)^{(p-1)/p} \left(\int_{\partial\Omega} |u_n - u|^p \mathrm{d}\sigma \right)^{1/p} \\
&\leq C_3 \|u_n - u\|_{L^2(\partial\Omega)} + C_4 \|u_n - u\|_{L^p(\partial\Omega)} \to 0, \text{ as } n \to \infty.
\end{aligned} \tag{18}
$$

Similarly, we have

$$\left|\int_{\partial\Omega} f(x,u)(u_n-u)\mathrm{d}\sigma\right| \to 0, \text{ as } n\to\infty. \tag{19}$$

Now, using (16),(18) and (19) we deduce from (15) that

$$\int_\Omega |\nabla(u_n-u)|^p \mathrm{d}x \to 0, \text{ as } n\to\infty.$$

Hence, $u_n\to u$ in $W^{1,p}(\Omega)$. The proof is completed. □

LEMMA 2.6. *There exists a sequence* $\{\kappa_n\}\subset U$ *with* $\kappa_n\to 1^-$ *as* $n\to\infty$ *and* $\{u_{\kappa_n}\}\subset W^{1,p}(\Omega)$ *such that* $J_{\kappa_n}(u_{\kappa_n})=c_{\kappa_n}, J'_{\kappa_n}(u_{\kappa_n})=0$.

Proof. We only need to show that for almost every $\kappa\in U$ there exists $u^\kappa\in W^{1,p}(\Omega)$ such that $J_\kappa(u^\kappa)=c_\kappa, J'_\kappa(u^\kappa)=0$. By Theorem 2.1, for almost each $\kappa\in U$, there exists a bounded sequence $\{u_n^\kappa\}\subset W^{1,p}(\Omega)$ such that

$$J_\kappa(u_n^\kappa)\to c_\kappa,\ J'_\kappa(u_n^\kappa)\to 0\ ,\ n\to\infty. \tag{20}$$

By, Lemma 2.5, we may assume that $u_n^\kappa\to u^\kappa$ in $W^{1,p}(\Omega)$ as $n\to\infty$. Then the continuity of J_κ and J'_κ implies that $J_\kappa(u^\kappa)=c_\kappa$, and $J'_\kappa(u^\kappa)=0$. The proof is completed. □

LEMMA 2.7. *Suppose* $H(f)_1$ *and* $H(f)_2$ *hold, then*

$$\frac{Lu-f_\infty Ku}{\|u\|_{1,p}^{p-1}}\to 0,\ u\in P, \tag{21}$$

where $\langle Lu,v\rangle=\int_{\partial\Omega} f(x,u)v\mathrm{d}\sigma$ *and* $\langle Ku,v\rangle=\int_\Omega |u|^{p-2}uv\mathrm{d}x,\ u,v\in W^{1,p}(\Omega)$.

Proof. By $H(f)_1$ and $H(f)_2$ for every $\varepsilon>0$, there is a constant $C_\varepsilon>0$ such that

$$|f(x,t)-f_\infty t^{p-1}|\le C_\varepsilon+\varepsilon t^{p-1},\ x\in\partial\Omega,\ t\ge 0. \tag{22}$$

For $\in P\backslash\{0\}$, letting $w=u/\|u\|_{1,p}$, by Holder's inequality and trace embedding $W^{1,p}(\Omega)\to L^p(\partial\Omega)$, we have

$$\begin{aligned}
&\sup_{\|u\|_{1,p}\le 1}\left|\left\langle\frac{Lu-f_\infty Ku}{\|u\|_{1,p}^{p-1}},v\right\rangle\right|\le \sup_{\|u\|_{1,p}\le 1}\int_{\partial\Omega}\frac{f(x,u)-f_\infty u^{p-1}}{\|u\|_{1,p}^{p-1}}|v|\mathrm{d}\sigma\\
&\le \sup_{\|u\|_{1,p}\le 1}\int_{\partial\Omega}(C_\varepsilon\|u\|_{1,p}^{-(p-1)}|v|+\varepsilon w^{p-1}|v|\mathrm{d}\sigma\\
&\le C_6\|u\|_{1,p}^{-(p-1)}+\varepsilon C_5,
\end{aligned} \tag{23}$$

where C_5 is independent of ε. The proof is completed. □

3. MAIN RESULTS

Our main result is the following theorem.

THEOREM 3.1. *Suppose that f satisfies $H(f)_1$ and $H(f)_2$ with $f_0 < \mu_1$ and $f_\infty > \lambda_1$. Then $(S_{p,2})$ has a positive solution.*

Proof. By Lemma 2.6, there exists a sequence $\{\kappa_n\} \subset U$ with $\kappa_n \to 1^-$ as $n \to \infty$ and $\{u_{\kappa_n}\} \subset W^{1,p}(\Omega)$ such that

$$J_{\kappa_n}(u_{\kappa_n}) = c_{\kappa_n}, \; J'_{\kappa_n}(u_{\kappa_n}) = 0. \tag{24}$$

By Lemma 2.4 and (24), we have $c_{\kappa_n} \geq c > 0$ and $\langle J'_{\kappa_n}(u_{\kappa_n}), u^-_{\kappa_n}\rangle = 0$. Hence, $u_{\kappa_n} \in P\backslash\{0\}$. In the following, we first claim that $\{\kappa_n\}$ in $W^{1,p}(\Omega)$. Assume by contradiction that, for a subsequence, $\|u_{\kappa_n}\|_{1,p} \to \infty$. Put $w_n = \frac{u_{\kappa_n}}{\|u_{\kappa_n}\|_{1,p}}$. Hence we have, for $v \in W^{1,p}(\Omega)$,

$$\begin{aligned} &\frac{1}{\|u_{\kappa_n}\|^{p-2}_{1,p}} \int_\Omega (\nabla w_n \nabla v + w_n v)\mathrm{d}x \\ &+ \int_\Omega |\nabla w_n|^{p-2}\nabla w_n \cdot \nabla v \mathrm{d}x + \int_\Omega |w_n|^{p-2} w_n v \mathrm{d}x \\ &= \kappa_n f_\infty \int_{\partial\Omega} w_n^{p-2} v + \kappa_n \int_{\partial\Omega} \frac{f(x,u_{\kappa_n}) - f_\infty u^{p-1}_{\kappa_n}}{\|u_{\kappa_n}\|^{p-2}_{1,p}} v \mathrm{d}\sigma. \end{aligned} \tag{25}$$

Since $\{\kappa_n\}$ is bounded in $W^{1,p}(\Omega)$, for a further subsequence, $w_n \rightharpoonup w$ in $P \subset W^{1,p}(\Omega)$, $w_n \to w$ in $L^p(\Omega)$ and by the trace embedding $W^{1,p}(\Omega) \to L^p(\partial\Omega)$, $w_n \to w$ in $L^p(\partial\Omega)$. Letting $v = w_n - w$ in (25), we get

$$\int_\Omega |\nabla w_n|^{p-2}\nabla w_n \cdot \nabla(w_n - w)\mathrm{d}x + \int_\Omega |w_n|^{p-2} w_n (w_n - w)\mathrm{d}x = 0. \tag{26}$$

Thus by (S_+) property, we have $w_n \to w$ in $W^{1,p}(\Omega)$. Passing to the limit in (25), we obtain by Lemma 2.7 that

$$\begin{aligned} &\int_\Omega |\nabla w|^{p-2}\nabla w \cdot \nabla v \mathrm{d}x + \int_\Omega |w|^{p-2} w v \mathrm{d}x \\ &= f_\infty \int_{\partial\Omega} w^{p-1} v \mathrm{d}\sigma \; \forall v \in W^{1,p}(\Omega). \end{aligned} \tag{27}$$

From (27) and the fact that $\|w\|_{1,p} = 1$, we get that $f_\infty = \lambda_1$, which contradicts the assumption $f_\infty > \lambda_1$. Since $\kappa_n \to 1^-$, we can show that

$$J'_1(u_{\kappa_n}) \to 0 \text{ in } W^{1,p}(\Omega)^*, \; n \to \infty. \tag{28}$$

In fact, for any $v \in W^{1,p}(\Omega)$, it follows from (17), Holder's inequality, and trace embedding $W^{1,p}(\Omega) \to L^p(\partial\Omega)$ that

$$\begin{aligned} \left|\int_{\partial\Omega} f(x,u_{\kappa_n}) v \mathrm{d}\sigma\right| &\leq C_1 \int_{\partial\Omega} |u_{\kappa_n}||v|\mathrm{d}\sigma + C_2 \int_{\partial\Omega} |u_{\kappa_n}|^{p-1}|v|\mathrm{d}\sigma \\ &\leq C_7 \|v\|_{1,p}. \end{aligned} \tag{29}$$

Furthermore, (24) implies that

$$\begin{aligned}\langle J_1'(u_{\kappa_n}),v\rangle + (1-u_{\kappa_n})\int_{\partial\Omega} f(x,u_{\kappa_n})v\mathrm{d}\sigma \\ = \langle J_1'(u_{\kappa_n}),v\rangle = 0,\ v\in W^{1,p}(\Omega).\end{aligned} \tag{30}$$

Hence, $J_1'(u_{\kappa_n}) \to 0$ in $W^{1,p}(\Omega)^*$, as $n\to\infty$. By Lemma 2.5, $\{u_{\kappa_n}\}$ has a convergent subsequence. Without loss of generality, we may assume that $u_{\kappa_n} \to u$ as $n\to\infty$. According to Lemma 2.5, (24) and

$$\left|\int_{\partial\Omega} F(x,u_{\kappa_n})v\mathrm{d}\sigma\right| \le C_8, \tag{31}$$

we have

$$\begin{aligned} J_1(u) &= \lim_{n\to\infty} J_1(u_{\kappa_n}) = \lim_{n\to\infty} J_{\kappa_n}(u_{\kappa_n}) \ge c > 0,\\ J_1'(u) &= \lim_{n\to\infty} J_1'(u_{\kappa_n}) = 0.\end{aligned} \tag{32}$$

The standard process shows that u is a positive solution to $(S_{p,2})$. The proof is completed. □

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University Moulay Ismail of Meknes
FST Errachidia
LMIMA Laboratory, ROLALI Group
Errachidia, Morocco
E-mail: abdelmajidboukhsas@gmail.com
https://orcid.org/0000-0002-9317-8232

Regional Centre of Trades Education and Training
Department of Mathematics
Oujda, Morocco
E-mail: abdellahzerouali@yahoo.fr
https://orcid.org/0000-0001-9090-4094

University Mohammed First
Faculty of Sciences
Department of Mathematics
Oujda, Morocco
E-mail: chakrone@yahoo.fr
https://orcid.org/0000-0002-2208-4220

University Moulay Ismail of Meknes
FST Errachidia
Department of Mathematics
Errachidia, Morocco
E-mail: karembelf@gmail.com
https://orcid.org/0000-0002-7455-5434