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WEAK OPENNESS AND WEAK CONTINUITY IN IDEAL TOPOLOGICAL SPACES

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Abstract. Our purpose is to introduce the concepts of weakly $\star$-open functions and weakly $\star$-continuous functions. Moreover, some characterizations of weakly $\star$-continuous functions and $\theta(\star)$-continuous functions are investigated. In particular, the relationships between weakly $\star$-continuous functions and $\theta(\star)$-continuous functions are established.

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1. INTRODUCTION

In 1984, Rose [21] introduced the notion of weakly open functions. Rose and Janković [23] have defined the notion of weakly closed functions and investigated some of the fundamental properties of weakly open and weakly closed functions. Caldas and Navalagi [4] introduced the notions of weakly semi-open and weakly semi-closed functions as a new generalization of weakly open and weakly closed functions, respectively. Noiri et al. [17] introduced a new class of functions called weakly b-open functions which is a generalization of weakly semi-open functions and investigated some characterizations concerning weakly b-open functions. Ekici [7] introduced the notion of weakly BR-continuous functions and obtained some characterizations of weakly BR-continuous functions and the relationships among weakly BR-continuous functions, strongly θ-b-continuous functions, weakly clopen functions and the other related functions. Caldas et al. [3] introduced the concept of weakly BR-closed functions and investigated some characterizations of weakly BR-closed functions. In 2011, Caldas et al. [2] introduced and studied a new class of functions by using the notions of b-θ-open sets and b-θ-closure operator called weakly BR-open functions. In [18], the present author introduced a new notion of weakly M-open functions as functions defined between sets satisfying some minimal conditions and obtained some characterizations of such functions.

The concept of weak continuity due to Levine [16] is one of the most important weak forms of continuity in topological spaces. Rose [22] has introduced

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the notion of subweakly continuous functions and investigated the relationships between subweak continuity and weak continuity. Ekici et al. [8] established a new class of functions called λ-continuous functions which is weaker than λ-continuous functions and investigated some fundamental properties of weakly λ-continuous functions. Popa and Noiri [19] introduced the notion of weakly (τ, m)-continuous functions as functions from a topological space into a set satisfying some minimal conditions and investigated several characterizations of such functions. Moreover, the present author [20] introduced the concept of weakly M-continuous functions as functions from a set satisfying some minimal conditions into a set satisfying some minimal conditions and investigated some characterizations of weakly M-continuous functions. The notion of ideals in topological spaces has been studied by Kuratowski [15] and Vaidyanathaswamy [24]. Janković and Hamlett [14] investigated further properties of ideal topological spaces. Hatir and Noiri [13] have introduced the notion of semi-$\mathscr{I}$-open sets to obtain decomposition of continuity. In [10], the present author introduced the notions of weakly semi-$\mathscr{I}$-open sets and weakly semi-$\mathscr{I}$-continuous functions.

The paper is organized as follows. In Section 3, we introduce and study the notion of weakly $\star$-open functions. Section 4 is devoted to introducing and studying weakly $\star$-continuous functions and $\theta(\star)$-continuous functions. Moreover, the relationships between weakly $\star$-continuous functions and $\theta(\star)$-continuous functions are discussed.

2. PRELIMINARIES

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated.

In a topological space (X, τ), the closure and the interior of any subset A of X will denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

An ideal $\mathscr{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties:

(1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$;
(2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$.

A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathscr{I})$. For an ideal topological space $(X, \tau, \mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X \mid U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$.

In [15], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ and $\mathrm{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$. For any ideal topological space $(X, \tau, \mathscr{I})$, there exists a topology $\tau^\star(\mathscr{I})$ finer than τ, generated by $\mathscr{B}(\mathscr{I}, \tau) = \{U - I_0 \mid U \in \tau \text{ and } I_0 \in \mathscr{I}\}$, but in general

$\mathscr{B}(\mathscr{I},\tau)$ is not always a topology [14]. A subset A is said to be $\star$-closed [14] if $A^\star \subseteq A$. The complement of a $\star$-closed set is called $\star$-open. The interior of a subset A in $(X,\tau^\star(\mathscr{I}))$ is denoted by $\mathrm{Int}^\star(A)$.

DEFINITION 2.1 ([25]). Let A be a subset of an ideal topological space $(X,\tau,\mathscr{I})$. A point $x \in X$ is called a *θ-$\mathscr{I}$-cluster point* of A if $\mathrm{Cl}^\star(U)\cap A \neq \emptyset$ for every $U \in \tau$ containing x. The set of all θ-$\mathscr{I}$-cluster points of A is called the *θ-$\mathscr{I}$-closure* of A and is denoted by $\mathrm{Cl}_{\theta_\imath}(A)$. A point $x \in X$ is called a *θ-$\mathscr{I}$-interior point* of A if $\mathrm{Cl}^\star(U) \subseteq A$ for some $U \in \tau$ containing x. The set of all θ-$\mathscr{I}$-interior points of A is called the *θ-$\mathscr{I}$-interior* of A and is denoted by $\mathrm{Int}_{\theta_\imath}(A)$.

LEMMA 2.2. *For subsets A and B of an ideal topological space $(X,\tau,\mathscr{I})$, the following properties hold:*

(1) $Cl_{\theta_\imath}(Cl_{\theta_\imath}(A)) = Cl_{\theta_\imath}(A)$.
(2) *If $A \subseteq B$, then $Cl_{\theta_\imath}(A) \subseteq Cl_{\theta_\imath}(B)$.*
(3) $Cl_{\theta_\imath}(X-A) = X - Int_{\theta_\imath}(A)$.
(4) $Int_{\theta_\imath}(X-A) = X - Cl_{\theta_\imath}(A)$.

DEFINITION 2.3. Let A be a subset of an ideal topological space $(X,\tau,\mathscr{I})$. A point $x \in X$ is called

(i) a *θ-$\star$-cluster point* of A if $\mathrm{Cl}(U)\cap A \neq \emptyset$ for every $\star$-open set U containing x,
(ii) a *θ-$\star$-interior point* of A if $\mathrm{Cl}(U) \subseteq A$ for some $\star$-open set U containing x.

The set of all θ-$\star$-cluster points of A is called the *θ-$\star$-closure* of A and is denoted by $\star\mathrm{Cl}_\theta(A)$. If $A = \star\mathrm{Cl}_\theta(A)$, then A is called *θ-$\star$-closed.* The complement of a θ-$\star$-closed set is said to be *θ-$\star$-open.* The set of all θ-$\star$-interior points of A is called the *θ-$\star$-interior* of A and is denoted by $\star\mathrm{Int}_\theta(A)$.

LEMMA 2.4. *For a subset A of an ideal topological space $(X,\tau,\mathscr{I})$, the following properties are hold:*

(1) *If A is open, then $Cl^\star(A) = \star Cl_\theta(A)$.*
(2) *$\star Cl_\theta(A)$ is $\star$-closed.*

Proof. (1) In general, $\mathrm{Cl}^\star(A) \subseteq \star\mathrm{Cl}_\theta(A)$ holds. Suppose that $x \notin \mathrm{Cl}^\star(A)$. Then, there exists a $\star$-open set U containing x such that $A\cap U = \emptyset$; hence $A\cap\mathrm{Cl}(U) = \emptyset$ since A is open. Thus, $x \notin \star\mathrm{Cl}_\theta(A)$ and hence $\star\mathrm{Cl}_\theta(A) \subseteq \mathrm{Cl}^\star(A)$. This shows that $\mathrm{Cl}^\star(A) = \star\mathrm{Cl}_\theta(A)$.

(2) Let $x \in X - \star\mathrm{Cl}_\theta(A)$. Then $x \notin \star\mathrm{Cl}_\theta(A)$. There exists a $\star$-open set U_x containing x such that $\mathrm{Cl}(U_x)\cap A = \emptyset$. Thus, $\star\mathrm{Cl}_\theta(A)\cap U_x = \emptyset$ and hence $x \in U_x \subseteq X - \star\mathrm{Cl}_\theta(A)$. Thus, $X - \star\mathrm{Cl}_\theta(A) = \cup_{x\in X-\star\mathrm{Cl}_\theta(A)} U_x$ is $\star$-open. This shows that $\star\mathrm{Cl}_\theta(A)$ is $\star$-closed. □

3. CHARACTERIZATIONS OF WEAKLY $\star$-OPEN FUNCTIONS

In this section, we introduce the concept of weakly $\star$-open functions and investigate some characterizations of weakly $\star$-open functions.

Definition 3.1. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be *weakly $\star$-open* if $f(U) \subseteq \mathrm{Int}^{\star}(f(\mathrm{Cl}^{\star}(U)))$ for each $U \in \tau$.

Theorem 3.2. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is weakly $\star$-open;*
(2) *$f(Int_{\theta\iota}(A)) \subseteq Int^{\star}(f(A))$ for every subset A of X;*
(3) *$Int_{\theta\iota}(f^{-1}(B)) \subseteq f^{-1}(Int^{\star}(B))$ for every subset B of Y;*
(4) *$f^{-1}(Cl^{\star}(B)) \subseteq Cl_{\theta\iota}(f^{-1}(B))$ for every subset B of Y;*
(5) *for each $x \in X$ and each open set U of X containing x, there exists an open set V of Y containing $f(x)$ such that $V \subseteq f(Cl^{\star}(U))$.*

Proof. (1) $\Rightarrow$ (2): Let A be any subset of X and $x \in \mathrm{Int}_{\theta\iota}(A)$. Then, there exists $U \in \tau$ such that $x \in U \subseteq \mathrm{Cl}^{\star}(U) \subseteq A$ and hence $f(x) \in f(U) \subseteq f(\mathrm{Cl}^{\star}(U)) \subseteq f(A)$. Since f is weakly $\star$-open, $f(U) \subseteq \mathrm{Int}^{\star}(f(\mathrm{Cl}^{\star}(U))) \subseteq \mathrm{Int}^{\star}(f(A))$ and $x \in f^{-1}(\mathrm{Int}^{\star}(f(A)))$. Thus, $\mathrm{Int}_{\theta\iota}(A) \subseteq f^{-1}(\mathrm{Int}^{\star}(f(A)))$ and $f(\mathrm{Int}_{\theta\iota}(A)) \subseteq \mathrm{Int}^{\star}(f(A))$.

(2) $\Rightarrow$ (3): Let B be any subset of Y. By (2), we have $f(\mathrm{Int}_{\theta\iota}(f^{-1}(B))) \subseteq \mathrm{Int}^{\star}(f(f^{-1}(B))) \subseteq \mathrm{Int}^{\star}(B)$. Thus, $\mathrm{Int}_{\theta\iota}(f^{-1}(B)) \subseteq f^{-1}(\mathrm{Int}^{\star}(B))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y. By (3),

$$\begin{aligned} X - \mathrm{Cl}_{\theta\iota}(f^{-1}(B)) &= \mathrm{Int}_{\theta\iota}(X - f^{-1}(B)) \\ &= \mathrm{Int}_{\theta\iota}(f^{-1}(Y - B)) \\ &\subseteq f^{-1}(\mathrm{Int}^{\star}(Y - B)) \\ &= f^{-1}(Y - \mathrm{Cl}^{\star}(B)) = X - f^{-1}(\mathrm{Cl}^{\star}(B)) \end{aligned}$$

and hence $f^{-1}(\mathrm{Cl}^{\star}(B)) \subseteq \mathrm{Cl}_{\theta\iota}(f^{-1}(B))$.

(4) $\Rightarrow$ (5): Let $x \in X$ and $U \in \tau$ containing x. By (4), we have

$$f^{-1}(\mathrm{Cl}^{\star}(Y - \mathrm{Cl}^{\star}(U))) \subseteq \mathrm{Cl}_{\theta\iota}(f^{-1}(Y - \mathrm{Cl}^{\star}(U))).$$

Since $f^{-1}(\mathrm{Cl}^{\star}(Y - \mathrm{Cl}^{\star}(U))) = X - f^{-1}(\mathrm{Int}^{\star}(f(\mathrm{Cl}^{\star}(U))))$ and

$$\begin{aligned} \mathrm{Cl}_{\theta\iota}(f^{-1}(Y - f(\mathrm{Cl}^{\star}(U)))) &= \mathrm{Cl}_{\theta\iota}(X - f^{-1}(f(\mathrm{Cl}^{\star}(U)))) \\ &\subseteq \mathrm{Cl}_{\theta\iota}(X - \mathrm{Cl}^{\star}(U)) \\ &= X - \mathrm{Int}_{\theta\iota}(\mathrm{Cl}^{\star}(U)) \subseteq X - U, \end{aligned}$$

$X - f^{-1}(\mathrm{Int}^{\star}(f(\mathrm{Cl}^{\star}(U)))) \subseteq X - U$. Thus, $U \subseteq f^{-1}(\mathrm{Int}^{\star}(f(\mathrm{Cl}^{\star}(U))))$ and hence $f(U) \subseteq \mathrm{Int}^{\star}(f(\mathrm{Cl}^{\star}(U)))$. Since $f(x) \in \mathrm{Int}^{\star}(f(\mathrm{Cl}^{\star}(U)))$, there exists a $\star$-open set V of Y such that $f(x) \in V \subseteq f(\mathrm{Cl}^{\star}(U))$.

(5) $\Rightarrow$ (1): Let $U \in \tau$ and $x \in U$. By (5), there exists a $\star$-open set V of Y containing $f(x)$ such that $V \subseteq f(\mathrm{Cl}^\star(U))$. Hence, we have

$$f(x) \in V \subseteq \mathrm{Int}^\star(f(\mathrm{Cl}^\star(U)))$$

for each $x \in U$. Consequently, we obtain $f(U) \subseteq \mathrm{Int}^\star(f(\mathrm{Cl}^\star(U)))$. This shows that f is weakly $\star$-open. □

Definition 3.3 ([6]). A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be $\star$-dense if $\mathrm{Cl}^\star(A) = X$.

Definition 3.4 ([9]). An ideal topological space $(X, \tau, \mathscr{I})$ is said to be $\star$-hyperconnected if V is $\star$-dense for every nonempty open set V of X.

Theorem 3.5. *Let $(X, \tau, \mathscr{I})$ be a $\star$-hyperconnected space. Then a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is weakly $\star$-open if and only if $f(X)$ is $\star$-open in $(Y, \sigma, \mathscr{J})$.*

Proof. Let f be weakly $\star$-open. Since $X \in \tau$, $f(X) \subseteq \mathrm{Int}^\star(f(\mathrm{Cl}^\star(X))) = \mathrm{Int}^\star(f(X))$ and hence $f(X) \subseteq \mathrm{Int}^\star(f(X))$. Thus, $f(X)$ is $\star$-open in $(Y, \sigma, \mathscr{J})$.

Conversely, suppose that $f(X)$ is $\star$-open in $(Y, \sigma, \mathscr{J})$. Let $U \in \tau$. Then $f(U) \subseteq f(X) = \mathrm{Int}^\star(f(X)) = \mathrm{Int}^\star(f(\mathrm{Cl}^\star(U)))$. Consequently, we obtain $f(U) \subseteq \mathrm{Int}^\star(f(\mathrm{Cl}^\star(U)))$. This shows that f is weakly $\star$-open. □

4. ON WEAKLY $\star$-CONTINUOUS FUNCTIONS

In this section, we introduce the concepts of weakly $\star$-continuous functions and $\theta(\star)$-continuous functions. Some characterizations of weakly $\star$-continuous functions and $\theta(\star)$-continuous functions are investigated. Moreover, the relationships between weakly $\star$-continuous functions and $\theta(\star)$-continuous functions are discussed.

Definition 4.1. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be *weakly $\star$-continuous at $x \in X$* if for each $\star$-open set V of Y containing $f(x)$, there exists a $\star$-open set U of X containing x such that $f(U) \subseteq \mathrm{Cl}(V)$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be *weakly $\star$-continuous* if it has that property at each point $x \in X$.

Theorem 4.2. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is weakly $\star$-continuous at $x \in X$ if and only if for each $\star$-open set V of Y containing $f(x)$, $x \in Int^\star(f^{-1}(Cl(V)))$.*

Proof. Let f be weakly $\star$-continuous at $x \in X$ and V be any $\star$-open set of Y containing $f(x)$. Then, there exists a $\star$-open set U of X containing x such that $f(U) \subseteq \mathrm{Cl}(V)$. Thus, $x \in U \subseteq f^{-1}(\mathrm{Cl}(V))$ and hence $x \in \mathrm{Int}^\star(f^{-1}(\mathrm{Cl}(V)))$.

Conversely, let $x \in X$ and V be any $\star$-open set of Y containing $f(x)$. By the hypothesis, we have $x \in \mathrm{Int}^\star(f^{-1}(\mathrm{Cl}(V)))$. There exists a $\star$-open set U of

X such that $x \in U \subseteq f^{-1}(\mathrm{Cl}(V))$; hence $f(U) \subseteq \mathrm{Cl}(V)$. This shows that f is weakly $\star$-continuous at x. □

THEOREM 4.3. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is weakly $\star$-continuous if and only if $f^{-1}(V) \subseteq Int^\star(f^{-1}(Cl(V)))$ for every $\star$-open set V of Y.*

Proof. Let V be any $\star$-open set of Y and $x \in f^{-1}(V)$. Then $f(x) \in V$. Since f is weakly $\star$-continuous at x, by Theorem 4.2, $x \in \mathrm{Int}^\star(f^{-1}(\mathrm{Cl}(V)))$ and hence $f^{-1}(V) \subseteq \mathrm{Int}^\star(f^{-1}(\mathrm{Cl}(V)))$.

Conversely, let $x \in X$ and V be any $\star$-open set of Y containing $f(x)$. Then $x \in f^{-1}(V) \subseteq \mathrm{Int}^\star(f^{-1}(\mathrm{Cl}(V)))$ and hence $x \in \mathrm{Int}^\star(f^{-1}(\mathrm{Cl}(V)))$. By Theorem 4.2, f is weakly $\star$-continuous at x. This shows that f is weakly $\star$-continuous. □

THEOREM 4.4. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is weakly $\star$-continuous;*
(2) *$f(Cl^\star(A)) \subseteq \star Cl_\theta(f(A))$ for every subset A of X;*
(3) *$Cl^\star(f^{-1}(B)) \subseteq f^{-1}(\star Cl_\theta(B))$ for every subset B of Y;*
(4) *$Cl^\star(f^{-1}(V)) \subseteq f^{-1}(Cl^\star(V))$ for every open set V of Y.*

Proof. (1) ⇒ (2): Let A be any subset of X. Suppose that $x \in \mathrm{Cl}^\star(A)$ and G be any $\star$-open set of Y containing $f(x)$. Since f is weakly $\star$-continuous, there exists a $\star$-open set U of X containing x such that $f(U) \subseteq \mathrm{Cl}(G)$. Since $x \in \mathrm{Cl}^\star(A)$, we have $U \cap A \neq \emptyset$. It follows that $\emptyset \neq f(U) \cap f(A) \subseteq \mathrm{Cl}(G) \cap f(A)$. Thus, $\mathrm{Cl}(G) \cap f(A) \neq \emptyset$ and $f(x) \in \star\mathrm{Cl}_\theta(f(A))$. This shows that $f(\mathrm{Cl}^\star(A)) \subseteq \star\mathrm{Cl}_\theta(f(A))$.

(2) ⇒ (3): Let B be any subset of Y. By (2),

$$f(\mathrm{Cl}^\star(f^{-1}(B))) \subseteq \star\mathrm{Cl}_\theta(f(f^{-1}(B))) \subseteq \star\mathrm{Cl}_\theta(B)$$

and hence $\mathrm{Cl}^\star(f^{-1}(B)) \subseteq f^{-1}(\star\mathrm{Cl}_\theta(B))$.

(3) ⇒ (4): Let V be any open set of Y. By Lemma 2.4, $\mathrm{Cl}^\star(V) = \star\mathrm{Cl}_\theta(V)$. Thus, the proof is obvious.

(4) ⇒ (1): Let $x \in X$ and V be any $\star$-open set of Y containing $f(x)$. Since

$$V \cap (Y - \mathrm{Cl}(V)) = \emptyset,$$

$f(x) \notin \mathrm{Cl}^\star(Y - \mathrm{Cl}(V))$ and hence $x \notin f^{-1}(\mathrm{Cl}^\star(Y - \mathrm{Cl}(V)))$. By (4), we have $x \notin \mathrm{Cl}^\star(f^{-1}(Y - \mathrm{Cl}(V)))$. Therefore, there exists a $\star$-open set U of X containing x such that $U \cap f^{-1}(Y - \mathrm{Cl}(V)) = \emptyset$; hence $f(U) \cap (Y - \mathrm{Cl}(V)) = \emptyset$. This implies that $f(U) \subseteq \mathrm{Cl}(V)$. Thus, f is weakly $\star$-continuous. □

THEOREM 4.5. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is weakly $\star$-continuous;*
(2) *$f^{-1}(V) \subseteq Int^\star(f^{-1}(Cl(V)))$ for every $\star$-open set V of Y;*
(3) *$Cl^\star(f^{-1}(Int(F))) \subseteq f^{-1}(F)$ for every $\star$-closed set F of Y;*
(4) *$Cl^\star(f^{-1}(Int(Cl^\star(B)))) \subseteq f^{-1}(Cl^\star(B))$ for every subset B of Y;*
(5) *$f^{-1}(Int^\star(B)) \subseteq Int^\star(f^{-1}(Cl(Int^\star(B))))$ for every subset B of Y.*

Proof. (1) ⇒ (2): This follows from Theorem 4.3.

(2) ⇒ (3): Let F be any $\star$-closed set of Y. Then $Y - F$ is $\star$-open in Y and by (2),

$$\begin{aligned} X - f^{-1}(F) = f^{-1}(Y - F) &\subseteq \mathrm{Int}^\star(f^{-1}(\mathrm{Cl}(Y - F))) \\ &= \mathrm{Int}^\star(f^{-1}(Y - \mathrm{Int}(F))) \\ &= X - \mathrm{Cl}^\star(f^{-1}(\mathrm{Int}(F))). \end{aligned}$$

Thus, $\mathrm{Cl}^\star(f^{-1}(\mathrm{Int}(F))) \subseteq f^{-1}(F)$.

(3) ⇒ (4): Let B be any subset of Y. Since $\mathrm{Cl}^\star(B)$ is $\star$-closed and by (3),

$$\mathrm{Cl}^\star(f^{-1}(\mathrm{Int}(\mathrm{Cl}^\star(B)))) \subseteq f^{-1}(\mathrm{Cl}^\star(B)).$$

(4) ⇒ (5): Let B be any subset of Y. By (4),

$$\begin{aligned} f^{-1}(\mathrm{Int}^\star(B)) &= X - f^{-1}(\mathrm{Cl}^\star(Y - B)) \\ &\subseteq X - \mathrm{Cl}^\star(f^{-1}(\mathrm{Int}(\mathrm{Cl}^\star(Y - B)))) \\ &= \mathrm{Int}^\star(f^{-1}(\mathrm{Cl}(\mathrm{Int}^\star(B)))). \end{aligned}$$

Thus, we get the result.

(5) ⇒ (1): Let V be any $\star$-open set of Y. By (5), we have $f^{-1}(V) = f^{-1}(\mathrm{Int}^\star(V)) \subseteq \mathrm{Int}^\star(f^{-1}(\mathrm{Cl}(\mathrm{Int}^\star(V)))) = \mathrm{Int}^\star(f^{-1}(\mathrm{Cl}(V)))$. Thus, by Theorem 4.3, f is weakly $\star$-continuous. □

DEFINITION 4.6. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be:

(1) *$\mathscr{I}$-R closed* [1] if $A = \mathrm{Cl}^\star(\mathrm{Int}(A))$;
(2) *pre-$\mathscr{I}$-open* [5] if $A \subseteq \mathrm{Int}(\mathrm{Cl}^\star(A))$;
(3) *semi-$\mathscr{I}$-open* [12] if $A \subseteq \mathrm{Cl}^\star(\mathrm{Int}(A))$;
(4) *strong β-$\mathscr{I}$-open* [11] if $A \subseteq \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A)))$.

THEOREM 4.7. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is weakly $\star$-continuous;*
(2) *$Cl^\star(f^{-1}(Int(F))) \subseteq f^{-1}(F)$ for every $\mathscr{J}$-R closed set F of Y;*

(3) $Cl^{\star}(f^{-1}(Int(Cl^{\star}(V)))) \subseteq f^{-1}(Cl^{\star}(V))$ *for every strong* β-$\mathscr{J}$*-open set* V *of* Y*;*
(4) $Cl^{\star}(f^{-1}(Int(Cl^{\star}(V)))) \subseteq f^{-1}(Cl^{\star}(V))$ *for every semi-*$\mathscr{J}$*-open set* V *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let F be any $\mathscr{J}$-R closed set of Y. Then $\text{Int}(F)$ is open, by Theorem 4.4, $\text{Cl}^{\star}(f^{-1}(\text{Int}(F))) \subseteq f^{-1}(\text{Cl}^{\star}(\text{Int}(F)))$. Since F is $\mathscr{J}$-R closed, we have $\text{Cl}^{\star}(f^{-1}(\text{Int}(F))) \subseteq f^{-1}(F)$.

(2) $\Rightarrow$ (3): Let V be any strong β-$\mathscr{J}$-open set of Y. Then $\text{Cl}^{\star}(V) = \text{Cl}^{\star}(\text{Int}(\text{Cl}^{\star}(V)))$ and hence $\text{Cl}^{\star}(V)$ is $\mathscr{J}$-R closed. By (2),

$$\text{Cl}^{\star}(f^{-1}(\text{Int}(\text{Cl}^{\star}(V)))) \subseteq f^{-1}(\text{Cl}^{\star}(V)).$$

(3) $\Rightarrow$ (4): The proof is obvious.

(4) $\Rightarrow$ (1): Let V be any open set of Y. Then V is strong β-$\mathscr{J}$-open and by (4), we have $\text{Cl}^{\star}(f^{-1}(V)) \subseteq \text{Cl}^{\star}(f^{-1}(\text{Int}(\text{Cl}^{\star}(V)))) \subseteq f^{-1}(\text{Cl}^{\star}(V))$. Hence, by Theorem 4.4, f is weakly $\star$-continuous. □

THEOREM 4.8. *For a function* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$*, the following properties are equivalent:*
(1) f *is weakly* $\star$*-continuous;*
(2) $Cl^{\star}(f^{-1}(Int(Cl^{\star}(V)))) \subseteq f^{-1}(Cl^{\star}(V))$ *for every pre-*$\mathscr{J}$*-open set* V *of* Y*;*
(3) $Cl^{\star}(f^{-1}(V)) \subseteq f^{-1}(Cl^{\star}(V))$ *for every pre-*$\mathscr{J}$*-open set* V *of* Y*;*
(4) $Cl^{\star}(f^{-1}(V)) \subseteq f^{-1}(Cl^{\star}(V))$ *for every open set* V *of* Y*.*

Proof. (1) $\Rightarrow$ (2): Let V be any pre-$\mathscr{J}$-open set of Y. Then $\text{Cl}^{\star}(V) = \text{Cl}^{\star}(\text{Int}(\text{Cl}^{\star}(V)))$ and hence $\text{Cl}^{\star}(V)$ is $\mathscr{J}$-R closed. By Theorem 4.7,

$$\text{Cl}^{\star}(f^{-1}(\text{Int}(\text{Cl}^{\star}(V)))) \subseteq f^{-1}(\text{Cl}^{\star}(V)).$$

(2) $\Rightarrow$ (3): Let V be any pre-$\mathscr{J}$-open set of Y. Then $V \subseteq \text{Int}(\text{Cl}^{\star}(V))$ and by (2), $\text{Cl}^{\star}(f^{-1}(V)) \subseteq \text{Cl}^{\star}(f^{-1}(\text{Int}(\text{Cl}^{\star}(V)))) \subseteq f^{-1}(\text{Cl}^{\star}(V))$.

(3) $\Rightarrow$ (4): The proof is obvious.

(4) $\Rightarrow$ (1): It follows from Theorem 4.4. □

DEFINITION 4.9. An ideal topological space $(X, \tau, \mathscr{I})$ is called $\star$*-Hausdorff* (resp. $\star$*-Urysohn*) if for each distinct points $x, y \in X$, there exist $\star$-open sets U and V containing x and y, respectively, such that $U \cap V = \emptyset$ (resp. $\text{Cl}(U) \cap \text{Cl}(V) = \emptyset$).

THEOREM 4.10. *If* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ *is a weakly* $\star$*-continuous injection and* $(Y, \sigma, \mathscr{J})$ *is* $\star$*-Urysohn, then* $(X, \tau, \mathscr{I})$ *is* $\star$*-Hausdorff.*

Proof. Let x, y be any distinct points of X. Then $f(x) \neq f(y)$. Since $(Y, \sigma, \mathscr{J})$ is $\star$-Urysohn, there exist $\star$-open sets U and V of Y containing $f(x)$ and $f(y)$, respectively, such that $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$. Since f is weakly $\star$-continuous, there exist $\star$-open sets G and W of X containing x and y, respectively, such that $f(G) \subseteq \mathrm{Cl}(U)$ and $f(W) \subseteq \mathrm{Cl}(V)$. This implies that $G \cap W = \emptyset$. Thus, $(X, \tau, \mathscr{I})$ is $\star$-Hausdorff. □

Definition 4.11. A subset K of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *$\mathscr{I}(\star)$-closed* (resp. *$\star$-compact*) *relative to* $(X, \tau, \mathscr{I})$ if for each cover $\{V_\gamma \mid \gamma \in \Gamma\}$ of K by $\star$-open sets of X, there exists finite subset Γ_0 of Γ such that $K \subseteq \cup\{\mathrm{Cl}(V_\gamma) \mid \gamma \in \Gamma_0\}$ (resp. $K \subseteq \cup\{V_\gamma \mid \gamma \in \Gamma\}$). If X is $\mathscr{I}(\star)$-closed (resp. $\star$-compact) relative to $(X, \tau, \mathscr{I})$, then $(X, \tau, \mathscr{I})$ is said to be *$\mathscr{I}(\star)$-closed* (resp. *$\star$-compact*).

Theorem 4.12. *If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is a weakly $\star$-continuous function and K is $\star$-compact relative to $(X, \tau, \mathscr{I})$, then $f(K)$ is $\mathscr{J}(\star)$-closed relative to $(Y, \sigma, \mathscr{J})$.*

Proof. Let K be $\star$-compact relative to $(X, \tau, \mathscr{I})$. Let $\{V_\gamma \mid \gamma \in \Gamma\}$ be any cover of $f(K)$ by $\star$-open sets of $(Y, \sigma, \mathscr{J})$. For each $x \in K$, there exists $\gamma(x) \in \Gamma$ such that $f(x) \in V_{\gamma(x)}$. Since f is weakly $\star$-continuous, there exists a $\star$-open set $U(x)$ containing x such that $f(U(x)) \subseteq \mathrm{Cl}(V_{\gamma(x)})$. The family $\{U(x) \mid x \in K\}$ is a cover of K by $\star$-open sets of X. Since K is $\star$-compact relative to $(X, \tau, \mathscr{I})$, there exist a finite number of points, say, $x_1, x_2, ..., x_n$ in K such that $K \subseteq \cup\{U(x_k) \mid x_k \in K, 1 \leq k \leq n\}$. Thus,

$$\begin{aligned} f(K) &\subseteq \cup\{f(U(x_k)) \mid x_k \in K, 1 \leq k \leq n\} \\ &\subseteq \cup\{\mathrm{Cl}(V_{\gamma(x_k))} \mid x_k \in K, 1 \leq k \leq n\}. \end{aligned}$$

This shows that $f(K)$ is $\mathscr{J}(\star)$-closed relative to $(Y, \sigma, \mathscr{J})$. □

Corollary 4.13. *If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is a weakly $\star$-continuous surjection and $(X, \tau, \mathscr{I})$ is $\star$-compact, then $(Y, \sigma, \mathscr{J})$ is $\mathscr{J}(\star)$-closed.*

Definition 4.14. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be *$\theta(\star)$-continuous at $x \in X$* if for each $\star$-open set V of Y containing $f(x)$, there exists a $\star$-open set U of X containing x such that $f(\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(V)$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be *$\theta(\star)$-continuous* if it has that property at each point $x \in X$.

Remark 4.15. For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following implication holds:

$$\theta(\star)\text{-continuity} \Rightarrow \text{weak } \star\text{-continuity}.$$

The converse of the implication is not true in general. We give an example for the implication as follows.

Example 4.16. Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, \{a\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset, \{b\}\}$. Let $Y = \{1, 2, 3\}$ with a topology

$$\sigma = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, Y\}$$

and an ideal $\mathscr{J} = \{\emptyset\}$. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is defined as follows: $f(a) = 1$ and $f(b) = f(c) = 3$. Then f is weakly $\star$-continuous but f is not $\theta(\star)$-continuous.

Theorem 4.17. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is $\theta(\star)$-continuous at $x \in X$ if and only if for each $\star$-open set V of Y containing $f(x)$, $x \in \star Int_\theta(f^{-1}(Cl(V)))$.*

Proof. Let f be $\theta(\star)$-continuous at $x \in X$ and V be any $\star$-open set of Y containing $f(x)$. Then, there exists a $\star$-open set U of X containing x such that $f(\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(V)$. Thus, $x \in U \subseteq \mathrm{Cl}(U) \subseteq f^{-1}(\mathrm{Cl}(V))$ and hence $x \in \star\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}(V)))$.

Conversely, let V be any $\star$-open set of Y containing $f(x)$. Then, by the hypothesis we have $x \in \star\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}(V)))$. There exists a $\star$-open set U of X such that $x \in U \subseteq \mathrm{Cl}(U) \subseteq f^{-1}(\mathrm{Cl}(V))$; hence $f(\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(V)$. This shows that f is $\theta(\star)$-continuous at $x \in X$. □

Theorem 4.18. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is $\theta(\star)$-continuous if and only if $f^{-1}(V) \subseteq \star Int_\theta(f^{-1}(Cl(V)))$ for every $\star$-open set V of Y.*

Proof. Let V be any $\star$-open set of Y and $x \in f^{-1}(V)$. Then $f(x) \in V$. Since f is $\theta(\star)$-continuous at x, by Theorem 4.17 we have $x \in \star\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}(V)))$ and hence $f^{-1}(V) \subseteq \star\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}(V)))$.

Conversely, let $x \in X$ and V be any $\star$-open set of Y containing $f(x)$. Then $x \in f^{-1}(V) \subseteq \star\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}(V)))$ and hence $x \in \star\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}(V)))$. Thus, by Theorem 4.17, f is $\theta(\star)$-continuous. □

Theorem 4.19. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is $\theta(\star)$-continuous;*
(2) *$f(\star Cl_\theta(A)) \subseteq \star Cl_\theta(f(A))$ for every subset A of X;*
(3) *$\star Cl_\theta(f^{-1}(B)) \subseteq f^{-1}(\star Cl_\theta(B))$ for every subset B of Y.*

Proof. (1) ⇒ (2): Let A be any subset of X. Let $x \in \star\mathrm{Cl}_\theta(A)$ and G be any $\star$-open set of Y containing $f(x)$. Since f is $\theta(\star)$-continuous, there exists a $\star$-open set U of X containing x such that $f(\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(G)$. Since $x \in \star\mathrm{Cl}_\theta(A)$, we have $\mathrm{Cl}(U) \cap A \neq \emptyset$. It follows that $\emptyset \neq f(\mathrm{Cl}(U)) \cap f(A) \subseteq \mathrm{Cl}(G) \cap f(A)$. Hence, $\mathrm{Cl}(G) \cap f(A) \neq \emptyset$ and $f(x) \in \star\mathrm{Cl}_\theta(f(A))$. This shows that $f(\star\mathrm{Cl}_\theta(A)) \subseteq \star\mathrm{Cl}_\theta(f(A))$.

(2) ⇒ (3): Let B be any subset of Y. By (2), we have $f(\star\mathrm{Cl}_\theta(f^{-1}(B))) \subseteq \star\mathrm{Cl}_\theta(f(f^{-1}(B))) \subseteq \star\mathrm{Cl}_\theta(B)$ and hence $\star\mathrm{Cl}_\theta(f^{-1}(B)) \subseteq f^{-1}(\star\mathrm{Cl}_\theta(B))$.

(3) ⇒ (1): Let $x \in X$ and V be any $\star$-open set of Y containing $f(x)$. Since

$$\mathrm{Cl}(V) \cap (Y - \mathrm{Cl}(V)) = \emptyset,$$

$f(x) \notin \star\mathrm{Cl}_\theta(Y - \mathrm{Cl}(V))$ and hence $x \notin f^{-1}(\star\mathrm{Cl}_\theta(Y - \mathrm{Cl}(V)))$. By (3), we have $x \notin \star\mathrm{Cl}_\theta(f^{-1}(Y - \mathrm{Cl}(V)))$. There exists a $\star$-open set U of X containing x such that $\mathrm{Cl}(U) \cap f^{-1}(Y - \mathrm{Cl}(V)) = \emptyset$; hence $f(\mathrm{Cl}(U)) \cap (Y - \mathrm{Cl}(V)) = \emptyset$. This shows that $f(\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(V)$. Thus, f is $\theta(\star)$-continuous. □

THEOREM 4.20. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is $\theta(\star)$-continuous if and only if $\star Cl_\theta(f^{-1}(V)) \subseteq f^{-1}(\star Cl_\theta(V))$ for every $\star$-open set V of Y.*

Proof. This is obvious from Theorem 4.19.

Conversely, let V be any $\star$-open set of Y containing $f(x)$. Since

$$\mathrm{Cl}^\star(V) \cap (Y - \mathrm{Cl}^\star(V)) = \emptyset,$$

$f(x) \notin \star\mathrm{Cl}_\theta(Y - \mathrm{Cl}^\star(V))$ and hence $x \notin f^{-1}(\star\mathrm{Cl}_\theta(Y - \mathrm{Cl}^\star(V)))$. By the hypothesis, $x \notin \star\mathrm{Cl}_\theta(f^{-1}(Y - \mathrm{Cl}^\star(V)))$ and there exists a $\star$-open set U of X containing x such that $\mathrm{Cl}(U) \cap f^{-1}(Y - \mathrm{Cl}^\star(V)) = \emptyset$. This shows that $f(\mathrm{Cl}(U)) \subseteq \mathrm{Cl}^\star(V) \subseteq \mathrm{Cl}(V)$. Therefore, f is $\theta(\star)$-continuous. □

THEOREM 4.21. *If $(X, \tau, \mathscr{I})$ is an ideal topological space and for any distinct points $x_1, x_2 \in X$, there exists a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ such that*

(1) *$(Y, \sigma, \mathscr{J})$ is $\star$-Urysohn,*
(2) *$f(x_1) \neq f(x_2)$ and*
(3) *f is $\theta(\star)$-continuous at x_1 and x_2, then $(X, \tau, \mathscr{I})$ is $\star$-Urysohn.*

Proof. Let x_1, x_2 be any distinct points of X. Then, by the hypothesis there exists a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ which satisfies the three conditions. Now let $y_i = f(x_i)$ for $i = 1, 2$. Then $y_1 \neq y_2$.

Since $(Y, \sigma, \mathscr{J})$ is $\star$-Urysohn, there exist $\star$-open sets $V_i, i = 1, 2$ such that $y_i \in V_i$ and $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$. Since f is $\theta(\star)$-continuous at x_i, there exists a $\star$-open set U_i containing x such that $f(\mathrm{Cl}(U_i)) \subseteq \mathrm{Cl}(V_i)$ for $i = 1, 2$. This implies that $\mathrm{Cl}(U_1) \cap \mathrm{Cl}(U_2) = \emptyset$. Thus, $(X, \tau, \mathscr{I})$ is $\star$-Urysohn. □

COROLLARY 4.22. *If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is a $\theta(\star)$-continuous injection and $(Y, \sigma, \mathscr{J})$ is $\star$-Urysohn, $(X, \tau, \mathscr{I})$ is $\star$-Urysohn.*

DEFINITION 4.23. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to have a *strongly $\theta(\star)$-closed graph* if for each $(x, y) \in (X \times Y) - G(f)$, there exist a $\star$-open set U of X containing x and a $\star$-open set V of Y containing y such that $[\mathrm{Cl}(U) \times \mathrm{Cl}(V)] \cap G(f) = \emptyset$.

Lemma 4.24. *A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ has a strongly $\theta(\star)$-closed graph if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist a $\star$-open set U of X containing x and a $\star$-open set V of Y containing y such that $f(Cl(U)) \cap Cl(V) = \emptyset$.*

Theorem 4.25. *If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is a $\theta(\star)$-continuous function and $(Y, \sigma, \mathscr{J})$ is $\star$-Urysohn, then $G(f)$ is strongly $\theta(\star)$-closed.*

Proof. Suppose that $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$. Since $(Y, \sigma, \mathscr{J})$ is $\star$-Urysohn, there exist $\star$-open sets V and W of Y containing y and $f(x)$, respectively, such that $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \emptyset$.

Since f is $\theta(\star)$-continuous, there exists a $\star$-open set U of X containing x such that $f(\mathrm{Cl}(U)) \subseteq \mathrm{Cl}(W)$. This implies that $f(\mathrm{Cl}(U)) \cap \mathrm{Cl}(V) = \emptyset$ and by Lemma 4.24, $G(f)$ is strongly $\theta(\star)$-closed. □

Theorem 4.26. *If $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is an injective $\theta(\star)$-continuous function with a strongly $\theta(\star)$-closed graph, then $(X, \tau, \mathscr{I})$ is $\star$-Urysohn.*

Proof. Let x and y be any distinct points of X. Then, since f is injective, we have $f(x) \neq f(y)$. Thus, $(x, f(y)) \in (X \times Y) - G(f)$. Since $G(f)$ is strongly $\theta(\star)$-closed, by Lemma 4.24 there exist a $\star$-open set U of X containing x and a $\star$-open set V of Y containing $f(y)$ such that $f(\mathrm{Cl}(U)) \cap \mathrm{Cl}(V) = \emptyset$.

Since f is $\theta(\star)$-continuous, there exists a $\star$-open set W of X containing y such that $f(\mathrm{Cl}(W)) \subseteq \mathrm{Cl}(V)$. Therefore, we have $f(\mathrm{Cl}(U)) \cap f(\mathrm{Cl}(W)) = \emptyset$ and hence $\mathrm{Cl}(U) \cap \mathrm{Cl}(W) = \emptyset$. This shows that $(X, \tau, \mathscr{I})$ is $\star$-Urysohn. □

REFERENCES

[1] A. Açikgöz and Ş. Yüksel, *Some new sets and decompositions of $A_{I\text{-}R}$-continuity, α-I-continuity, continuity via idealization*, Acta Math. Hungar., **114** (2007), 79–89.

[2] M. Caldas, E. Ekici, S. Jafari and R. M. Latif, *On weakly BR-open functions and their characterizations in topological spaces*, Demonstr. Math., **44** (2011), 159–168.

[3] M. Caldas, E. Ekici, S. Jafari and S. P. Moshokoa, *On weakly BR-closed functions between topological spaces*, Math. Commun., **14** (2009), 67–73.

[4] M. Caldas and G. Navalagi, *On weak forms of semi-open and semi-closed functions*, Missouri J. Math. Sci., **18** (2006), 165–178.

[5] J. Dontchev, *Idealization of Ganster-Reilly decomposition theorems*, arXiv:9901017, 1999.

[6] J. Dontchev, M. Ganster and D. Rose, *Ideal resolvability*, Topology Appl., **93** (1999), 1–16.

[7] E. Ekici, *Generalization of weakly clopen and strongly θ-b-continuous functions*, Chaos Solitons Fractals, **38** (2008), 79–88.

[8] E. Ekici, S. Jafari, M. Caldas and T. Noiri, *Weakly λ-continuous functions*, Novi Sad J. Math., **38** (2008), 47–56.

[9] E. Ekici and T. Noiri, *$\star$-hyperconnected ideal topological spaces*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.), **58** (2012), 121–129.

[10] E. Hatir and S. Jafari, *On weakly semi-$\mathscr{I}$-open sets and another decomposition of continuity via ideals*, Sarajevo J. Math., **2** (2006), 107–114.

[11] E. Hatir, A. Keskin and T. Noiri, *On a new decomposition of continuity via idealization*, JP J. Geom. Topol., **3** (2003), 53–64.
[12] E. Hatir and T. Noiri, *On decompositions of continuity via idealization*, Acta Math. Hungar., **96** (2002), 341–349.
[13] E. Hatir and T. Noiri, *On semi-$\mathscr{I}$-open sets and semi-$\mathscr{I}$-continuous functions*, Acta Math. Hungar., **107** (2005), 345–353.
[14] D. Janković and T. R. Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly, **97** (1990), 295–310.
[15] K. Kuratowski, *Topology*, Vol. I, Academic Press, New York, 1966.
[16] N. Levine, *A decomposition of continuity in topological spaces*, Amer. Math. Monthly, **68** (1961), 44–46.
[17] T. Noiri, A. Al-Omari and Mohd. Salmi Md. Noorani, *Weakly b-open functions*, Math. Balkanica (N.S.), **23** (2009), 1–13.
[18] T. Noiri and V. Popa, *A unified theory of weakly open functions*, Missouri J. Math. Sci., **18** (2006), 179–196.
[19] V. Popa and T. Noiri, *On weakly (τ, m)-continuous functions*, Rend. Circ. Mat. Palermo (2), **51** (2002), 295–316.
[20] V. Popa and T. Noiri, *A unified theory of weak continuity for functions*, Rend. Circ. Mat. Palermo (2), **51** (2002), 439–464.
[21] D. A. Rose, *Weak openness and almost openness*, Int. J. Math. Math. Sci., **7** (1984), 35–40.
[22] D. A. Rose, *Weak continuity and almost continuity*, Int. J. Math. Math. Sci., **7** (1984), 311–318.
[23] D. A. Rose and D. S. Janković, *Weakly closed functions and Hausdorff spaces*, Math. Nachr., **130** (1987), 105–110.
[24] R. Vaidyanathaswamy, *The localisation theory in set topology*, Proc. Indian Acad. Sci. Sect. A, **20** (1944), 51–61.
[25] Ş. Yüksel, T. H. Simsekler, Z. G. Ergul and T. Noiri, *Strongly θ-pre-$\mathscr{I}$-continuous functions*, Sci. Stud. Res. Ser. Math. Inform., **20** (2010), 111–126.

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