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# STRONGLY DEFERRED ALMOST CONVERGENCE AND DEFERRED ALMOST STATISTICAL CONVERGENCE

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**Abstract.** This paper introduces the concepts of deferred almost convergence, strongly deferred almost convergence and deferred almost statistical convergence, and investigates the relationship between these concepts. Also, it gives the notions of asymptotical deferred almost equivalence and asymptotical deferred almost statistical equivalence.

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**Key words.** Statistical convergence, almost convergence, strongly almost convergence, almost statistical convergence, deferred Cesàro mean, deferred almost convergence, deferred almost statistical convergence, asymptotical equivalence.

## 1. INTRODUCTION

A sequence $(x_k)$ is said to be strongly Cesàro summable to the number $\ell$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - \ell| = 0.$$

Nearly all of the transformations used in the theory of summability have undesirable features. For example, the Cesàro transformation of any given positive order increases ultimate bounds and oscillations of certain sequences of functions, and does not always preserve uniform convergence, or continuous convergence, of sequences of functions. Deferred Cesàro means have useful properties not possessed by Cesàro's and other well known transformations. R.P. Agnew [1] defined the deferred Cesàro mean as a generalization of Cesàro mean of real (or complex) valued sequence $(x_k)$ by

$$(D_{p,q}x)_n := \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k, \qquad n = 1, 2, 3, ...,$$

where $(p(n))$ and $(q(n))$ are sequences of nonnegative integers satisfying $p(n) < q(n)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} q(n) = \infty$.

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A sequence $(x_k)$ is said to be strongly $D_{p,q}$-convergent to $\ell \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} |x_k - \ell| = 0.$$

A sequence $(x_k)$ is said to be statistically convergent to the number $\ell \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : \quad |x_k - \ell| \ge \varepsilon\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set.

The concept of statistical convergence was first introduced by Fast [5]. In 1953 the concept arose as an example of convergence in density as introduced by Buck [2]. Schoenberg [21] studied statistical convergence as a summability method, and Zygmund [22] established a relation between it and strong summability. This idea has grown a little faster after the papers of Šalàt [20], Fridy [7], Connor [3,4]. The relation between statistical convergence and strong Cesàro summability is known (see e.g. [3]):

If a sequence $(x_k)$ is strongly Cesàro convergent to $\ell$, then $(x_k)$ is statistically convergent to $\ell$, and the converse is true if $(x_k)$ is a bounded sequence.

The idea of statistical convergence depends upon the density of subsets of the set $\mathbb{N}$ of natural numbers. The density of a subset $K$ of $\mathbb{N}$ is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_K(k)$$

provided that the limit exists, where $\chi_K$ is the characteristic function of the set $K$.

The deferred density of $K$ is defined by

$$\delta_{p,q}(K) = \frac{1}{q(n)-p(n)} |\{p(n) < k \le q(n) : \quad k \in K\}|$$

provided the limit exists.

The concept of deferred statistical convergence was introduced in [10] as follows:

A sequence $(x_k)$ is said to be deferred statistically convergent to the number $\ell \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{p(n)-q(n)} |\{p(n) < k \le q(n) : \quad |x_k - \ell| \ge \varepsilon\}| = 0.$$

A continuous linear functional $\phi$ on $\ell_\infty$, the space of real bounded sequences, is said to be a Banach limit if

(a) $\phi(x) \ge 0$ when the sequence $x = (x_k)$ has $x_k \ge 0$ for all $k$,
(b) $\phi(e) = 1$ where $e = (1, 1, 1, ...)$ and
(c) $\phi(x_{k+1}) = \phi(x_k)$ for all $x \in l_\infty$.

A sequence $(x_k) \in l_\infty$ is said to be almost convergent to the value $\ell$ if all of its Banach limits equal to $\ell$. Lorentz [12] has given the following characterization.

A bounded sequence $(x_k)$ is almost convergent to $\ell$ if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_{m+k} = \ell$$

uniformly in $m$.

Almost convergence and statistical convergence are incompatible. For example; the sequence $(x_k)$ defined by $x_k = 1$ if $k$ is even and $x_k = 0$ if $k$ is odd, is almost convergent to $1/2$, but is not statistically convergent. Now consider the sequence of 0's and 1's defined as follows:

$$\underset{\to 100 \leftarrow}{0, 0, ..., 0}, \underset{\to 10 \leftarrow}{1, 1, ..., 1}, 0, 0, ...$$

where the blocks of 0's are increasing by factors of 100 and the blocks of 1's are increasing by factors of 10. This sequence is not almost convergent but is statistically convergent to zero.

In [14], Maddox has defined the notion of strongly almost convergent sequence as follows:

A bounded sequence $(x_k)$ is said to be strongly almost convergent to $\ell$ if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_{m+k} - \ell| = 0$$

uniformly in $m$.

By a lacunary sequence (see e.g. [6]) we mean an increasing integer sequence $\theta = (k_n)$ such that $k_0 = 0$ and $h_n = k_n - k_{n-1} \to \infty$ as $n \to \infty$. The intervals determined by $\theta$ will be denoted by $I_n = (k_{n-1}, k_n]$.

A bounded sequence $(x_k)$ is said to be strongly lacunary almost convergent to $\ell$ if

$$\lim_{n\to\infty} \frac{1}{h_n} \sum_{k\in I_n} |x_{k+m} - \ell| = 0$$

uniformly in $m$.

Let $\theta = (k_n)$ be a lacunary sequence. $(x_k)$ is said to be lacunary statistically almost convergent to $\ell$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{h_n} |\{k \in I_n : \quad |x_{k+m} - \ell| \geq \varepsilon\}| = 0$$

uniformly in $m$.

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers tending to $\infty$, and $\lambda_{n+1} - \lambda_n \leq 1$, $\lambda_1 = 1$. The generalized de la Vallée-Poussin mean [11] is defined by

$$\frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^{n} x_k.$$

A sequence $(x_k)$ is said to be strongly $\lambda$-almost convergent to $\ell$ if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k=n-\lambda_n+1}^{n} |x_{k+m} - \ell| = 0$$

uniformly in $m$.

$(x_k)$ is said to be $\lambda$-statistically almost convergent to $\ell$ if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{n-\lambda_n+1 \leq k \leq n : \quad |x_{k+m} - \ell| \geq \epsilon\}| = 0$$

uniformly in $m$.

Let $A = (a_{nk})$ be an infinite matrix and $(x_k)$ be a sequence. Let $X$ and $Y$ be two nonempty subset of the space $w$ of all real or complex number sequences. We write $Ax = (A_n x)$ if $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ converges for each $n$. If $(x_k) \in X$ implies $Ax \in Y$, we say that $A$ defines a matrix transformation from $X$ to $Y$. A matrix $A$ is said to be regular if $A$ transforms every convergent sequence to a convergent sequence by preserving the limit. The following conditions are, by the Silverman-Toeplitz Theorem, necessary and sufficient conditions for the regularity of $A = (a_{nk})$:

(i) $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$;
(ii) $\lim_{n\to\infty} a_{nk} = 0$ for each $k \in \mathbb{N}$;
(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$.

The reader can refer to [6, 8, 11–14, 16, 17] for more detailed information on these subjects.

## 2. DEFERRED ALMOST AND DEFERRED ALMOST STATISTICAL CONVERGENCE

Definition 2.1. A sequence $(x_k)$ is said to be deferred almost convergent to $\ell \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x_{k+m} = \ell$$

uniformly in $m$.

Definition 2.2. A sequence $(x_k)$ is said to be strongly deferred almost convergent to $\ell \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} |x_{k+m} - \ell| = 0$$

uniformly in $m$. In this case we write $x_k \to \ell(\hat{D}[p,q])$.

A strongly deferred almost convergent sequence is deferred almost convergent, but the converse may not be true.

EXAMPLE 2.3. Let the sequence $(x_k)$ be defined by

$$x_k = \begin{cases} 0, & k \text{ is even} \\ 2, & k \text{ is odd} \end{cases}$$

and suppose that $p(n) = 2n - 1$ and $q(n) = 4n - 1$ for all $n \in \mathbb{N}$. Then, we have

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_{k+m} = \lim_{n\to\infty} \frac{1}{2n} \sum_{k=2n}^{4n-1} x_{k+m} = 1$$

uniformly in $m$. That is, this sequence is deferred almost convergent to 1, but not strongly deferred almost convergent to 1.

DEFINITION 2.4. A sequence $(x_k)$ is said to be strongly $r$-deferred almost convergent to $\ell \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} |x_{k+m} - \ell|^r = 0$$

uniformly in $m$ where $0 < r < \infty$. In this case we write $x_k \to \ell(\hat{D}^r[p, q])$.

It is clear that;

- If $q(n) = n$ and $p(n) = 0$ for all $n \in \mathbb{N}$, then Definition 2.2 coincides with the definition of strong almost convergence.
- Let $\theta = (k_n)$ be a lacunary sequence. If we consider $q(n) = k_n$ and $p(n) = k_{n-1}$ for all $n \in \mathbb{N}$, then Definition 2.2 coincides with the strong lacunary almost convergence.
- If $q(n) = n$ and $p(n) = n - \lambda_n$ for all $n \in \mathbb{N}$, then Definition 2.2 coincides with the strong $\lambda$-almost convergence.

DEFINITION 2.5. A sequence $(x_k)$ is said to be deferred almost statistically convergent to the number $\ell \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} |\{p(n) < k \leq q(n) : |x_{k+m} - \ell| \geq \varepsilon\}| = 0$$

uniformly in $m$. In this case we write $x_k \to \ell(\hat{DS}[p, q])$.

It is clear that;

- If $q(n) = n$ and $p(n) = 0$ for all $n \in \mathbb{N}$, then Definition 2.5 coincides with the definition of almost statistical convergence.
- Let $\theta = (k_n)$ be a lacunary sequence. If we consider $q(n) = k_n$ and $p(n) = k_{n-1}$ for all $n \in \mathbb{N}$, then Definition 2.5 coincides with the lacunary almost statistical convergence.
- If $q(n) = n$ and $p(n) = n - \lambda_n$ for all $n \in \mathbb{N}$, then Definition 2.5 coincides with the almost $\lambda$-statistical convergence of sequences.

### 3. INCLUSION RELATIONS

THEOREM 3.1. *Let* $(p(n)),(q(n)),(p'(n))$ *and* $(q'(n))$ *be sequences of of non-negative integers satisfying* $p(n)\le p'(n)<q'(n)\le q(n)$ *for all* $n\in\mathbb{N}$ *and*

$$\limsup_{n\to\infty}\frac{q(n)-p(n)}{q'(n)-p'(n)}<\infty,$$

*then* $x_k\to\ell(\hat{DS}[p,q])$ *implies* $x_k\to\ell(\hat{DS}[p',q'])$.

*Proof.* From the inclusion

$$\{p'(n)<k\le q'(n):\quad |x_{k+m)}-\ell|\ge\varepsilon\}\subset\{p(n)<k\le q(n):\quad |x_{k+m}-\ell|\ge\varepsilon\}$$

we can write

$$\frac{1}{q'(n)-p'(n)}|\{p'(n)<k\le q'(n):\quad |x_{k+m}-\ell|\ge\varepsilon\}|$$
$$\le\frac{q(n)-p(n)}{q'(n)-p'(n)}\frac{1}{q(n)-p(n)}|\{p(n)<k\le q(n):|x_{k+m}-\ell|\ge\varepsilon\}|.$$

After taking limit when $n\to\infty$, the desired result is obtained. □

THEOREM 3.2. *Let* $(p(n)),(q(n)),(p'(n))$ *and* $(q'(n))$ *be sequences of of non-negative integers satisfying* $p(n)\le p'(n)<q'(n)\le q(n)$ *for all* $n\in\mathbb{N}$ *and*

$$\limsup_{n\to\infty}\frac{q(n)-p(n)}{q'(n)-p'(n)}<\infty,$$

*then* $x_k\to\ell(\hat{D}[p,q])$ *implies* $x_k\to\ell(\hat{D}[p',q'])$.

Since the proof is similar to the proof of Theorem 3.1, we omit it.

THEOREM 3.3. *If* $(x_k)$ *is strongly deferred almost convergent to* $\ell$, *then* $(x_k)$ *is deferred almost statistically convergent to* $\ell$, *that is, if* $x_k\to\ell(\hat{D}[p,q])$, *then* $x_k\to\ell(\hat{DS}[p,q])$.

*Proof.* Let $x_k\to\ell(\hat{D}[p,q])$. For an arbitrary $\varepsilon>0$, we get

$$\begin{aligned}
&\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}|x_{k+m}-\ell|\\
&= \frac{1}{q(n)-p(n)}\left(\sum_{\substack{k=p(n)+1\\|x_{k+m}-\ell|\ge\varepsilon}}^{q(n)}|x_{k+m}-\ell|+\sum_{\substack{k=p(n)+1\\|x_{k+m}-\ell|<\varepsilon}}^{q(n)}|x_{k+m}-\ell|\right)\\
&\ge \frac{1}{q(n)-p(n)}\sum_{\substack{k=p(n)+1\\|x_{k+m)}-\ell|\ge\varepsilon}}^{q(n)}|x_{k+m}-\ell|\\
&\ge \frac{\varepsilon}{q(n)-p(n)}|\{p(n)<k\le q(n):|x_{k+m}-\ell|\ge\varepsilon\}|
\end{aligned}$$

for each $m$. Hence, we have

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}|\{p(n)<k\le q(n):|x_{k+m}-\ell|\ge\varepsilon\}|=0$$

uniformly in $m$, that is $x_k \to \ell(\hat{DS}[p,q])$. □

THEOREM 3.4. *If $(x_k)$ is bounded and deferred almost statistically convergent to $\ell$, then $(x_k)$ is strongly deferred almost convergent to $\ell$, that is, if $(x_k)$ is bounded and $x_k \to \ell(\hat{DS}[p,q])$, then $x_k \to \ell(\hat{D}[p,q])$.*

*Proof.* Suppose that $x_k \to \ell(\hat{DS}[p,q])$ and $(x_k)$ is bounded, say $|x_{k+m}-\ell| \le M$ for all $k$ and $m$. Given $\varepsilon > 0$, we get

$$\begin{aligned}
&\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}|x_{k+m}-\ell|\\
= \ &\frac{1}{q(n)-p(n)}\sum_{\substack{k=p(n)+1\\|x_{k+m}-\ell|\ge\varepsilon}}^{q(n)}|x_{k+m}-\ell| + \frac{1}{q(n)-p(n)}\sum_{\substack{k=p(n)+1\\|x_{k+m}-\ell|<\varepsilon}}^{q(n)}|x_{k+m}-\ell|\\
\le \ &\frac{M}{q(n)-p(n)}\sum_{\substack{k=p(n)+1\\|x_{k+m}-\ell|\ge\varepsilon}}^{q(n)}1 + \frac{\varepsilon}{q(n)-p(n)}\sum_{\substack{k=p(n)+1\\|x_{k+m}-\ell|<\varepsilon}}^{q(n)}1\\
\le \ &\frac{M}{q(n)-p(n)}|\{p(n)<k\le q(n):|x_{k+m}-\ell|\ge\varepsilon\}|\\
&+\frac{\varepsilon}{q(n)-p(n)}|\{p(n)<k\le q(n):|x_{k+m}-\ell|<\varepsilon\}|
\end{aligned}$$

for each $m$, hence we have

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}|x_{k+m}-\ell|=0$$

uniformly in $m$. □

EXAMPLE 3.5. Let the sequence $(x_k)$ be defined by

$$x_k=\begin{cases}0, & k \text{ is not square}\\ 2, & k \text{ is square}\end{cases}$$

and suppose that $p(n)=2n-1$ and $q(n)=4n-1$ for all $n\in\mathbb{N}$. Then, we have

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}|\{p(n)<k\le q(n):|x_{k+m}-0|\ge\varepsilon\}|=0$$

uniformly in m. That is, this sequence is deferred almost statistically convergent to 0. Since $(x_k)$ is bounded, by Theorem 3.4, this sequence is strongly deferred almost convergent to 0.

THEOREM 3.6. *If the sequence* $\left(\frac{p(n)}{q(n)-p(n)}\right)$ *is bounded, then* $x_k \to \ell(\hat{S})$ *implies* $x_k \to \ell(\hat{DS}[p,q])$.

*Proof.* Let $x_k \to \ell(\hat{S})$ then for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : \quad |x_{k+m} - \ell| \geq \varepsilon\}| = 0$$

uniformly in $m$. Hence, for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{q(n)}|\{k \leq q(n) : \quad |x_{k+m} - \ell| \geq \varepsilon\}| = 0$$

uniformly in $m$. From the following inclusion

$$\{p(n) < k \leq q(n) : \quad |x_{k+m} - \ell| \geq \varepsilon\} \subseteq \{k \leq q(n) : \quad |x_{k+m} - \ell| \geq \varepsilon\}$$

and the inequality

$$|\{p(n) < k \leq q(n) : |x_{k+m} - \ell| \geq \varepsilon\}| \leq |\{k \leq q(n) : |x_{k+m} - \ell| \geq \varepsilon\}|$$

we have

$$\begin{aligned} & \frac{1}{q(n)-p(n)}|\{p(n) < k \leq q(n) : |x_{k+m} - \ell| \geq \varepsilon\}| \\ = \quad & \frac{q(n)-p(n)+p(n)}{q(n)-p(n)} \frac{1}{q(n)}|\{p(n) < k \leq q(n) : |x_{k+m} - \ell| \geq \varepsilon\}| \\ \leq \quad & \left(1 + \frac{p(n)}{q(n)-p(n)}\right) \frac{1}{q(n)}|\{k \leq q(n) : |x_{k+m} - \ell| \geq \varepsilon\}| \end{aligned}$$

for each $m$ and hence, we obtain $x_k \to \ell(\hat{DS}[p,q])$. □

THEOREM 3.7. *Let* $q(n) = n$ *for all* $n \in \mathbb{N}$. *Then,* $x_k \to \ell(\hat{DS}[p,n])$ *if and only if* $x_k \to \ell(\hat{S})$.

*Proof.* Assume that $x_k \to \ell(\hat{DS}[p,n])$. By using the technique which was used by Agnew in [1], for each $n \in \mathbb{N}$, letting $p(n) = n^{(1)}$, $p(n^{(1)}) = n^{(2)}$, $p(n^{(2)}) = n^{(3)}, \ldots$, we may write

$$\{k \leq n : \quad |x_{k+m} - \ell| \geq \varepsilon\} =$$
$$\left\{k \leq n^{(1)} : \quad |x_{k+m} - \ell| \geq \varepsilon\right\} \cup \left\{n^{(1)} < k \leq n : \quad |x_{k+m)} - \ell| \geq \varepsilon\right\},$$

$$\left\{k \leq n^{(1)} : \quad |x_{k+m} - \ell| \geq \varepsilon\right\} =$$
$$\left\{k \leq n^{(2)} : \quad |x_{k+m} - \ell| \geq \varepsilon\right\} \cup \left\{n^{(2)} < k \leq n^{(1)} : \quad |x_{k+m} - \ell| \geq \varepsilon\right\}$$

and

$$\left\{k \leq n^{(2)} : \quad |x_{k+m} - \ell| \geq \varepsilon\right\} =$$
$$\left\{k \leq n^{(3)} : \quad |x_{k+m} - \ell| \geq \varepsilon\right\} \cup \left\{n^{(3)} < k \leq n^{(2)} : \quad |x_{k+m} - \ell| \geq \varepsilon\right\}$$

for each $m$. This process may be continued until for some positive integer $h$ depending on $n$, we obtain

$$\left\{k \leq n^{(h-1)} : \quad |x_{k+m} - \ell| \geq \varepsilon\right\} =$$
$$\left\{k \leq n^{(h)} : \quad |x_{k+m} - \ell| \geq \varepsilon\right\} \cup \left\{n^{(h)} < k \leq n^{(h-1)} : \quad |x_{k+m} - \ell| \geq \varepsilon\right\}$$

for each $m$ where $n^{(h)} \geq 1$ and $n^{(h+1)} = 0$. Therefore, we can write

$$\frac{1}{n}|\{k \leq n : \quad |x_{k+m} - \ell| \geq \varepsilon\}| = \sum_{r=0}^{h} \frac{n^{(r)} - n^{(r+1)}}{n} t_{rm}$$

for every $n$ and $m$ where

$$t_{rm} = \frac{1}{n^{(r)} - n^{(r+1)}} |\{n^{(r+1)} < k \leq n^{(r)} : |x_{k+m} - \ell| \geq \varepsilon\}|.$$

If we take the matrix $A = (a_{nr})$ as follows

$$a_{nr} = \begin{cases} \dfrac{n^{(r)} - n^{(r+1)}}{n}, & r = 0, 1, 2, ..., h \\ 0, & \text{otherwise} \end{cases}$$

where $n^{(0)} = n$, then the sequence

$$\left\{\frac{1}{n}|\{k \leq n : |x_{k+m} - \ell| \geq \varepsilon\}|\right\}$$

is the $(a_{nr})$ transformation of the sequence $(t_{rm})$. Since $n^{(r)} > n^{(r+1)}, r = 1, 2, ..., h$, and $n^{(h+1)} = 0$, this transformation evidently satisfies (i) and (iii); and for fixed $k$, $\frac{n^{(k)} - n^{(k+1)}}{n}$ is either zero or a fraction of which the denominator is $n$ and the numerator is less than or equal $k$ so that (ii) holds. Therefore, the matrix $(a_{nr})$ is a regular, and since the sequence

$$\left\{\frac{1}{n^{(r)} - n^{(r+1)}} |\{n^{(r+1)} < k \leq n^{(r)} : |x_{k+m} - \ell| \geq \varepsilon\}|\right\}$$

is convergent to zero for each $m$, we have

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_{k+m} - \ell| \geq \varepsilon\}| = 0$$

uniformly in $m$.

Conversely, since $\left(\frac{p(n)}{q(n)-p(n)}\right)$ is bounded for $q(n) = n$, by Theorem 3.6, we obtain that $x_k \to \ell(\hat{S})$ implies $x_k \to \ell(\hat{DS}[p, q])$. $\square$

## 4. DEFERRED ALMOST STATISTICAL EQUIVALENCE

In [15], Marouf introduced the definitions of asymptotically equivalent sequences and asymptotic regular matrices. Patterson [18] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. In [9], a generalization of asymptotical statistical equivalence of nonnegative sequences by using deferred statistical density, was given. This section shall provide definitions of asymptotical strongly $r$-deferred almost equivalence and deferred almost statistical equivalence of nonnegative sequences. We will examine the relation between asymptotical deferred almost statistical equivalence and strongly $r$-deferred almost equivalence.

Firstly in [19], Pobyvanets introduced the concept of asymptotical equivalence as follows:

Two nonnegative sequences $(x_k)$ and $(y_k)$ are said to be asymptotically equivalent with multiple $L$ if

$$\lim_{n\to\infty} \frac{x_k}{y_k} = L.$$

For example, if we take $x_k = k^2 + k$ and $y_k = k^2$, then $\lim_{n\to\infty} \frac{x_k}{y_k} = \lim_{n\to\infty} \frac{k^2+k}{k^2} = 1$, that is the sequences $(x_k)$ and $(y_k)$ are asymptotically equivalent with multiple 1.

Later in [18], Patterson introduced the concept of asymptotical statistical equivalence as follows:

Two nonnegative sequences $(x_k)$ and $(y_k)$ are said to be asymptotically statistical equivalent with multiple $L$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : \quad \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0.$$

Two nonnegative sequences $(x_k)$ and $(y_k)$ are said to be asymptotically almost statistical equivalent with multiple $L$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : \quad \left| \frac{x_{k+m}}{y_{k+m}} - L \right| \geq \varepsilon \right\} \right| = 0$$

uniformly in $m$. In this case we write $x \overset{\hat{S}_L}{\sim} y$. If $L = 1$, then we say that $(x_k)$ and $(y_k)$ are asymptotically almost statistical equivalent and we write $x \overset{\hat{S}}{\sim} y$.

Definition 4.1. Two nonnegative sequences $(x_k)$ and $(y_k)$ are said to be asymptotically deferred almost equivalent with multiple $L$ if

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \frac{x_{k+m}}{y_{k+m}} = L$$

uniformly in $m$. In this case we write $x \overset{\hat{D}_L}{\sim} y$. If $L = 1$, then we say that $(x_k)$ and $(y_k)$ are asymptotically deferred almost equivalent and we write $x \overset{\hat{D}}{\sim} y$.

Definition 4.2. Two nonnegative sequences $(x_k)$ and $(y_k)$ are said to be asymptotically strongly deferred almost equivalent with multiple $L$ if

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left| \frac{x_{k+m}}{y_{k+m}} - L \right| = 0$$

uniformly in $m$. In this case we write $x \overset{[\hat{D}]_L}{\sim} y$. If $L=1$, then we say that $(x_k)$ and $(y_k)$ are asymptotically strongly deferred almost equivalent and we write $x \overset{[\hat{D}]}{\sim} y$.

Definition 4.3. Two nonnegative sequences $(x_k)$ and $(y_k)$ are said to be asymptotically strongly r-deferred almost equivalent with multiple $L$ if

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \left| \frac{x_{k+m}}{y_{k+m}} - L \right|^r = 0$$

uniformly in $m$, where $0 < r < \infty$. In this case we write $x \overset{[\hat{D_r}]_L}{\sim} y$. If $L=1$, then we say that $(x_k)$ and $(y_k)$ are asymptotically strongly $r$-deferred almost equivalent and we write $x \overset{[\hat{D_r}]}{\sim} y$.

Definition 4.4. Two nonnegative sequences $(x_k)$ and $(y_k)$ are said to be asymptotically deferred almost statistical equivalent with multiple $L$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{p(n)-q(n)} \left| \left\{ p(n) < k \le q(n) : \quad \left| \frac{x_{k+m}}{y_{k+m}} - \ell \right| \ge \varepsilon \right\} \right| = 0$$

uniformly in $m$. In this case we write $x \overset{\hat{DS}_L}{\sim} y$. If $L=1$, then we say that $(x_k)$ and $(y_k)$ are asymptotically deferred almost statistical equivalent and we write $x \overset{\hat{DS}}{\sim} y$.

Since the proofs of the following theorems are similar to the proofs of the theorems given in Section 3, we give the theorems without proof in order not to repeat them.

Theorem 4.5. *Let $(x_k)$ and $(y_k)$ be two nonnegative sequences. Then*

$$x \overset{[\hat{D_r}]_L}{\sim} y \text{ implies } x \overset{\hat{DS}_L}{\sim} y.$$

Theorem 4.6. *If $\left(\frac{x_k}{y_k}\right)$ is a bounded sequence, then*

$$x \overset{\hat{DS}_L}{\sim} y \text{ implies } x \overset{[\hat{D_r}]_L}{\sim} y.$$

THEOREM 4.7. *Let $q(n) = n$ for all $n \in \mathbb{N}$. Then*

$$x \overset{\hat{DS}_L}{\sim} y \text{ implies } x \overset{\hat{S}_L}{\sim} y.$$

THEOREM 4.8. *Let $(p(n)), (q(n)), (p'(n))$ and $(q'(n))$ be sequences of of non-negative integers satisfying $p(n) \leq p'(n) < q'(n) \leq q(n)$ for all $n \in \mathbb{N}$ and*

$$\limsup_{n\to\infty} \frac{q(n) - p(n)}{q'(n) - p'(n)} < \infty,$$

*then*

$$x \overset{[\hat{D}_r]_L[p,q]}{\sim} y \text{ implies } x \overset{[\hat{D}_r]_L[p',q']}{\sim} y.$$

THEOREM 4.9. *Let $q(n) = n$ for all $n \in \mathbb{N}$. Then*

$$x \overset{\hat{DS}_L}{\sim} y \text{ implies } x \overset{\hat{S}_L}{\sim} y.$$

THEOREM 4.10. *Let $(p(n)), (q(n)), (p'(n))$ and $(q'(n))$ be sequences of non-negative integers satisfying $p(n) \leq p'(n) < q'(n) \leq q(n)$ for all $n \in \mathbb{N}$ and*

$$\limsup_{n\to\infty} \frac{q(n) - p(n)}{q'(n) - p'(n)} = 0,$$

*then*

$$x \overset{[\hat{DS}_r]_L[p,q]}{\sim} y \text{ implies } x \overset{[\hat{DS}_r]_L[p',q']}{\sim} y.$$

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