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COEFFICIENT INEQUALITIES FOR SUBCLASSES OF ANALYTIC FUNCTIONS BASED ON QUASI-SUBORDINATION AND MAJORIZATION RELATED WITH SIGMOID FUNCTIONS

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Dedicated to Prof. Dr. Hari Mohan Srivastava on the occasion of his 80th birthday.

Abstract. In the present paper, we estimate the coefficient bounds for certain subclasses of the starlike and convex functions using quasi-subordination and majorization relating with sigmoid functions.
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1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots , \tag{1}$$

which are *analytic* in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0$, $f'(0) = 1$. Also, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions which are *univalent* in $\mathbb{U}$. Here and subsequently, $\Omega(z)$ denotes the class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots , \tag{2}$$

and satisfying a condition $|w(z)| < 1$ in $\mathbb{U}$, known as a class of *Schwarz functions*. To recall the principle of subordination between analytic functions, let the functions $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is *subordinate* to $g(z)$, if there exists a Schwarz function $w(z)$, such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. We denote this subordination by $f \prec g$ (or $f(z) \prec g(z),\ z \in \mathbb{U}$). In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, the above subordination is equivalent to the conditions $f(0) = g(0),\ f(\mathbb{U}) \subset g(\mathbb{U})$. Due to Ma-Minda[8] we state the following subordination principle:

DEFINITION 1.1. Suppose ϕ is an analytic function such that

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(1) $\Re\ (\phi) > 0$ in $\mathbb{U}$
(2) $\phi(0) = 1, \quad \phi'(0) > 0$
(3) ϕ maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis.

Such a function has a series expansion of the following form:

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots, \quad B_1 > 0, \tag{3}$$

Let $\mathcal{S}^*(\phi)$ be the class of function $f \in \mathcal{S}$ for which $\frac{zf'(z)}{f(z)} \prec \phi(z), (z \in \mathbb{U})$ and $\mathcal{C}(\phi)$ be the class of function $f \in \mathcal{S}$ for which $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$, $(z \in \mathbb{U})$. These classes were introduced and studied by Ma-Minda [8].

An extension of the notion of the subordination is the quasi-subordination introduced by Robertson in [15]. We call a function $f(z)$ *quasi-subordinate* to a function $g(z)$ in $\mathbb{U}$ if there exist the Schwarz function $\omega(z)$ and an analytic functions $\varphi(z)$ satisfying $|\varphi(z)| < 1$ such that $f(z) = \varphi(z)g(w(z))$ in $\mathbb{U}$. We then write$f \prec_q g$. Where

$$\varphi(z) = d_0 + d_1 z + d_2 z^2 + \cdots \text{ and } |\ d_n\ | \leq 1. \tag{4}$$

If $\varphi(z) \equiv 1$ then the quasi-subordination reduces to the subordination. If we set $w(z) = z$, then $f(z) = \varphi(z)g(z)$ and we say that $f(z)$ is *majorized* by $g(z)$ and it is written as $f(z) \ll g(z)$ in $\mathbb{U}$. Therefore *quasi-subordination* is a generalization of the notion of the subordination as well as the *majorization* that underline its importance. Related works of quasi-subordination may be found in [1, 3, 5, 6, 14].

Now we shall recollect about the definition and properties of sigmoid functions and its applications in the coefficient problems which have been studied recently by Function theorist. Special functions concentrates on information process after adapting the concept of brain process information associated with central nervous system. It includes a huge amount of highly interconnected processing elements (neurones) working together so as to find solution for a specific problem. The functions are being appreciated by other fields like real analysis, algebra, topology, functional analysis, differential equations and so on because it resembles the way a human brain functions. With the help of examples the same can be trained. Special functions have three main categories such as the threshold function, the ramp function and the logistic sigmoid function. Logistic sigmoid function is considered as the most important among three functions due to its gradient descendent learning algorithm. It can be estimated using different methods, especially through truncated series expansion. The logistic sigmoid function is an novel concept in the area of univalent function theory. Recently, Fadipe-Joseph[4] introduced and studied the sigmoid function which is given by

$$h(z) = \frac{1}{1 + \mathrm{e}^{-z}}.$$

This function is differentiable and has the following properties:

(1) It outputs real numbers between 0 and 1.
(2) It maps a very large input domain to a small range of outputs.
(3) It never loses information because it is an injective function.
(4) It increases monotonically.

From the above properties, it is clear that sigmoid function plays a key role in geometric functions theory for details [9, 11, 12, 13].

We shall need the following definitions and lemmas in order to state and prove our main results.

Lemma 1.2 ([4]). *Let $h(z)$ be a sigmoid function and*

$$\Phi(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m,$$

then $\Phi(z) \in \mathcal{P}, |z| < 1$ where $\Phi(z)$ is a modified sigmoid function.

Lemma 1.3 ([4]). *Let*

$$\Phi_{n,m} = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m$$

then $|\Phi_{n,m}(z)| < 2$.

Setting $m = 1$, from Fadipe-Joseph[4] remarked that $\Phi(z) = 1 + \sum\limits_{n=1}^{\infty} c_n z^n$, where $c_n = \dfrac{(-1)^{n+1}}{2n!}$. Such that $|c_n| \leq 2$, $n \in \mathbb{N} = \{1, 2, 3, ...\}$ and the result is sharp for each n. It is given that

$$\Phi_{n,m}(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{17}{40320}z^7 + O\left(z^9\right)$$

Definition 1.4. Let $\phi(z)$ be as assumed in (3) and a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{H}_q(\alpha, \beta, \Phi)$ for $0 \leq \alpha \leq 1; 0 \leq \beta \leq 1$ if it satisfies the following subordination condition

$$\begin{aligned} &(1-\beta)\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \\ + \quad &\beta\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)} - 1 \prec_q \phi(z) - 1 \quad (z \in \mathbb{U}). \end{aligned}$$

Remark 1.5. Let $\phi(z)$ be as assumed in (3) and a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{H}_q(\alpha, 0, \Phi) \equiv \mathcal{M}_q(\alpha, \Phi)$, and $0 \leq \alpha \leq 1$ if it satisfies the following subordination condition

$$\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] - 1 \prec_q \phi(z) - 1 \quad (z \in \mathbb{U}).$$

REMARK 1.6. Let $\phi(z)$ be as assumed in (3) and a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{H}_q(\alpha, 1, \Phi) \equiv \mathcal{N}_q(\alpha, \Phi)$, and $0 \leq \alpha \leq 1$ if it satisfies the following subordination condition

$$\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)} - 1 \prec_q \phi(z) - 1 \quad (z \in \mathbb{U}).$$

By fixing $\alpha = 0$ and $\alpha = 1$ respectively, in above remarks we state the following:

REMARK 1.7. Let $\phi(z)$ be as assumed in (3) we let

$$\mathcal{S}_q(\Phi) = \{f \inf \in \mathcal{A} : \frac{zf'(z)}{f(z)} - 1 \prec_q \phi(z) - 1\}$$

$$\mathcal{K}_q(\Phi) = \{f \inf \in \mathcal{A} : \frac{zf''(z)}{f'(z)} \prec_q \phi(z) - 1\}.$$

DEFINITION 1.8. Let $\phi(z)$ be as assumed in (3) and a function $f \in \mathcal{S}$ is said to be in the class $\mathcal{R}_q(\alpha, \Phi)$, $0 \leq \alpha \leq 1$ if

$$\left(f'(z)\right)^{\alpha} \left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} - 1 \prec_q \phi(z) - 1, \quad (z \in \mathbb{U}).$$

DEFINITION 1.9. Let $\phi(z)$ be as assumed in (3) and a function $f \in \mathcal{S}$ is said to be in the class $\mathcal{L}_q(\alpha, \Phi)$, $0 \leq \alpha \leq 1$ if

$$\left(f'(z)\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} - 1 \prec_q \phi(z) - 1, \quad (z \in \mathbb{U}).$$

By fixing $\alpha = 1$, in above Definitions we state the following:

DEFINITION 1.10. Let $\phi(z)$ be as assumed in (3) and a function $f \in \mathcal{S}$ is said to be in the class $\mathcal{N}_q(\Phi)$, if

$$f'(z) - 1 \prec_q \phi(z) - 1, \quad (z \in \mathbb{U}).$$

By fixing $\alpha = 0$, in Definitions 1.8 and 1.9 yields we state the subclasses given in Remark 1.7. In this paper, we obtain coefficient estimates for the functions in the above defined class associated with quasi-subordination and majorization.

The following lemma regarding the coefficients of functions in $\Omega(z)$ are needed to prove our main results.

LEMMA 1.11 ([7]). *If $\omega(z)$ is analytic function in $\mathbb{U}$ with $\omega(0) = 0$, $|\omega(z)| < 1$ and*

$$\omega(z) = \omega_1 z + \omega_2 z^2 + \omega_3^z 3 + \cdots . \tag{5}$$

Then $\left|\omega_2 - \mu\omega_1^2\right| \leq \max\{1, |\mu|\}$ for any complex number μ. The result is sharp for the function $\omega(z) = z$ or $\omega(z) = z^2$.

Lemma 1.12 ([2]). *If $\omega(z)$ is analytic function in $\mathbb{U}$ with $\omega(0) = 0, |\omega(z)| < 1$ and given by* (5) *then*

$$|\omega_n| \leq \begin{cases} 1 & \text{for} \quad n = 1, \\ 1 - |\omega_1|^2 & \text{for} \quad n \geq 2. \end{cases}$$

The result is sharp for the function $\omega(z) = z$ or $\omega(z) = z^n$.

Lemma 1.13 ([7]). *Let $\varphi(z)$ given in (4) be analytic function in $\mathbb{U}$ with the condition $|\varphi(z)| < 1$. Then $|d_0| \leq 1$ and $|d_n| \leq 1 - |d_0|^2 \leq 1$ for $n > 0$.*

2. THE FEKETE-SZEGÖ FUNCTIONAL ASSOCIATED WITH QUASI-SUBORDINATION

In this section, we first state and prove the following theorem.

Theorem 2.1. *Let α, β be positive real numbers, let $\phi(z)$ be as assumed in* (3), *$\varphi(z)$ is given in* (4) *and if $f \in \mathcal{H}_q(\alpha, \beta, \Phi)$ given by* (1), *then*

$$|a_2| \leq \frac{B_1}{4(1+\alpha)}$$

and

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{8(1+2\alpha)}$$
$$\times \max\left\{1, \frac{1}{4}\left|\frac{\left[2\mu(1+2\alpha) - (1+3\alpha-\alpha\beta+\alpha^2\beta)\right]B_1}{(1+\alpha)^2} - \frac{B_2 - B_1}{B_1}\right|\right\}.$$

Proof. If $f \in \mathcal{H}_{\alpha,\beta}(\phi)$, then by Definition (1.4) we have

$$(1-\beta)\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] + \beta\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)}$$
$$-1 = \varphi(z)\left(\phi\left(\frac{\Phi(\omega(z)) - 1}{\Phi(\omega(z)) + 1}\right) - 1\right)$$
$$1 + (1+\alpha)a_2 z + \left[2(1+2\alpha)a_3 - (1+3\alpha-\alpha\beta+\alpha^2\beta)a_2^2\right]z^2 + \cdots$$
$$= \frac{d_0 B_1 \omega_1}{4}z + \left(\frac{d_0 B_1 \omega_2}{4} + \frac{(B_2 - B_1)d_0\omega_1^2}{4} + \frac{d_1 B_1 \omega_1}{4}\right)z^2 + \cdots$$

Equating the coefficient of z and z^2 from the above equation

$$a_2 = \frac{d_0 B_1 \omega_1}{4(1+\alpha)} \tag{6}$$

and

$$a_3 = \frac{B_1}{8(1+2\alpha)}(\omega_1 d_1 \tag{7}$$
$$+ d_0\left[\omega_2 + \frac{1}{4}\left(\frac{B_2 - B_1}{B_1} + \frac{(1+3\alpha-\alpha\beta+\alpha^2\beta)d_0 B_1}{(1+\alpha)^2}\right)\omega_1^2\right]\right). \tag{8}$$

Since μ is a complex number, from (6) and (8), we get

$$\begin{aligned} a_3-\mu a_2^2 = \frac{B_1}{8(1+2\alpha)}\Bigg(\omega_1 d_1 + d_0\omega_2 - \frac{1}{4}\Bigg[\frac{2\mu(1+2\alpha)B_1 d_0^2}{(1+\alpha)^2} \\ -\frac{(B_2-B_1)d_0}{B_1} - \frac{(1+3\alpha-\alpha\beta+\alpha^2\beta)d_0^2 B_1}{(1+\alpha)^2}\Bigg]\omega_1^2\Bigg). \end{aligned} \tag{9}$$

Since $\varphi(z)$ given in (4) is analytic and bounded in $\mathbb{U}$, Hence applying the result in [10, p. 172], for some $y(|y|\leq 1)$. We have

$$|d_0|\leq 1 \ \text{ and } \ d_1=(1-d_0^2)y. \tag{10}$$

Substituting the value of d_1 in (9), which yields

$$\begin{aligned} a_3-\mu a_2^2 =& \frac{B_1}{8(1+2\alpha)}\Bigg(y\omega_1 + d_0\omega_2 + \frac{(B_2-B_1)d_0}{4B_1}\omega_1^2 \\ &-\Bigg(\frac{1}{4}\Bigg[\frac{2\mu(1+2\alpha)B_1}{(1+\alpha)^2} - \frac{(1+3\alpha-\alpha\beta+\alpha^2\beta)B_1}{(1+\alpha)^2}\Bigg) \\ &\times\omega_1^2 + y\omega_1\Bigg] d_0^2\Bigg). \end{aligned} \tag{11}$$

If $d_0=0$, the equation (11) becomes

$$a_3-\mu a_2^2 = \frac{B_1 y\omega_1}{8(1+2\alpha)}.$$

Using Lemma 1.12 and Lemma 1.13

$$|a_3-\mu a_2^2| \leq \frac{B_1}{8(1+2\alpha)}.$$

If $d_0\neq 0$, from equation (11), let

$$\begin{aligned} F(d_0) \ = \ & y\omega_1 + d_0\omega_2 + \frac{(B_2-B_1)d_0}{4B_1}\omega_1^2 - \Bigg(\frac{1}{4}\Bigg[\frac{2\mu(1+2\alpha)B_1}{(1+\alpha)^2} \\ &-\frac{(1+3\alpha-\alpha\beta+\alpha^2\beta)B_1}{(1+\alpha)^2}\Bigg)\omega_1^2 + y\omega_1\Bigg] d_0^2, \end{aligned}$$

which is a polynomial in d_0 and therefore it is analytic in $|d_0|\leq 1$, and hence maximum of $|F(d_0)|$ is attained at $d_0 = \mathrm{e}^{\mathrm{i}\theta}, (0\leq\theta<2\pi)$. Therefore we obtained that

$$\max_{0\leq\theta<2\pi}|F(\mathrm{e}^{\mathrm{i}\theta})| = |F(1)|$$

and

$$\begin{aligned} |a_3-\mu a_2^2| \ = \ & \frac{B_1}{8(1+2\alpha)}\Bigg|\omega_2 - \frac{1}{4}\Bigg[\frac{2\mu(1+2\alpha)B_1}{(1+\alpha)^2} - \frac{(B_2-B_1)}{B_1} \\ &-\frac{(1+3\alpha-\alpha\beta+\alpha^2\beta)B_1}{(1+\alpha)^2}\Bigg]\omega_1^2\Bigg|. \end{aligned}$$

By Lemma 1.11

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{8(1+2\alpha)} \max\left\{1, \frac{1}{4}\left|\frac{\left[2\mu(1+2\alpha) - (1+3\alpha-\alpha\beta+\alpha^2\beta)\right]B_1}{(1+\alpha)^2} - \frac{B_2 - B_1}{B_1}\right|\right\}$$

which completes the proof. $\square$

Applying the same technique as in Theorem 2.1 for the classes $\mathcal{R}_q(\alpha, \Phi)$ and $\mathcal{L}_q(\alpha, \Phi)$, we get the following Theorems.

THEOREM 2.2. *Let* $0 \leq \alpha \leq 1$*,* $\phi(z)$ *be as assumed in* (3)*,* $\varphi(z)$ *is given in* (4) *and if* $f(z)$ *given by* (1) *belongs to* $\mathcal{R}_q(\alpha, \Phi)$*, then* $|a_2| \leq \frac{B_1}{4(1+\alpha)}$ *and*

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{4(2+\alpha)} \max\left\{1, \left|\frac{1}{4}\left(\frac{\mu B_1}{(1+\alpha)^2} - \frac{B_2 - B_1}{B_1} - \frac{(\alpha-1)(\alpha+2)B_1}{2(\alpha+1)^2}\right)\right|\right\}.$$

THEOREM 2.3. *Let* $0 \leq \alpha \leq 1$*,* $\phi(z)$ *be as assumed in* (3)*,* $\varphi(z)$ *is given in* (4) *and if* $f(z)$ *given by* (1) *belongs to* $\mathcal{L}_q(\alpha, \Phi)$*, then* $|a_2| \leq \frac{B_1}{8}$ *and*

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{12(2-\alpha)} \max\left\{1, \left|\frac{1}{4}\left(\frac{3\mu B_1(2-\alpha)}{4} - \frac{B_2 - B_1}{B_1} - (\alpha-1)B_1\right)\right|\right\}.$$

3. THE FEKETE-SZEGÖ FUNCTIONAL ASSOCIATED WITH MAJORIZATION

THEOREM 3.1. *Let* α, β *be positive real numbers, let* $\phi(z)$ *be as assumed in* (3) *and if* $f(z)$ *given by* (1) *and satisfies the condition*

$$(1-\beta)\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] + \beta\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)} - 1 \ll \phi\left(\Phi(\omega(z))\right) - 1,$$

then $|a_2| \leq \frac{B_1}{4(1+\alpha)}$ *and*

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{8(1+2\alpha)} \max\left\{1, \frac{1}{4}\left|\frac{\left[2\mu(1+2\alpha) - (1+3\alpha-\alpha\beta+\alpha^2\beta)\right]B_1}{(1+\alpha)^2} - \frac{B_2 - B_1}{B_1}\right|\right\}.$$

Proof. From the proof of Theorem 2.1, put $\omega(z) = z$ in (5) we get $a_2 = \frac{d_0 B_1}{4(1+\alpha)}$. Then, by Lemma 1.13, $a_2 \leq \frac{B_1}{4(1+\alpha)}$. From (9) and putting $\omega(z) = z$

in (5), we obtain

$$\begin{aligned} a_3-\mu a_2^2 = &\frac{B_1}{8(1+2\alpha)}\left(d_1+\frac{(B_2-B_1)d_0}{4B_1}\right. \\ &\left. -\frac{1}{4}\left[\frac{2\mu(1+2\alpha)B_1}{(1+\alpha)^2}-\frac{(1+3\alpha-\alpha\beta+\alpha^2\beta)B_1}{(1+\alpha)^2}\right]d_0^2\right). \end{aligned} \tag{12}$$

Substituting the value of d_1 from (10) in (12), which implies that

$$\begin{aligned} a_3-\mu a_2^2 = &\frac{B_1}{8(1+2\alpha)}\left(y+\frac{(B_2-B_1)d_0}{4B_1}\right. \\ &\left. -\left(\frac{1}{4}\left[\frac{2\mu(1+2\alpha)B_1}{(1+\alpha)^2}-\frac{(1+3\alpha-\alpha\beta+\alpha^2\beta)B_1}{(1+\alpha)^2}\right]+y\right)d_0^2\right). \end{aligned}$$

If $d_0=0$, the equation (12) becomes

$$|a_3-\mu a_2^2| \leq \frac{B_1}{8(1+2\alpha)}. \tag{13}$$

If $d_0 \neq 0$, let

$$\begin{aligned} H(d_0) \;=\; & y+\frac{(B_2-B_1)d_0}{4B_1} \\ & -\left(\frac{1}{4}\left[\frac{2\mu(1+2\alpha)B_1}{(1+\alpha)^2}-\frac{(1+3\alpha-\alpha\beta+\alpha^2\beta)B_1}{(1+\alpha)^2}\right]+y\right)d_0^2. \end{aligned}$$

which is a polynomial in d_0 and therefore it is analytic in $|d_0| \leq 1$, and hence maximum of $|H(d_0)|$ is attained at $d_0 = \mathrm{e}^{\mathrm{i}\theta}, (0 \leq \theta < 2\pi)$. Therefore we obtained that $\max\limits_{0\leq\theta<2\pi}|H(\mathrm{e}^{\mathrm{i}\theta})| = |H(1)|$ and consequently

$$\begin{aligned} |a_3-\mu a_2^2| \leq &\frac{B_1}{8(1+2\alpha)}\left|\frac{1}{4}\left[\frac{2\mu(1+2\alpha)B_1}{(1+\alpha)^2}-\frac{(B_2-B_1)}{B_1}\right.\right. \\ &\left.\left. -\frac{(1+3\alpha-\alpha\beta+\alpha^2\beta)B_1}{(1+\alpha)^2}\right]\right|. \end{aligned} \tag{14}$$

From (13) and (14) we get

$$\begin{aligned} |a_3-\mu a_2^2| \leq &\frac{B_1}{8(1+2\alpha)}\max\left\{1,\frac{1}{4}\left|\frac{\left[2\mu B_1(1+2\alpha)-(1+3\alpha-\alpha\beta+\alpha^2\beta)\right]B_1}{(1+\alpha)^2}\right.\right. \\ &\left.\left. -\frac{B_2-B_1}{B_1}\right|\right\}, \end{aligned}$$

which is the required proof. □

Implementing the same method as in Theorem 3.1 for the classes $\mathcal{R}_q(\alpha,\Phi)$ and $\mathcal{L}_q(\alpha,\Phi)$, we get the following theorems.

THEOREM 3.2. *Let $0 \leq \alpha \leq 1$, $\phi(z)$ be as assumed in (3) and if $f(z)$ given by (1) satisfying the majorization condition*

$$\left(f'(z)\right)^{\alpha}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} - 1 \ll \phi\left(\Phi(\omega(z))\right) - 1,$$

then $|a_2| \leq \frac{B_1}{4(1+\alpha)}$ and

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{4(2+\alpha)} \max\left\{1, \left|\frac{1}{4}\left(\frac{\mu B_1}{(1+\alpha)^2} - \frac{B_2 - B_1}{B_1} - \frac{(\alpha-1)(\alpha+2)B_1}{2(\alpha+1)^2}\right)\right|\right\}.$$

THEOREM 3.3. *Let $0 \leq \alpha \leq 1$, $\phi(z)$ be as assumed in (3), $\varphi(z)$ is given in (4) and if $f(z)$ given by (1) and satisfying the condition that*

$$\left(f'(z)\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} - 1 \ll \phi\left(\Phi(\omega(z))\right) - 1,$$

then $|a_2| \leq \frac{B_1}{8}$ and

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{12(2-\alpha)} \max\left\{1, \left|\frac{1}{4}\left(\frac{3\mu B_1(2-\alpha)}{4} - \frac{B_2 - B_1}{B_1} - (\alpha-1)B_1\right)\right|\right\}.$$

REMARK 3.4. By fixing the parameters α and β as mentioned in Remarks 1.5 to 1.7, one can easily estimate the coefficient bounds for the subclasses using quasi-subordination and majorization relating with sigmoid functions.

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