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ON FLAT EPIMORPHISMS OF RINGS AND POINTWISE LOCALIZATIONS

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Abstract. In this paper all rings are commutative. We prove some new results on flat epimorphisms of rings and pointwise localizations. Especially among them, it is proved that a ring R is an absolutely flat (von-Neumann regular) ring if and only if it is isomorphic to the pointwise localization $R^{(-1)}R$, or equivalently, each R-algebra is R-flat. For a given minimal prime ideal $\mathfrak{p}$ of a ring R, the surjectivity of the canonical map $R \to R_{\mathfrak{p}}$ is characterized. Finally, we give a new proof to the fact that in a flat epimorphism of rings, the contraction-extension of an ideal equals the same ideal.

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1. INTRODUCTION AND PRELIMINARIES

In this paper, all rings are commutative. Let $\varphi : R \to S$ be a morphism of rings and J an ideal of S. Then it is easy to see that the contraction and then extension of J under φ is contained in J, i.e., $J^{ce} \subseteq J$, see [2, Proposition 1.17]. In this paper, in Theorem 2.5, we give a new proof to the fact that if φ is a flat epimorphism of rings, then the equality holds.

Absolutely flat rings play a major role throughout this paper. We give two new characterizations for absolutely flat rings. The first one states that a given ring R is absolutely flat if and only if each R-algebra is R-flat, see Theorem 2.2. The second characterization states that a ring R is absolutely flat if and only if it is canonically isomorphic to the pointwise localization $R^{(-1)}R$, see Theorem 3.13.

By an epimorphism of rings $\varphi : R \to S$ we mean it is an epimorphism in the category of commutative rings. It is important to notice that surjective ring maps are special cases of epimorphisms. As an example, the canonical ring map $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism of rings which is not surjective. It is well known that a morphism of rings $R \to S$ is an epimorphism if and only if in the ring $S \otimes_R S$, $s \otimes 1 = 1 \otimes s$ for all $s \in S$. It is also well known that any faithfully flat epimorphism of rings is an isomorphism. In particular, an epimorphism

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of rings with source a field is an isomorphism if and only if the target is a nonzero ring. We refer the interested reader to [3, Tag 04VM], [4], [5], [7], [9], [10] and [11] for a comprehensive discussion of epimorphisms of commutative rings.

By a flat epimorphism of rings we mean a ring map which is both a flat ring map and an epimorphism of rings. If S is a multiplicative subset of a ring R, then the canonical ring map $R \to S^{-1}R$ is a typical example of flat epimorphisms of rings.

For a given ring R, the quotient ring $R/\mathfrak{N}$ is denoted by R_{red} where $\mathfrak{N}$ is the nil-radical of R. For any ring map $\varphi : R \to S$ the induced map $R_{\text{red}} \to S_{\text{red}}$ is denoted by φ_{red}.

We shall freely use the above facts in this paper.

2. ABSOLUTELY FLAT RINGS AND EPIMORPHISMS OF RINGS

Recall that a ring R is said to be an absolutely flat ring if each R-module is R-flat. It is well known that a ring R is absolutely flat if and only if it is von-Neumann regular ring (i.e., each $r \in R$ can be written as $r = r^2 s$ for some $s \in R$). In the following result we give a new and quite elementary proof for this well known fact.

THEOREM 2.1. *Let R be a ring. Then R is an absolutely flat ring if and only if each $r \in R$ can be written as $r = r^2 s$ for some $s \in R$.*

Proof. If R is an absolutely flat ring then R/I is R-flat where $I = (r)$. Then by [13, Remark 2.2], $I = I^2$ and so $r = r^2 s$ for some $s \in R$. To prove the reverse implication, it will be enough to show that for each R-module M and for each ideal I of R, then the canonical map $I \otimes_R M \to M$ which sends each pure tensor $a \otimes m$ of $I \otimes_R M$ into am is injective. Assume $\sum\limits_{i=1}^{n} a_i m_i = 0$ where $a_i \in I$ and $m_i \in M$ for all i. By hypothesis, each $a_i = r_i a_i^2$ for some $r_i \in R$. For $i = 1, 2$ we have $a_i = a_i b'$ where $b' = r_1 a_1 + r_2 a_2 - r_1 r_2 a_1 a_2 \in I$. Thus by induction on n, we may find some $b \in I$ such that $a_i = a_i b$ for all $i = 1, \dots, n$. It follows that $\sum\limits_{i=1}^{n} a_i \otimes m_i = b \otimes (\sum\limits_{i=1}^{n} a_i m_i) = 0$. Hence, the above map is injective. $\square$

In the following result we provide a new characterization for absolutely flat rings.

THEOREM 2.2. *Let R be a ring. Then R is absolutely flat if and only if each R-algebra is R-flat.*

Proof. The implication "$\Rightarrow$" is clear. Conversely, let M be a R-module. Then consider the ring $S = R \times M$ whose the addition and multiplication are defined as $(r, m) + (r', m') = (r + r', m + m')$ and $(r, m).(r', m') = (rr', rm' + r'm)$, respectively. Clearly S is a commutative ring whose identity element is

$(1,0)$ and the map $\varphi : R \to S$ given by $r \rightsquigarrow (r,0)$ is a morphism of rings, (this construction is due to Nagata and in the literature, the ring S is called the "idealization" or the trivial extension of R by M). The R-module structure induced via φ on S is the same as the usual R-module structure on the direct sum $R \oplus M$. By hypothesis, φ is a flat morphism. Thus $S = R \oplus M$ is a flat R-module. It it well known that the direct sum of a family of R-modules is R-flat if and only if each factor is R-flat. Hence, M is a flat R-module. $\square$

LEMMA 2.3. *Let S and T be two multiplicative subsets of a ring R. Then $S^{-1}R \otimes_R T^{-1}R = 0$ if and only if there exist $f \in S$ and $g \in T$ such that $fg = 0$.*

Proof. It is proved exactly like [1, Lemma 3.1]. $\square$

THEOREM 2.4. *Let $\mathfrak{p}$ be a minimal prime ideal of a ring R. Then the canonical map $\pi : R \to R_{\mathfrak{p}}$ is surjective if and only if the canonical map $R_{\mathfrak{m}} \to R_{\mathfrak{p}}$ is surjective for all $\mathfrak{m} \in \mathrm{Max}(R) \cap V(\mathfrak{p})$.*

Proof. The implication "$\Rightarrow$" is clear, since the map π factors as:

$$R \longrightarrow R_{\mathfrak{m}} \longrightarrow R_{\mathfrak{p}}.$$

To see the converse, it suffices to show that the induced map $\pi_{\mathfrak{m}} : R_{\mathfrak{m}} \to (R_{\mathfrak{p}})_{\mathfrak{m}}$ is surjective for all $\mathfrak{m} \in \mathrm{Max}(R)$. If $\mathfrak{p} \subseteq \mathfrak{m}$, then $(R_{\mathfrak{p}})_{\mathfrak{m}} \simeq R_{\mathfrak{p}}$. Thus by the hypothesis, $\pi_{\mathfrak{m}}$ is surjective. But if $\mathfrak{p} \nsubseteq \mathfrak{m}$, then choose $f \in \mathfrak{p} \setminus \mathfrak{m}$. Clearly $\mathfrak{p}R_{\mathfrak{p}}$ is the nil-radical of $R_{\mathfrak{p}}$. So there exists some $g \in R \setminus \mathfrak{p}$ such that fg is nilpotent. Then by Lemma 2.3 or by [1, Lemma 3.1], $(R_{\mathfrak{p}})_{\mathfrak{m}} \simeq R_{\mathfrak{p}} \otimes_R R_{\mathfrak{m}} = 0$. Hence, $\pi_{\mathfrak{m}}$ is surjective. $\square$

THEOREM 2.5. *Let $\varphi : R \to S$ be a flat epimorphism of rings. Then the following statements hold:*

(i) *If $\mathfrak{q}$ is a prime ideal of S, then the induced map $\varphi_{\mathfrak{q}} : R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ is an isomorphism of rings where $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$.*

(ii) *If J is an ideal of S, then $J^{ce} = J$.*

(iii) *The induced map $\varphi^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a homeomorphism onto its image.*

(iv) *If R is a Noetherian ring, then S is as well.*

(v) *If R is an Artinian ring, then S is as well.*

Proof. (i) See [12, Lemma 2.1].

(ii) Clearly $IS \subseteq J$ where $I := \varphi^{-1}(J)$. To see the reverse inclusion, we have $S/J \cong S/J \otimes_R S$ as S-modules. On the other hand, since S is flat over R, thus from the exact sequence $0 \longrightarrow R/I \xrightarrow{\overline{\varphi}} S/J$ we obtain the following exact sequence $0 \longrightarrow R/I \otimes_R S \xrightarrow{\overline{\varphi} \otimes 1_S} S/J \otimes_R S$. Furthermore, the following diagram is commutative:

$$\begin{array}{ccc} R/I \otimes_R S & \xrightarrow{\overline{\varphi}\otimes 1_S} & S/J \otimes_R S \\ \downarrow & & \downarrow \\ S/IS & \longrightarrow & S/J \end{array}$$

therefore $S/IS \to S/J$ is injective. Thus, $J \subseteq IS$.

(iii) By (ii), the function φ^* is a closed map onto its image.

(iv) Take an arbitrary ideal J of S, then $I = \varphi^{-1}(J) = (a_1, ..., a_n)$ is a finitely generated ideal, since R is a noetherian ring. By (ii), $J = IS = \big(\varphi(a_1), ..., \varphi(a_n)\big)$. Hence, S is a Noetherian ring.

(v) Using (ii), then every descending chain of ideals of S stabilizes. $\square$

Let $\varphi : R \to S$ be a morphism of rings and let J be the kernel of the canonical ring map $S \otimes_R S \to S$ given by $s \otimes s' \rightsquigarrow ss'$. Then it is well known that J/J^2 as S-module is canonically isomorphic to $\Omega_R(S)$, the module of Kähler differentials of S over R, (it is also denoted by $\Omega_{S/R}$). In particular, J is an idempotent ideal if and only if $\Omega_R(S) = 0$. Then we provide a new proof to the following well known result.

Theorem 2.6. *A ring map $\varphi : R \to S$ is an epimorphism of rings if and only if the following three conditions hold.*

(i) *The induced map $\varphi^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is injective.*

(ii) *If $\mathfrak{q}$ is a prime ideal of S, then the induced map $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$ is an isomorphism where $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$.*

(iii) *The kernel of the canonical ring map $p : S \otimes_R S \to S$ is a finitely generated and idempotent ideal.*

Proof. Assume $\varphi : R \to S$ is an epimorphism of rings. To prove (i), it will be enough to show that for each prime ideal $\mathfrak{p}$ of R, then $(\varphi^*)^{-1}(\mathfrak{p})$ has at most one element. It is well known that the fiber $(\varphi^*)^{-1}(\mathfrak{p})$ is homeomorphic to Spec $\big(S \otimes_R \kappa(\mathfrak{p})\big)$. On the other hand, the base change $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}) \otimes_R S$ is an epimorphism of rings. Hence, $\kappa(\mathfrak{p}) \otimes_R S$ is a field whenever it is a non-zero ring. Therefore, $(\varphi^*)^{-1}(\mathfrak{p})$ has at most one element. To prove (ii), let $\mathfrak{q}$ be a prime ideal of S laying over $\mathfrak{p}$, i.e., $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$. Consider the following commutative diagram of rings:

$$\begin{array}{ccc} R & \xrightarrow{\varphi=epic} & S \\ \downarrow & & \downarrow{\scriptstyle epic} \\ \kappa(\mathfrak{p}) & \longrightarrow & \kappa(\mathfrak{q}). \end{array}$$

The composition $R \longrightarrow \kappa(\mathfrak{p}) \longrightarrow \kappa(\mathfrak{q})$ is an epimorphism hence the induced ring map $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$ is also an epimorphism (in fact it is an isomorphism). The statement (iii) is obvious, since φ is an epimorphism if and only if p is an isomorphism. To prove the reverse implication we act as follows. Let T be a reduced ring and let $f, g : S \to T$ be two ring maps such that $f \circ \varphi = g \circ \varphi$. We claim that $f = g$. We have $\varphi^* \circ f^* = (f \circ \varphi)^* = (g \circ \varphi)^* = \varphi^* \circ g^*$. Therefore $f^* = g^*$, since φ^* is injective. If P is a prime ideal of T, then setting $\mathfrak{q} := f^{-1}(\mathfrak{P})$, also setting $\mathfrak{p} := \varphi^{-1}(\mathfrak{q})$. Denote by $\widetilde{\varphi} : \prod\limits_{P \in \operatorname{Spec}(T)} \kappa(\mathfrak{p}) \to \prod\limits_{P \in \operatorname{Spec}(T)} \kappa(\mathfrak{q})$ the ring map induced via $\varphi : R \to S$. By the hypothesis (ii), $\widetilde{\varphi}$ is an isomorphism. Similarly above, denote by $\widetilde{f}, \widetilde{g} : S' = \prod\limits_{P \in \operatorname{Spec}(T)} \kappa(\mathfrak{q}) \to T' = \prod\limits_{P \in \operatorname{Spec}(T)} \kappa(P)$ the ring maps induced by f and g, respectively. From $f \circ \varphi = g \circ \varphi$ we conclude that $\widetilde{f} \circ \widetilde{\varphi} = \widetilde{g} \circ \widetilde{\varphi}$. Thus $\widetilde{f} = \widetilde{g}$, since $\widetilde{\varphi}$ is an isomorphism. The following diagram is commutative:

$$\begin{array}{ccc} S & \xrightarrow{f,g} & T \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho'} \\ S' & \xrightarrow{\widetilde{f},\widetilde{g}} & T' \end{array}$$

i.e. $\widetilde{f} \circ \rho = \rho' \circ f$ and $\widetilde{g} \circ \rho = \rho' \circ g$. Thus $\rho' \circ f = \rho' \circ g$. Since T is reduced, therefore ρ' is injective and so $f = g$. This establishes the claim. Now to conclude the assertion, since $i_1 \circ \varphi = i_2 \circ \varphi$ therefore $\eta \circ i_1 \circ \varphi = \eta \circ i_2 \circ \varphi$ where $i_1, i_2 : S \to S \otimes_R S$ and $\eta : S \otimes_R S \to (S \otimes_R S)_{\text{red}}$ are the canonical maps. By applying what we have just proved for a morphism with target a reduced ring, we conclude that $\eta \circ i_1 = \eta \circ i_2$. Therefore for each $s \in S$, $s \otimes 1 - 1 \otimes s$ is nilpotent. By the hypothesis (iii), $J = \operatorname{Ker}(p)$ is generated by a finite number of nilpotent elements of the form $s \otimes 1 - 1 \otimes s$. Hence, J is a nilpotent ideal. Thus $J = 0$, since it is idempotent. Therefore, φ is an epimorphism of rings. $\square$

3. POINTWISE RINGS WITH APPLICATIONS

In this section we study the theory of pointwise localizations and some of its applications. This theory was originally introduced and studied during the Séminaire Samuel [9].

If R is an absolutely flat ring, then by Theorem 2.1, each element $a \in R$ can be written as $a = a^2 b$ for some $b \in R$. This leads us to the following definition.

Definition 3.1. Let R be a ring and let $a \in R$. If there is an element $b \in R$ such that $a = a^2 b$ and $b = b^2 a$, then b is said to be a pointwise inverse of a.

Lemma 3.2. *Let $a, b \in R$. Then b is a pointwise inverse of a if and only if $a \in Ra^2$. Moreover, if b is a pointwise inverse of a then there is an idempotent element $e \in R$ such that $(e + a)(e + b) = 1$. Finally, the pointwise inverse, if it exists, is unique.*

Proof. Suppose $a \in Ra^2$. We have $a = ra^2$ for some $r \in R$. Let $b = r^2 a$. Then b is a pointwise inverse of a. Clearly $e = 1 - ab$ is an idempotent element and $(e+a)(e+b)=1$. Let $c \in R$ be another pointwise inverse of a. We have $b = ab^2 = (ac)(ab^2) = a^2c^2b = ac^2 = c$. $\square$

The pointwise inverse of $a \in R$, if it exists, is usually denoted by $a^{(-1)}$. The pointwise inverse has appeared in the literature also under other names, e.g. outer inverse or 2-inverse.

LEMMA 3.3. *Let $\varphi : R \to S$ be a ring map. Suppose $a, b \in R$ have pointwise inverses in R. Then the pointwise inverses of $\varphi(a)$ and ab exist. Moreover $\varphi(a)^{(-1)} = \varphi(a^{(-1)})$ and $(ab)^{(-1)} = a^{(-1)}b^{(-1)}$.*

Proof. It is an easy exercise. $\square$

The following result establishes the universal property of the poinwise rings.

PROPOSITION 3.4. *Let R be a ring and let S be a subset of R. Then there exist a ring $S^{(-1)}R$ and a canonical ring map $\eta : R \to S^{(-1)}R$ such that for each $s \in S$, the pointwise inverse of $\eta(s)$ in $S^{(-1)}R$ exists and the pair $(S^{(-1)}R, \eta)$ satisfies in the following universal property: if there is a ring map $\varphi : R \to R'$ such that for each $s \in S$ the pointwise inverse of $\varphi(s)$ in R' exists then there is a unique ring map $\psi : S^{(-1)}R \to R'$ such that $\varphi = \psi \circ \eta$.*

Proof. Consider the polynomial ring $A = R[x_s : s \in S]$ and let $S^{(-1)}R = A/I$ where the ideal I is generated by elements of the form $sx_s^2 - x_s$ and $s^2x_s - s$ with $s \in S$. Let $\eta : R \to S^{(-1)}R$ be the canonical ring map. For each $s \in S$, the element $x_s + I$ is the pointwise inverse of $\eta(s) = s + I$. Let $\varphi : R \to R'$ be a ring map such that for each $s \in S$, the pointwise inverse of $\varphi(s)$ exists in R'. By the universal property of the polynomial rings, there is a (unique) homomorphism of R-algebras $\widetilde{\varphi} : R[x_s : s \in S] \to R'$ such that $x_s \rightsquigarrow \varphi(s)^{(-1)}$ for all $s \in S$. We have $\widetilde{\varphi}(I) = 0$. Denote by $\psi : S^{(-1)}R \to R'$ the ring map induced by $\widetilde{\varphi}$. Clearly ψ is the unique ring homomorphism such that $\varphi = \psi \circ \eta$. Because suppose there is another such ring map $\psi' : S^{(-1)}R \to R'$. Then we have $\psi(x_s + I) = \widetilde{\varphi}(x_s) = \varphi(s)^{(-1)} = \psi'\big(\eta(s)\big)^{(-1)} = \psi'\big(\eta(s)^{(-1)}\big) = \psi'(x_s + I)$ for all $s \in S$. Therefore $\psi = \psi'$. $\square$

We call $S^{(-1)}R$ the pointwise localization of R with respect to S.

PROPOSITION 3.5. *Let R be a ring and let S be a subset of R. Then the following statements hold:*

(i) *The canonical ring map $\eta : R \to S^{(-1)}R$ is an epimorphism.*

(ii) *The map $\eta^* : \mathrm{Spec}\big(S^{(-1)}R\big) \to \mathrm{Spec}(R)$ is bijective.*

(iii) *For each $s \in S$, then $(\eta^*)^{-1}\big(V(s)\big)$ is a clopen (both open and closed) subset of* $\mathrm{Spec}\big(S^{(-1)}R\big)$.

(iv) *The ring $S^{(-1)}R$ is nonzero if and only if R is as well.*

(v) $\mathrm{Ker}(\eta) \subseteq \mathfrak{N}$ *where $\mathfrak{N}$ is the nil-radical of R.*

Proof. (i) It follows from Proposition 3.4.

(ii) By Theorem 2.6, the map η^* is injective. To see surjectivity, let $\mathfrak{p}$ be a prime ideal of R and consider the canonical ring map $\pi : R \to \kappa(\mathfrak{p})$. The image of every element of R under π has a pointwise inverse in $\kappa(\mathfrak{p})$. Thus, by Proposition 3.4, there is a (unique) ring map $\psi : S^{(-1)}R \to \kappa(\mathfrak{p})$ such that $\pi = \psi \circ \eta$. Then $\mathfrak{p} = \eta^*(\mathfrak{q})$ where $\mathfrak{q} = \psi^{-1}(0)$.

(iii) We have $(\eta^*)^{-1}\big(V(s)\big) = V\big(\eta(s)\big)$. Moreover, we have $V\big(\eta(s)\big) = D\big(1-\eta(s)\eta(s)^{(-1)}\big)$.

(iv) and (v) These are immediate consequences of (ii). □

LEMMA 3.6. *Let $\varphi : R \to S$ be an epimorphism of rings where S is a nonzero ring with trivial idempotents. Suppose $\varphi(r)$ has a pointwise inverse in S for all $r \in R$. Then $A := \mathrm{Im}(\varphi)$ is an integral domain and S is its field of fractions.*

Proof. Suppose $\varphi(r)\varphi(r') = 0$ for some elements $r, r' \in R$. If $\varphi(r) \neq 0$ then $\varphi(r)\varphi(r)^{(-1)} = 1$ since $\varphi(r)\varphi(r)^{(-1)}$ is an idempotent element. Therefore A is an integral domain. Let K be the field of fractions of A. Since every non-zero element of A is invertible in S therefore by the universal property of the localization, there is a (unique) ring map $\psi : K \to S$ such that $i = \psi \circ j$ where $i : A \to S$ and $j : A \to K$ are the canonical injections. The map φ factors as $\varphi = i \circ \varphi'$ where $\varphi' : R \to A$ is the ring map induced by φ. Since φ is an epimorphism thus i and so ψ are epimorphisms. Hence, ψ is an isomorphism. □

THEOREM 3.7. *Let R be a ring and let $\eta : R \to R' = R^{(-1)}R$ be the canonical ring map. Then the following statements hold:*

(i) *If $\mathfrak{q}$ is a prime ideal of R', then $R'_{\mathfrak{q}}$ is canonically isomorphic to $\kappa(\mathfrak{p})$ where $\mathfrak{p} = \eta^{-1}(\mathfrak{q})$.*

(ii) *The ring $R^{(-1)}R$ is absolutely flat.*

Proof. (i) For each prime ideal $\mathfrak{q}$ of $R^{(-1)}R$, the map:

$$R \xrightarrow{\eta} R^{(-1)}R \longrightarrow R'_{\mathfrak{q}}$$

satisfies all of the hypotheses of Lemma 3.6. Therefore $R'_{\mathfrak{q}}$ is a field. Now consider the following commutative diagram:

$$\begin{tikzcd} R_{\mathfrak{p}} \arrow[r, "\eta_{\mathfrak{q}}=epic"] \arrow[d] & R'_{\mathfrak{q}} \arrow[d, "\simeq"] \\ \kappa(\mathfrak{p}) \arrow[r] & \kappa(\mathfrak{q}) \end{tikzcd}$$

where $\mathfrak{p} = \eta^*(\mathfrak{q})$. The ring map $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$ is an isomorphism, since it is an epimorphism.

(ii) It is deduced from (i) and the fact that the absolutely flatness is a local property. □

By Proposition 3.4 and Theorem 3.7, the assignment $R \rightsquigarrow R^{(-1)}R$ is a covariant functor form the category of commutative rings into the category of absolutely flat rings.

LEMMA 3.8. *Let $\varphi : R \to S$ be a ring map, let M and N be S-modules and consider the canonical map $\eta : M \otimes_R N \to M \otimes_S N$ which maps each pure tensor $m \otimes_R n$ into $m \otimes_S n$. Then $\mathrm{Ker}(\eta)$ is generated by elements of the form $sm \otimes_R n - m \otimes_R sn$ with $s \in S \setminus \mathrm{Im}(\varphi)$, $m \in M$ and $n \in N$. In particular, if φ is an epimorphism of rings then η is an isomorphism.*

Proof. Let K be the R-submodule of $M \otimes_R N$ generated by elements of the form $sm \otimes_R n - m \otimes_R sn$ with $s \in S \setminus \mathrm{Im}(\varphi)$, $m \in M$ and $n \in N$. Clearly $K \subseteq \mathrm{Ker}(\eta)$. Consider the map $\overline{\eta} : P = M \otimes_R N/K \to M \otimes_S N$ induced by η. We have $\mathrm{Ker}(\overline{\eta}) = \mathrm{Ker}(\eta)/K$. The scalar multiplication $S \times P \to P$ which is defined on pure tensors by $s.(m \otimes_R n + K) = sm \otimes_R n + K$ is actually well-defined and puts a S-module structure over P. By the universal property of the tensor products, the S-bilinesr map $M \times N \to P$ defined by $(m, n) \rightsquigarrow m \otimes_R n + K$ induces a (unique) S-homomorphism $M \otimes_S N \to P$ which maps each pure tensor $m \otimes_S n$ into $m \otimes_R n + K$. This implies that $\overline{\eta}$ is bijective. Therefore $\mathrm{Ker}(\eta) = K$. □

LEMMA 3.9. *Let $\varphi : R \to S$ be a flat ring map which has a factorization $R \xrightarrow{\psi} A \xrightarrow{\varphi'} S$ such that φ' is an injective ring map and ψ is an epimorphism of rings. Then φ' is a flat ring map.*

Proof. For each A-module M, the canonical map $\eta_M : M \otimes_R S \to M \otimes_A S$ which maps each pure tensor $m \otimes_R s$ into $m \otimes_A s$ is injective because in $A \otimes_R A$-module $M \otimes_R S$ we have $am \otimes_R s = (a \otimes_R 1_A).(m \otimes_R s) = (1_A \otimes_R a).(m \otimes_R s) = m \otimes_R a.s$ then apply Lemma 3.8. In fact, it is bijective. Now suppose $0 \longrightarrow N \xrightarrow{f} M$ is an exact sequence of A-modules. The following diagram is commutative:

$$\begin{array}{ccc} N \otimes_R S & \xrightarrow{f \otimes_R 1} & M \otimes_R S \\ \big\downarrow{\scriptstyle \eta_N} & & \big\downarrow{\scriptstyle \eta_M} \\ N \otimes_A S & \xrightarrow{f \otimes_A 1} & M \otimes_A S \end{array}$$

and the map $f \otimes_R 1$ is injective since S is flat over R. Therefore $f \otimes_A 1$ is injective as well. Hence, S is a flat module over A, i.e., φ' is a flat ring map. □

LEMMA 3.10. *Let $\varphi : R \to S$ be a flat epimorphism of rings. Then for each prime $\mathfrak{p}$ of R we have either $\mathfrak{p}S = S$ or the canonical ring map $R_{\mathfrak{p}} \to T^{-1}S$ given by $r/s \rightsquigarrow \varphi(r)/\varphi(s)$ is an isomorphism where $T = \varphi(R \setminus \mathfrak{p})$.*

Proof. Suppose $\mathfrak{p}S \neq S$ for some prime $\mathfrak{p}$. The canonical map $R_{\mathfrak{p}} \to T^{-1}S$ is a flat epimorphism because flat morphisms and epics are stable under base change and composition (recall that the ring $T^{-1}S$ is canonically isomorphic to $S_{\mathfrak{p}}$). It is also faithfully flat since $\mathfrak{p}S \neq S$. Hence, it is an isomorphism. $\square$

It is worth mentioning that the converse of Lemma 3.10 also holds.

Theorems 3.11 and 3.12 are well known, we provide new proofs for them.

THEOREM 3.11. *Let $\varphi : R \to S$ be a flat epimorphism of rings. If φ_{red} is surjective, then φ is as well.*

Proof. The map φ factors as $R \xrightarrow{\pi} R/\operatorname{Ker}(\varphi) \xrightarrow{\varphi'} S$ where π is the canonical ring map and φ' is induced by φ. We have $\operatorname{Im}(\varphi) = \operatorname{Im}(\varphi')$, φ' is an epimorphism and φ'_{red} is surjective. Moreover, by Lemma 3.9, φ' is flat. Therefore, without loss of generality, we may assume that φ is injective. It follows that φ_{red} is an isomorphism and so $\varphi^* : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is bijective. Therefore $\mathfrak{p}S \neq S$ for all primes $\mathfrak{p}$ of R and so by Lemma 3.10, the canonical map $R_{\mathfrak{p}} \to S_{\mathfrak{p}}$ is bijective. It follows that $S/\varphi(R) \otimes_R R_{\mathfrak{p}} = 0$ for all primes $\mathfrak{p}$. $\square$

THEOREM 3.12. *Let $\varphi : R \to S$ be an epimorphism of rings such that R is absolutely flat. Then φ is surjective.*

Proof. The map φ factors as $R \xrightarrow{\pi} R/\operatorname{Ker}(\varphi) \xrightarrow{\varphi'} S$ where π is the canonical ring map and φ' is the injective ring map induced by φ. The quotient ring $R/\operatorname{Ker}(\varphi)$ is absolutely flat. Moreover, $\operatorname{Im}(\varphi) = \operatorname{Im}(\varphi')$ and yet φ' is an epimorphism. Hence, without loss of generality, we may assume that φ is injective. In this case, φ is a faithfully flat morphism. Because, suppose $S \otimes_R M = 0$ for some R-module M. From the following short exact sequence of R-modules

$$0 \longrightarrow R \xrightarrow{\varphi} S \xrightarrow{\pi} S/R \longrightarrow 0$$

we obtain the following long exact sequence of R-modules $\ldots \longrightarrow$

$$\operatorname{Tor}_1^R(S/R, M) \longrightarrow R \otimes_R M \xrightarrow{\varphi\otimes 1_M} S \otimes_R M \xrightarrow{\pi\otimes 1_M} S/R \otimes_R M \longrightarrow 0 \,.$$

But $\operatorname{Tor}_1^R(S/R, M) = 0$ since S/R is R-flat, see [8, Theorem 7.2]. Thus $M \simeq R \otimes_R M = 0$. Therefore φ is a faithfully flat epimorphism and so it is an isomorphism. This means that, in our factorization $\varphi = \varphi' \circ \pi$, φ' is an isomorphism. Therefore the original φ is surjective. $\square$

THEOREM 3.13. *Let R be a ring. Then R is an absolutely flat ring if and only if the canonical ring map $\eta : R \to R^{(-1)}R$ is an isomorphism.*

Proof. Suppose R is absolutely flat. Then, by Theorem 3.12, η is surjective. Pick $a \in \operatorname{Ker}(\eta)$. By Theorem 2.1, there exists some $b \in R$ such that $a = ba^2$. It follows that $a = b^{n-1}a^n$ for all $n \geqslant 1$. But a is a nilpotent element, see Proposition 3.5. Therefore $a = 0$. The reverse implication follows from Theorem 3.7. $\square$

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