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INTEGRAL INEQUALITIES IN AMALGAM SPACE

SUKET KUMAR

Abstract. Weighted norm inequalities for the Volterra integral operator with the kernel have been characterized in the amalgam space for certain ranges of indices.

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Key words. Hardy's inequality, amalgam space, weight function, weight sequence.

1. INTRODUCTION

Amalgam of L^p and ℓ^q on the real line is the space $\ell^q(L^p)$ consisting of functions which are locally integrable on $[n, n+1]$, $n \in Z$, the set of integers, where the integrals over the intervals $[n, n+1]$ form ℓ^q sequence. This space was introduced by N. Wiener [8]. Amalgam space is related to various area of analysis, such as, to almost periodic function, Tauberian theorem, Fourier transform, Fourier multipliers, approximation theory, algebras and modules and product convolution operator. For the study of amalgam space, we refer to [3, 4, 5] and references cited therein.

The study of the Volterra operator is an important area of analysis, differential equation and boundary value problems. For the study of the Volterra operator we refer to [1, 2, 6, 7] and references therein. Sufficient conditions for the validity of the Hardy's inequality for the Volterra operator in the amalgam space is available in [2, Theorem 3.1] for the operator $(Tf)(x) = \int_{-\infty}^{x} k(x,y)f(y)\mathrm{d}y$, $f \geq 0$, where kernel $k(x,y)$ is non-increasing in the first variable, non-decreasing in the second variable and defined on the set $S = \{(x,y) \in R^2 : y < x\}$.

Motivated by this, in Section 2, of this paper, we consider the operator T with the kernel (discussed in [1, 7]), which is different from those considered in [2] and give necessary as well as sufficient conditions for the validity of the Hardy's inequality in the amalgam space for such operator for four different cases such that $1 < p, \bar{p}, q, \bar{q} < \infty$.

Throughout the paper, u and v are weight functions, that is, a measurable function positive almost everywhere in the appropriate interval, $\chi_{[n,n+1]}$ is the characteristic function defined on $[n, n+1]$, $1 < p, q, \bar{p}, \bar{q} < \infty$, the conjugate index p' of p is given by $p' = p/(p-1)$ and the same is true for other indices. T

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is the Volterra operator with the Oinarov kernel, R is the set of real numbers. Z denote the set of integers.

We conclude this section by the following basic definitions and results:

DEFINITION 1.1. The kernel $k(x, y)$; $x, y \in R$ is called Oinarov kernel if it satisfies (i) $k(x, y) \geq 0$ (ii) $k(x, y)$ is increasing in x and decreasing in y and (iii) $k(x, y) \sim (k(x, z) + k(z, y))$, $0 < y < z < x$; $x, y, z \in R$. If $x, y, z \in Z$, $k(x, y)$ is called discrete Oinarov kernel.

DEFINITION 1.2. A function f defined on $(-\infty, \infty)$ belongs to weighted amalgam space, denoted by $\ell^q(L_u^p)$ if

$$\|f\|_{p,u,q} = \left(\sum_{n=-\infty}^{\infty}\left(\int_n^{n+1} | f(x) |^p \, u(x)\mathrm{d}x\right)^{q/p}\right)^{1/q} < \infty,$$

where $1 < p, q < \infty$ and $u(x)$ is a weight function defined on $(-\infty, \infty)$.

In the following result, we characterize the discrete Hardy's inequality:

PROPOSITION 1.3. *Suppose $\{u_n\}$ and $\{v_n\}$ are weight sequences and $k(m, n)$ is the discrete Oinarov kernel. The inequality*

$$\left(\sum_{n=-\infty}^{\infty} | \sum_{m=-\infty}^{n} k(n, m) a_m |^q \, u_n\right)^{1/q} \leq C\left(\sum_{n=-\infty}^{\infty} | a_n |^{\bar{q}} \, v_n\right)^{1/\bar{q}}$$

holds for all non-negative sequence $\{a_n\}$ belongs to sequence space $\ell^{\bar{q}}(v_n)$ and a suitable constant $C > 0$ if and only if

(a) *in case $1 < \bar{q} \leq q < \infty$*

$$\sup_{n\in \mathrm{Z}}\left(\sum_{m=n}^{\infty} k^q\,(m, n)\, u_m\right)^{1/q}\left(\sum_{m=-\infty}^{n} v_m^{1-\bar{q}'}\right)^{1/\bar{q}} < \infty,$$

$$\sup_{n\in \mathrm{Z}}\left(\sum_{m=n}^{\infty} u_m\right)^{1/q}\left(\sum_{m=-\infty}^{n} k^{\bar{q}'}(n, m) v_m^{1-\bar{q}'}\right)^{1/\bar{q}} < \infty,$$

(b) *in case $1 < q < \bar{q} < \infty$*

$$\left(\sum_{n\in \mathrm{Z}}\left(\sum_{m=n}^{\infty} k^q(m, n) u_m\right)^{r/q}\left(\sum_{m=-\infty}^{n} v_m^{1-\bar{q}'}\right)^{r/q'} v_n^{1-\bar{q}'}\right)^{1/r} < \infty,$$

$$\left(\sum_{n\in \mathrm{Z}}\left(\sum_{m=n}^{\infty} u_m\right)^{r/\bar{q}}\left(\sum_{m=-\infty}^{n} k^{\bar{q}'}(n, m) v_m^{1-\bar{q}'}\right)^{r/\bar{q}'} u_n\right)^{1/r} < \infty,$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{\bar{q}}$.

The proof of Proposition 1.3 is similar to the proof of the continuous case available in [1], [6, Theorems 2.10, 2.15]. We omit the details.

2. MAIN RESULTS

Consider the Volterra operator $(Tf)(x) = \int_{-\infty}^{x} k(x,y)f(y)\mathrm{d}y$, where $k(x,y)$ is the Oinarov kernel. In this section, we characterize the Hardy's inequality for this Volterra operator with the Oinarov kernel in the amalgam space for certain ranges of indices, which covers the four different cases. The following are the results:

THEOREM 2.1. *Suppose u and v are weight functions and $1 < p, \bar{p} < \infty$, $1 < \bar{q} \le q < \infty$. T is the operator with kernel k defined above. There exists a constant $B > 0$ such that the inequality*

$$\|Tf\|_{p,u,q} \le B\|f\|_{\bar{p},v,\bar{q}} \tag{1}$$

holds for all $f \in \ell^{\bar{q}}(L_v^{\bar{p}})$ if and only if

(i) *in case $\bar{p} \le p$, $\max(C_1, C_2, C_3, C_4) < \infty$, where*

$$C_1 = \sup_{n\in Z} \sup_{\alpha\in(n-1,n+1)} \left(\int_{\alpha}^{n+1} k^p(s,\alpha)u(s)ds\right)^{1/p} \left(\int_{n-1}^{\alpha} v^{1-\bar{p}'}(s)ds\right)^{1/\bar{p}'},$$

$$C_2 = \sup_{n\in Z} \sup_{\alpha\in(n-1,n+1)} \left(\int_{\alpha}^{n+1} u(s)ds\right)^{1/p} \left(\int_{n-1}^{\alpha} k^{\bar{p}'}(\alpha,s)v^{1-\bar{p}'}(s)ds\right)^{1/\bar{p}'},$$

$$C_3 = \sup_{n\in Z} \left(\sum_{m=n+1}^{\infty} k^q(m,n+1)\left(\int_{m-1}^{m} u\right)^{q/p}\right)^{1/q} \times \left(\sum_{m=-\infty}^{n} \left(\int_{m}^{m+1} v^{1-\bar{p}'}\right)^{\bar{q}'/\bar{p}'}\right)^{1/\bar{q}'},$$

$$C_4 = \sup_{n\in Z} \left(\sum_{m=n+1}^{\infty} \left(\int_{m-1}^{m} u\right)^{q/p}\right)^{1/q} \times \left(\sum_{m=-\infty}^{n} k^{\bar{q}'}(n+1,m)\left(\int_{m}^{m+1} v^{1-\bar{p}'}\right)^{\bar{q}'/\bar{p}'}\right)^{1/\bar{q}'},$$

(ii) *in case $p < \bar{p}$ $\max(C_3, C_4, C_5, C_6) < \infty$, where*

$$C_5 = \sup_{n\in Z} \left(\int_{n-1}^{n+1} \left(\int_{\alpha}^{n+1} k^p(s,\alpha)u(s)ds\right)^{\frac{r}{p}} \left(\int_{n-1}^{\alpha} v^{1-\bar{p}'}\right)^{\frac{r}{p'}} v^{1-\bar{p}'}\right)^{1/r},$$

$$C_6 = \sup_{n\in Z} \left(\int_{n-1}^{n+1} \left(\int_{\alpha}^{n+1} u\right)^{\frac{r}{\bar{p}}} \left(\int_{n-1}^{\alpha} k^{\bar{p}'}(\alpha,s)v^{1-\bar{p}'}(s)ds\right)^{\frac{r}{\bar{p}'}} u\right)^{1/r}$$

and $\frac{1}{r} = \frac{1}{p} - \frac{1}{\bar{p}}$.

Proof. Sufficiency. Since $|Tf| \leq T(|f|)$, we assume without the loss of generality that $f \geq 0$. For $n \leq x \leq n+1$, we note that

$$(Tf)(x) = \int_{-\infty}^{n-1} k(x,y)f(y)\mathrm{d}y + \int_{n-1}^{x} k(x,y)f(y)\mathrm{d}y = T_1(x,n) + T_2(x,n).$$

Define $U_{n+1} = \int_n^{n+1} u$ and $a_i = \int_i^{i+1} f$. Using monotonicity of the Oinarov kernel, $k \geq 0$, and $a_{i-1} \geq 0$ for $f \geq 0$, we find that

$$\begin{aligned}
\left(\int_n^{n+1} |T_1(x,n)|^p u(x)\mathrm{d}x\right)^{\frac{1}{p}} &\leq U_{n+1}^{\frac{1}{p}} T_1(n+1,n) \\
&= U_{n+1}^{\frac{1}{p}} \int_{-\infty}^{n-1} k(n+1,y)f(y)\mathrm{d}y \\
&\leq U_{n+1}^{\frac{1}{p}} \int_{-\infty}^{n} k(n+1,y)f(y)\mathrm{d}y \\
&= U_{n+1}^{\frac{1}{p}} \sum_{i=-\infty}^{n-1} \int_i^{i+1} k(n+1,y)f(y)\mathrm{d}y \\
&\leq U_{n+1}^{\frac{1}{p}} \sum_{i=-\infty}^{n-1} k(n+1,i)a_i \\
&\leq U_{n+1}^{1/p} \left[\sum_{i=-\infty}^{n-1} k(n+1,i)a_i + k(n+1,n)a_n\right] \\
&= U_{n+1}^{1/p} \left(\sum_{i=-\infty}^{n} k(n+1,i)a_i\right).
\end{aligned}$$

Using an application of the Proposition 1.3(a), we find that for a constant $B_1 > 0$

$$\|T_1(x,n)\|_{p,u,q} \leq \left(\sum_{n\in Z} |\sum_{i=-\infty}^{n} k(n+1,i)a_i|^q U_{n+1}^{q/p}\right)^{1/q} \leq B_1 \left(\sum_{n\in Z} a_n^{\bar{q}} V_n\right)^{1/\bar{q}}$$

holds, if $\max(C_3, C_4) < \infty$. For $V_n = \left(\int_n^{n+1} v^{1-\bar{p}'}\right)^{-\bar{q}/\bar{p}'}$, $\left(\sum_{n\in Z} a_n^{\bar{q}} V_n\right)^{1/\bar{q}} \leq \|f\|_{\bar{p},v,\bar{q}}$. Using an application of [6, Theorem 2.10], we find $\max(C_1, C_2) < \infty$ for $\bar{p} \leq p$ and by using [6, Theorem 2.15], $\max(C_5, C_6) < \infty$ for $p < \bar{p}$ holds, which implies that

$$\begin{aligned}
\left(\int_n^{n+1} |T_2(x,n)|^p u(x)\mathrm{d}x\right)^{1/p} &\leq \left(\int_{n-1}^{n+1} \left(\int_{n-1}^{x} k(x,y)f(y)\mathrm{d}y\right)^p u(x)\mathrm{d}x\right)^{1/p} \\
&\leq B_2 \left(\int_{n-1}^{n+1} f^{\bar{p}} v\right)^{1/\bar{p}}
\end{aligned}$$

holds for a constant $B_2 > 0$. Using [2, Proposition 1.1] (proved in [3]), we find that

$$\|T_2(.,n)\|_{p,u,q} \le B_2 \|f\|_{\bar{p},v,\bar{q}}$$

holds. Using Minkowski's inequality in the amalgam (see [3]),

$$\|Tf\|_{p,u,q} \le \|T_1(.,n)\|_{p,u,q} + \|T_2(.,n)\|_{p,u,q},$$

we find that the sufficiency of $\max(C_1, C_2, C_3, C_4) < \infty$ for $\bar{p} \le p$ and $\max(C_3, C_4, C_5, C_6) < \infty$ for $p < \bar{p}$ is established.

Necessity. For a non-negative sequence $\{a_k\}$, let $f = \sum_{k \in Z} a_k v^{1-\bar{p}'} \chi_{[k,k+1]}$. For $n \le x \le n+1$, using monotonicity of the Oinarov kernel, we find

$$\begin{aligned}
(Tf)(x) &= \int_{-\infty}^{x} k(x,y) f(y) \mathrm{d}y \\
&\ge \int_{-\infty}^{n-1} k(x,y) f(y) \mathrm{d}y \\
&\ge \sum_{\alpha=-\infty}^{n-2} \int_{\alpha}^{\alpha+1} k(n,y) f(y) \mathrm{d}y \\
&\ge \sum_{\alpha=-\infty}^{n-2} k(n, \alpha+1) A_\alpha,
\end{aligned}$$

where $A_\alpha = a_\alpha \int_\alpha^{\alpha+1} v^{1-\bar{p}'}(x) \mathrm{d}x$. Therefore,

$$\left(\int_n^{n+1} (Tf)^p(x) u(x) \mathrm{d}x \right)^{1/p} \ge \left(\sum_{\alpha=-\infty}^{n-2} k(n, \alpha+1) A_\alpha \right) U_{n+1}^{1/p}$$

holds for $U_{n+1} = \int_n^{n+1} u$. This implies

$$\|Tf\|_{p,u,q} \ge \left(\sum_{n \in Z} \left(\sum_{\alpha=-\infty}^{n} k(n+1, \alpha) A_\alpha \right)^q U_{n+1}^{q/p} \right)^{1/q},$$

whereas $\|f\|_{\bar{p},v,\bar{q}} = \left(\sum_{n \in Z} A_n^{\bar{q}} V_n \right)^{1/\bar{q}}$ holds for $V_n = \left(\int_n^{n+1} v^{1-\bar{p}'} \right)^{-\bar{q}/\bar{p}'}$. Therefore the inequality (1) holds for a constant $B > 0$, which implies that

$$\left(\sum_{n \in Z} \left(\sum_{\alpha=-\infty}^{n} k(n+1, \alpha) A_\alpha \right)^q U_{n+1}^{q/p} \right)^{1/q} \le B \left(\sum_{n \in Z} A_n^{\bar{q}} V_n \right)^{1/\bar{q}}$$

holds, which implies, using an application of the Proposition 1.3 (a), that $\max(C_3, C_4) < \infty$.

Next, we define $f = g\chi_{[m,m+1]}$, where $g \ge 0$, $m \in Z$ is fixed. Then the inequality (1) holds, which implies

$$\left(\int_m^{m+1} \left(\int_{m-1}^{x} k(x,y) g(y) \mathrm{d}y \right)^p u(x) \mathrm{d}x \right)^{1/p}$$

$$\leq \left(\int_{m-1}^{m+1}\left(\int_{m-1}^{x} k(x,y)g(y)\mathrm{d}y\right)^{p} u(x)\mathrm{d}x\right)^{1/p} \leq \|Tf\|_{p,u,q} \leq B\|f\|_{\bar{p},v,\bar{q}}$$

$$\leq B\left(\int_{m-1}^{m+1} g^{\bar{p}}v\right)^{1/\bar{p}}$$

holds for all $m \in Z$ and a constant $B > 0$ independent of m. This implies, using [6, Theorems 2.10, 2.15] that $\max(C_1, C_2) < \infty$ for $\bar{p} \leq p$ and $\max(C_5, C_6) < \infty$ for $p < \bar{p}$ holds. Therefore the necessity of $\max(C_1, C_2, C_3, C_4) < \infty$ for $\bar{p} \leq p$ and $\max(C_3, C_4, C_5, C_6) < \infty$ for $p < \bar{p}$ is established. □

REMARK 2.2. If T is the Hardy operator then $k(x,t) \equiv 1$ for $t < x$, $k(x,t) \equiv 0$ for $t > x$. Subsequently we find $C_1 \equiv C_2, C_3 \equiv C_4$ and C_6 is a multiple of C_5 [6, Remark 2.14]. Thus Theorem 3.1 becomes [2, Theorem 2.1].

For $1 < q < \bar{q} < \infty$, the embedding property of the involved sequence space differs from that used in Theorem 3.1. Consequently, the following result covers this case:

THEOREM 2.3. *Suppose u and v are weight functions $1 < p, \bar{p} < \infty$; $1 < q < \bar{q} < \infty$, $1/r = (1/q) - (1/\bar{q})$. Define*

$$C_7 = \left(\sum_{m\in Z}\left(\sum_{n=m+1}^{\infty} k^{q}(n, m+1)U_n^{q/p}\right)^{r/q}\left(\sum_{n=-\infty}^{m} V_n^{1-\bar{q}'}\right)^{r/q'} V_m^{1-\bar{q}'}\right)^{1/r},$$

$$C_8 = \left(\sum_{m\in Z}\left(\sum_{n=m+1}^{\infty} U_n^{q/p}\right)^{r/\bar{q}}\left(\sum_{n=-\infty}^{m} k^{\bar{q}'}(m+1, n)V_n^{1-\bar{q}'}\right)^{r/\bar{q}'} U_m^{q/p}\right)^{1/r}.$$

(i) *For $\bar{p} \leq p$, let*

$$A_n = \sup_{\alpha\in(n-1,n+1)}\left(\int_{\alpha}^{n+1} k^{p}(s,\alpha)u(t)\mathrm{d}t\right)^{1/p}\left(\int_{n-1}^{\infty} v^{1-\bar{p}'}(t)\mathrm{d}t\right)^{1/\bar{p}'},$$

$$B_n = \sup_{\alpha\in(n-1,n+1)}\left(\int_{\alpha}^{n+1} u(t)\mathrm{d}t\right)^{1/p}\left(\int_{n-1}^{\alpha} k^{\bar{p}'}(\alpha,t)v^{1-\bar{p}'}(t)\mathrm{d}t\right)^{1/\bar{p}'}.$$

(ii) *For $p < \bar{p}$, let*

$$D_n = \left(\int_{n-1}^{n+1}\left(\int_{\alpha}^{n+1} k^{p}(t,\alpha)u(t)\mathrm{d}t\right)^{\frac{s}{p}}\left(\int_{n-1}^{\alpha} v^{1-\bar{p}'}(t)\mathrm{d}t\right)^{\frac{s}{p'}} v^{1-\bar{p}'}(\alpha)\mathrm{d}\alpha\right)^{1/s},$$

$$E_n = \left(\int_{n-1}^{n+1}\left(\int_{\alpha}^{n+1} u(t)\mathrm{d}t\right)^{\frac{s}{\bar{p}}}\left(\int_{n-1}^{\alpha} k^{\bar{p}'}(\alpha,t)v^{1-\bar{p}'}(t)\mathrm{d}t\right)^{\frac{s}{\bar{p}'}} u(\alpha)\mathrm{d}\alpha\right)^{1/s},$$

where $\frac{1}{s} = \frac{1}{p} - \frac{1}{\bar{p}}$. Then the inequality (1) *holds for a constant $B > 0$ if* $\max(C_7, C_8) < \infty$, $\{A_n\} \in \ell^r$ *and* $\{B_n\} \in \ell^r$ (*respectively* $\max(C_7, C_8)$

$< \infty$, $\{D_n\} \in \ell^r$ and $\{E_n\} \in \ell^r$) in case $\bar{p} \le p$ (respectively, in case $p < \bar{p}$).

Conversely, $\max(C_7, C_8) < \infty$ *is necessary for* (1) *in case* $1 < p, \bar{p} < \infty$. *Also* $\sup_n(A_n, B_n) < \infty$ (*respectively* $\sup_n(D_n, E_n) < \infty$) *is necessary for* (1) *in case* $\bar{p} \le p$ (*respectively in case* $p < \bar{p}$).

Proof. Sufficiency. Without the loss of generality, we assume $f \ge 0$. Making similar arguments, as done in the proof of Theorem 2.1, we find

$$\begin{aligned}\|Tf\|_{p,u,q} &\le \left(\sum_{n\in Z} \mid \sum_{i=-\infty}^{n} k(n+1,i)a_i \mid^q U_{n+1}^{q/p}\right)^{1/q} \\ &+ \left(\sum_{n\in Z}\left(\int_n^{n+1} T_2^p(x,n)u(x)\mathrm{d}x\right)^{q/p}\right)^{1/q} \\ &= I_1 + I_2.\end{aligned}$$

Since $q < \bar{q}$, Proposition 1.3(b) yields $I_1 \le B\left(\sum_{n\in Z} a_n^{\bar{q}} V_n\right)^{1/\bar{q}}$, which is dominated by $B\|f\|_{\bar{p},v,\bar{q}}$ for $V_n = \left(\int_n^{n+1} v^{1-\bar{p}'}\right)^{-\bar{q}/\bar{p}'}$ and a constant $B > 0$ provided $\max(C_7, C_8) < \infty$.

(i) If $\bar{p} \le p$, then using [6, Theorem 2.10], we find

$$\begin{aligned}\left(\int_n^{n+1} T_2^p(x,n)u(x)\mathrm{d}x\right)^{1/p} &\le \left(\int_{n-1}^{n+1} T_2^p(x,n)u(x)\mathrm{d}x\right)^{1/p} \\ &\le C_n\left(\int_{n-1}^{n+1} f^{\bar{p}}v\right)^{1/\bar{p}}\end{aligned}$$

holds for $C_n \sim K_n = \max(A_n, B_n)$. Therefore, using Hölder's inequality with the index $\beta = \frac{\bar{q}}{q}$, we find that for a constant $A > 0$

$$\begin{aligned}\left(\sum_{n\in Z}\left(\int_{n-1}^{n+1} T_2^p(x,n)u(x)\mathrm{d}x\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} &\le A\left(\sum_{n\in Z} K_n^{\bar{q}}\left(\int_{n-1}^{n+1} f^{\bar{p}}v\right)^{\frac{q}{\bar{p}}}\right)^{\frac{1}{q}} \\ &\le A\|f\|_{\bar{p},v,\bar{q}}\left(\sum_{n\in Z} K_n^{q\beta'}\right)^{1/q\beta'},\end{aligned}$$

where in the last step, we use the embedding property given in [2, Proposition 1.1]. Since $q\beta' = r$ and $\{A_n\} \in \ell^r$, $\{B_n\} \in \ell^r$, the result follows.

(ii) If $p < \bar{p}$, the argument is similar as given in (i), only now, we apply [6, Theorem 2.15] in the place of [6, Theorem 2.10]. Therefore the sufficiency is established.

Necessity. The necessity of $\max(C_7, C_8) < \infty$ is established in the same way as done in the proof of Theorem 3.1, where we use Proposition 1.3 (b) in the place of Proposition 1.3 (a). To prove the necessity of $\sup_n(A_n, B_n) < \infty$

(respectively $\sup_n (D_n, E_n) < \infty$), we note that if $\|Tf\|_{p,u,q} \le B\|f\|_{\bar{p},v,\bar{q}}$ holds for all $f \in \ell^{\bar{q}}(L_v^{\bar{p}})$ with constant $B > 0$, then the inequality (1) holds in particular for all f, in the subspace $\ell^q(L_v^{\bar{p}})$ of $\ell^{\bar{q}}(L_v^{\bar{p}})$, since $q < \bar{q}$ [2, Proposition 1.1]. Theorem 2.1, is therefore applicable with $\bar{q} = q$, and thus the necessity is established. □

Remark 2.4. We define the dual operator of T as

$$(T^*f)(x) = \int_x^\infty k(y,x) f(y) \mathrm{d}y$$

[6] and the dual space of $\ell^q(L_u^p)$ as $\ell^{q'}(L_{u^{1-p'}}^{p'})$ [2, 3]. According to the duality argument, if we replace $p, \bar{p}, q, \bar{q}, u, v$ by, respectively, $\bar{p}', p', \bar{q}', q', v^{1-\bar{p}'}, u^{1-p'}$ in $C_i, i = 1,2,3,4,5,6,7,8,\ A_n, B_n, D_n, E_n$, as defined in the Theorems 2.1 and 2.3, we will get the conditions for the validity of (1) for T replaced by T^* for the four different cases as discussed in Theorems 2.1 and 2.3. We omit the details.

REFERENCES

[1] S. Bloom and R. Kerman, *Weighted norm inequalities for operators of Hardy type*, Proc. Amer. Math Soc., **113** (1991), 135–141.

[2] C. Carton-Lebrun, H.P. Heinig and S.C. Hofmann, *Integral operators on weighted amalgams*, Studia Math., **109** (1994), 133–157.

[3] J.J.F. Fournier and J. Stewart , *Amalgams of L^p and ℓ^q*, Bull. Amer. Math. Soc., **13** (1985), 1–21.

[4] H.P. Heinig and A. Kufner, *Weighted Friedrichs inequalities in amalgams*, Czechoslovak Math. J., **43** (1993), 285–308.

[5] F. Holland, *Harmonic analysis on amalgams of L^p and ℓ^q*, J. London Math. Soc., **10** (1975), 295–305.

[6] A. Kufner and L.-E. Persson, *Weighted Inequalities of Hardy Type*, World Scientific, 2003.

[7] R. Oinarov, *Weighted inequalities for a class of integral operators*, Soviet Math. Dokl., **44** (1992), 291–293.

[8] N. Wiener, *On the representation of functions by trigonometrical integrals*, Math. Z., **24** (1926), 575–616.

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National Institute of Technology, Hamirpur
Department of Mathematics
and Scientific Computing
Chakmoh, Galore, Hamirpur
Himachal Pradesh – 177005, India
E-mail: kumar.suket@gmail.com