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FIXED POINTS AND STABILITY OF A CLASS OF NONLINEAR DIFFERENTIAL SYSTEMS WITH SEVERAL DELAYS OF FEEDBACK CONTROL

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Abstract. In this work we offer existence criteria and sufficient conditions, so that the trivial solution of the differential system with several delays of feedback control is asymptotically stable. Here the fixed point technique is a practical method for this purpose. When these results are applied to some special delay mathematics models, some new results are obtained, and many known results are improved. Lastly, we provide an example that illustrates our results.

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Key words. Fixed points, stability, several delays, nonlinear differential systems.

1. INTRODUCTION

Delay differential equations (DDEs) have long played important roles in the real systems.Various delay differential equations, primarily taken from the biological sciences literature, are presented, along with necessary background from the application area, in order to motivate our study of delay differential equations. These range from models in population biology, physiology, epidemiology, economics, and neural networks, to control of mechanical systems also two-body problems of electrodynamics. For details of the derivation of these equations can be found in Mackey and Glass (1977), [1, 4, 15]. Existence, uniqueness, stability and positivity of solutions of DDEs and DDNSs are of great interest in mathematics and its applications, see [1]–[15]. For more examples on the use of Liapunov functionals, one can find them in Gopalsamy (1989), Burton (1985), Hale (1977) and the references cited therein. In this work we consider the following nonlinear systems modeling the dynamics of one predator species feeding exclusively on one prey species

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) + a(t) f_1(u(t-\tau(t))) + g_1(t, u(t), v(t-r(t))) = 0, \tag{1}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t}(t) + b(t) f_2(v(t-r(t))) + g_2(t, v(t), u(t-\tau(t))) = 0. \tag{2}$$

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Along with (1)–(2), we have to specify initial conditions as follows

$$u(t) = \psi_1(t) \text{ on } [m(t_0), t_0], \tag{3}$$

$$v(t) = \psi_2(t) \text{ on } [m(t_0), t_0], \tag{4}$$

where $f_1, f_2 : C(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$, $g_1, g_2 : \mathbb{R} \times C(\mathbb{R}, \mathbb{R}) \times C(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ and a, b are positive continuous functions. Here the functions $t - \tau(t)$, $t - r(t)$ are continuous and tend to infinity as $t \to +\infty$. For an elaborate discussion of the prey-predator association we refer to the (see [10, p. 197]). As before in [6, 8] we proved the existence of positive periodic solutions of the same class system. In the main results we will use a fixed point theorem to present a theorem that is very useful in obtaining stability of the zero solution.

2. PRELIMINARIES

In this section we begin our stability investigation with some preliminary definitions and notations.

2.1. DEFINITIONS AND NOTATIONS

For each $t_0 \geq 0$, define

$$\begin{aligned} m_1(t_0) &= : \inf\{t - \tau(t), t \geqslant t_0\}, \\ m_2(t_0) &= : \inf\{t - r(t), t \geqslant t_0\}, \\ m(t_0) &= \min\{m_1(t_0), m_2(t_0)\}. \end{aligned}$$

The functions p_1 and p_2 denote respectively, the inverse of $t-\tau_1(t)$ and $t-\tau_2(t)$. Let

$$C(t_0) := C([m(t_0), t_0], \mathbb{R}),$$

be the space of continuous functions endowed with function supremum norm $\|\cdot\|_0$, that is, for $\psi \in C(t_0)$,

$$\|\psi\|_0 := \sup\{|\psi(t)| : m(t_0) \leq t \leq t_0\}.$$

We will also use

$$\|\varphi\|_0 := \sup\{|\varphi(s)| : s \in [m(t_0), \infty)\},$$

to express the supremum of continuous bounded functions on $[m(t_0), \infty)$ later. It is well known (see [12]) that, for given continuous functions ψ_1 and ψ_2 there exists a solution for equation (1) on an interval $[m(t_0), T)$, and if the solution remains bounded, then $T = \infty$. To obtain stability of the zero solution of (1)–(4), we need the mapping P to map bounded functions into bounded functions. For that, we let $(\mathcal{C}, \|\cdot\|_0)$ to be the Banach space of real-valued bounded continuous functions on $[m(t_0), \infty)$ with the supremum norm $\|\cdot\|_0$, that is for $\varphi \in \mathcal{C}$

$$\|\varphi\|_0 := \sup\{|\varphi(t)|, \ t \in [m(t_0), \infty)\}.$$

Our investigations will be carried out on the complete metric space $(\mathcal{C}, \rho)$, where ρ is the uniform metric. That is, for $\varphi, \phi \in \mathcal{C}$ we set

$$\rho(\varphi, \phi) = \|\varphi - \phi\|_0 .$$

Let $\psi \in C([m(t_0), t_0], \mathbb{R})$ be fixed and define

$$S_\psi := \{\varphi : [m(t_0), \infty) \to \mathbb{R} \mid \varphi \in \mathcal{C},\ \varphi(t) = \psi(t) \text{ for } t \in [m(t_0), t_0]\},$$

being closed in $\mathcal{C}$, (S_ψ, ρ) is itself complete. There is no confusion if we use the norm $\|\cdot\|_0$ on $[m(t_0), t_0]$ or on $[m(t_0), \infty)$.

Our definitions follow ones in Hale and Lunel [12]. Let the system of delay differential equations

$$x'(t) = f(t, x_t),\ x \equiv \psi \text{ on } [t_0 - \tau(t_0), t_0] . \tag{5}$$

We denote by $x(t)$ the solution $x(t, t_0, \psi)$. Suppose that f satisfies $f(t, 0) = 0$, $t \in \mathbb{R}$ so that $x(t) = 0$ is a solution.

Remark 2.1. The stability of any other solution of (5) can be defined by changing variables such that the given solution is the zero solution. More precisely, given a solution y of (5) defined on $t \in \mathbb{R}$, its stability properties are those of the zero solution of

$$z'(t) = f(t, z + y) - f(t, y). \tag{6}$$

Indeed, if $x(t)$ is another solution of (5) and if we let $z(t) = x(t) - y(t)$ then $z_t = x_t - y_t$ so z satisfies (6). The special case that solution $y(t) = e$, an equilibrium, is of primary interest. In that case, let $\hat{e}$ be the constant function identically equal to e. Then equation (6) for the perturbation $z(t) = x(t) - e$ becomes

$$z'(t) = f(t, z_t + \hat{e}).$$

Note that by the change of variables the equilibrium $y(t) = e$ now becomes $z(t) = 0$.

Stability definitions, fixed point technique and more details on delay differential equations can be found in [11, 12, 14].

Definition 2.2. The zero solution of (2) is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $[\psi_1, \psi_2 \in \mathcal{C}(t_0)$ and $\|\psi_1\|_0 + \|\psi_2\|_0 < \delta]$ implies that $|u(t, t_0, \psi_1)| + |v(t, t_0, \psi_2)| < \varepsilon$ for $t \geq t_0$.

Definition 2.3. The zero solution of (2) is asymptotically stable if it is stable and there is a $\delta_1 = \delta_1(t_0) > 0$ such that $[\psi_1, \psi_2 \in \mathcal{C}(t_0)$ and $\|\psi_1\|_0 + \|\psi_2\|_0 < \delta_1]$ implies that $|u(t, t_0, \psi_1)| + |v(t, t_0, \psi_2)| \longrightarrow 0$ as $t \longrightarrow \infty$.

Definition 2.4. The zero solution is unstable if it is not stable.

2.2. BASIC IDEAS OF THE PROOF

To investigate the stability for both EDOs and DDEs by technique fixed point we need transform the problem into an abstract fixed point equation (as $P\varphi = \varphi$) in an appropriate normed. In order to apply this idea, one can use methods of the variation of parameters formula to our problem. A difficulty of this class of nonlinear system is the absence of a linear term in (1)–(4). So, for this situation we will now perturb the two terms $f_j(u(t-\tau(t)))$, $j = 1, 2$ in several different ways. On the other hand, to make (1)–(4) more tractable for this purpose, we transform it into a neutral functional differential equation of the form

$$u'(t) + A(t)u(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_t^{p_1(t)} \hat{f}(s, u(s))\,\mathrm{d}s - g_1(t, u(t), v(t-r(t))), \tag{7}$$

$$v'(t) + B(t)v(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_t^{p_2(t)} \tilde{f}_2(s, v(s))\,\mathrm{d}s - g_2(t, v(t), u(t-r(t))). \tag{8}$$

Next, from (7)–(8) we introduce a mapping P associated to (7)–(8) can be defined on a carefully chosen complete metric space X^0 in which P possesses a unique fixed point. The final result is an asymptotic stability theorem for the zero solution with a necessary and sufficient condition.

Lemma 2.5. *Let $\psi_1, \psi_2 \in C([m(0), 0], \mathbb{R})$ are a given continuous initial functions. Then, $(u(t), v(t))$ is a solution of equation (1)–(4) on an interval $[m(0), T)$ satisfying the initial condition $(u(t), v(t)) = (\psi_1(t), \psi_2(t))$ on $[m(0), 0]$ if and only if $(u(t), v(t))$ is a solution of the following integral equation.*

$$(u, v) = (P_1(u, v), P_2(u, v)), \tag{9}$$

where

$$\begin{aligned} u(t) &= -\left(u(0) + \int_0^{p_1(0)} a(s) f(\psi_1(s-\tau(s)))\,\mathrm{d}s\right) \mathrm{e}^{-\int_0^t A_1(t)\mathrm{d}t} \\ &+ \int_t^{p_1(t)} a(s) f_1(u(s-\tau(s)))\,\mathrm{d}s \\ &- \int_0^t A_1(s)\,\mathrm{e}^{-\int_s^t A_1(\theta)\mathrm{d}\theta} \int_s^{p_1(s)} a(y) f_1(u(y-\tau(y)))\,\mathrm{d}y\,\mathrm{d}s \\ &- \int_0^t \mathrm{e}^{-\int_s^t A_1(\theta)\mathrm{d}\theta} g_1(s, v(s-r(s)))\,\mathrm{d}s \\ &= P_1(u, v), \end{aligned} \tag{10}$$

and

$$
\begin{aligned}
v(t) = & -\left(v(0)+\int_0^{p_2(0)} a(s) f(\psi_2(s-\tau(s)))\,\mathrm{d}s\right)\mathrm{e}^{-\int_0^t A_2(t)\mathrm{d}t} \\
& + \int_t^{p_2(t)} a(s) f_2(u(s-\tau(s)))\,\mathrm{d}s \\
& - \int_0^t A_2(s)\,\mathrm{e}{-}\int_s^t A_2(\theta)\,\mathrm{d}\theta \int_s^{p_2(s)} b(y) f_2(u(y-\tau(y)))\,\mathrm{d}y\,\mathrm{d}s \\
& - \int_0^t \mathrm{e}^{-\int_s^t A_2(\theta)\mathrm{d}\theta} g_2(s, v(s-r(s)))\,\mathrm{d}s \\
& = P_2(u,v).
\end{aligned} \tag{11}
$$

Proof. Let the function p_1 denotes the inverse of $t-\tau_1(t)$. We have

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}\int_t^{p_1(t)} a(s) f_1(u(s-\tau(s)))\,\mathrm{d}s - p_1'(t)\,a(p_1(t))\,f_1(u(t)) \\
& = -a(t) f_1(u(t-\tau_1(t))).
\end{aligned}
$$

Then, the equation (1) can be written as

$$
\begin{aligned}
& u'(t) + A_1(t)\,u(t) \\
& = \frac{\mathrm{d}}{\mathrm{d}t}\int_t^{p_1(t)} a(s) f_1(u(s-r(s)))\,\mathrm{d}s - g_1(t, u(t), v(t-r(t))),
\end{aligned}
$$

with

$$
A_1(t) = p_1'(t)\,a(p_1(t))\,\frac{f_1(u(t))}{u(t)}.
$$

Applying the variation of parameters formula to the second equation of (14), we get

$$
\begin{aligned}
u(t) = & -\left(u(0)+\int_0^{p_1(0)} a(s) f_1(\psi_1(s-\tau(s)))\,\mathrm{d}s\right)\mathrm{e}^{-\int_0^t A_1(t)\mathrm{d}t} \\
& + \int_t^{p_1(t)} a(s) f_1(u(s-\tau(s)))\,\mathrm{d}s \\
& - \int_0^t A(s)\,\mathrm{e}^{-\int_s^t A_1(\theta)\mathrm{d}\theta} \int_s^{p_1(s)} a(y) f_1(u(y-\tau(y)))\,\mathrm{d}y\,\mathrm{d}s \\
& - \int_0^t \mathrm{e}^{-\int_s^t A_1(\theta)\mathrm{d}\theta} g_1(s, v(s-r(s)))\,\mathrm{d}s \\
& = P_1(u,v).
\end{aligned} \tag{12}
$$

A similar argumentation like in (12) we can deduce that $v = P_2(u,v)$. □

For that reason we assume that the followings conditions hold.

H1) There exist some functions R_1, $R_2 \in C(\mathbb{R}, \mathbb{R}^+)$ and constants $L_1 > 0$, $L_2 > 0$, $l > 0$ such that for $x, y \in [-l, l]$

$$\begin{aligned} |f_1(x) - f_1(y)| &\leq L_1 |x-y| \text{ and } |f_2(x) - f_2(y)| \leq L_2 |x-y|, \\ f_1(0) &= f_2(0) = g_1(t,0) = g_2(t,0) = 0 \text{ for } t \in \mathbb{R}^+, \\ |g_1(t,x) - g_1(t,y)| &\leq R_1(t) \|x-y\| \text{ for all } t \in \mathbb{R}, \\ |g_2(t,x) - g_2(t,y)| &\leq R_2(t) \|x-y\| \text{ for all } t \in \mathbb{R}. \end{aligned}$$

H2) There is a constant $0 < \alpha < 1$ satisfying

$$\begin{gathered} 2L_1 \sup \int_t^{p_1(t)} a(s)\,\mathrm{d}s + \sup \int_0^t R_1(s)\,\mathrm{e}^{-\int_s^t A_1(\theta)\mathrm{d}\theta}\mathrm{d}s \leq \alpha_1, \\ 2L_2 \sup \int_t^{p_2(t)} b(s)\,\mathrm{d}s + \sup \int_0^t R_2(s)\,\mathrm{e}^{-\int_s^t A_2(\theta)\mathrm{d}\theta} \leq \alpha_2, \\ \alpha = \max\{\alpha_1, \alpha_2\} < 1. \end{gathered}$$

H3) There exist constants $\beta_1 > 0$ and $\beta_2 > 0$ such that

$$\lim \frac{f_1(x)}{x} \geq \beta_1 > 0 \text{ and } \lim \frac{f_2(x)}{x} \geq \beta_2 > 0 \text{ as } x \to 0.$$

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS OF (1)–(4)

From (9) we shall derive a fixed point mapping P for (1)–(4). But the challenge here is to choose a suitable metric space of functions on which the map P can be defined on this set into itself but also is a contraction. Toward this, let C be the space of all continuous functions $\varphi : [m(0), +\infty) \to \mathbb{R}$. For a given initial function $\psi_j : [m(0), 0] \to [-l, l]$, $l > 0$, $j = 1, 2$ define the set

$$S_{\psi_j}^l := \{\varphi \in S_{\psi_j} \backslash \ \|\varphi(t)\|_0 \leq l\}, \ j = 1, 2,$$

and we introduce the set

$$X = S_{\psi_1}^l \times S_{\psi_2}^l,$$

with the norm

$$\|(\varphi, \phi)\|_X = \max\{\|\varphi\|_0, \|\phi\|_0\}, \text{ for all } (\varphi, \phi) \in X.$$

Next, we will define a mapping directly from (9). Remember that, by Lemma 2.5 a fixed point of that map will be a solution of the system (1)–(4). Define the mapping P on X as follows, for $t \leq 0$

$$P(\varphi_1(t), \varphi_2(t)) := (\psi_1(t), \psi_2(t)) \text{ on } [m(0), 0],$$

while for $t > 0$

$$P(\varphi_1(t), \varphi_2(t)) = (P_2(\varphi_1(t), \varphi_2(t)), P_1(\varphi_1(t), \varphi_2(t))). \tag{13}$$

We now establish the existence and uniqueness of solutions by showing that $P : X \to X$ if $\|\psi_1\|$ and $\|\psi_2\|$ are sufficiently small.

THEOREM 3.1. *Assume that the hypotheses* (H1)–(H3) *hold. Then:*
(i) P *maps* X *into itself and is a contraction mapping.*
(ii) *the system* (1)–(4) *has a unique solution in* X.

Proof. Obviously, for all $\varphi, \tilde{\varphi} \in X$ so that $\varphi = (\varphi_1, \varphi_2)$ and $\tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2)$ we have

$$\begin{aligned} \|P_1\varphi - P_1\tilde{\varphi}\| &< \alpha_1 \|\varphi - \tilde{\varphi}\|_X , \\ \|P_2\varphi - P_2\tilde{\varphi}\| &< \alpha_2 \|\varphi - \tilde{\varphi}\|_X . \end{aligned}$$

Thus

$$\begin{aligned} &\|P\varphi - P\tilde{\varphi}\|_X \\ &= \|(P_1(\varphi_1, \varphi_2) - P_1(\tilde{\varphi}_1, \tilde{\varphi}_2)), (P_2(\varphi_1, \varphi_2) - P_2(\tilde{\varphi}_1, \tilde{\varphi}_2))\| \\ &= \max\{\|(P_1(\varphi_1, \varphi_2) - P_1(\tilde{\varphi}_1, \tilde{\varphi}_2))\|, \|(P_2(\varphi_1, \varphi_2) - P_2(\tilde{\varphi}_1, \tilde{\varphi}_2))\|\} \\ &< \max\{\alpha_1 \|(\varphi_1, \varphi_2) - (\tilde{\varphi}_1, \tilde{\varphi}_2)\|_X , \alpha_2 \|(\varphi_1, \varphi_2) - (\tilde{\varphi}_1, \tilde{\varphi}_2)\|_X\} \\ &< \max\{\alpha_1, \alpha_2\} \|(\varphi_1, \varphi_2) - (\tilde{\varphi}_1, \tilde{\varphi}_2)\|_X \\ &< \alpha \|\varphi - \tilde{\varphi}\|_X . \end{aligned}$$

Nevertheless, if $\varphi \in X$ and we can always find

$$\|P_1\varphi(t)\| \leq \left(|u(0)| + \int_0^{p_1(0)} a(s) |f(\psi_1(s - \tau(s)))| \,\mathrm{d}s\right) + \alpha_1 \|\varphi\|_X .$$

We now establish the existence and uniqueness of solutions by showing that $P : X \to X$ if $\|\psi_1\|$ and $\|\psi_2\|$ are sufficiently small. Indeed, for $\delta > 0$ we choose $\|\psi_1\|_0 + \|\psi_2\|_0 \leq \delta$ so that

$$\left(\delta + L_1\delta \int_0^{p_1(0)} a(s) \,\mathrm{d}s\right) < (1 - \alpha) l, \tag{14}$$

$$\left(\delta + L_2\delta \int_0^{p_2(0)} b(s) \,\mathrm{d}s\right) < (1 - \alpha) l. \tag{15}$$

We have

$$\begin{aligned} \|P_1\varphi(t)\| &\leq \left(|u(0)| + \int_0^{p_1(0)} |a(s) f(\psi_1(s - \tau(s)))| \,\mathrm{d}s\right) + \alpha \|\varphi\|_X \\ &\leq \left(\delta + L_1\delta \int_0^{p_1(0)} a(s) \,\mathrm{d}s\right) + \alpha l < l. \end{aligned}$$

Also, from (14) we have

$$\|P_2\varphi(t)\| \leq \left(\delta + L_2\delta \int_0^{p_2(0)} b(s) \,\mathrm{d}s\right) + \alpha l < l.$$

it is clear that

$$\|P\varphi\|_X = \max\{\|P_1\varphi\|_0 , \|P_2\varphi\|_0\} \leq l$$

P is a contraction on the complete space $(X, \|\cdot\|_X)$. Therefore P maps X into itself then, P has a unique fixed point $(u, v) \in X$ such that $P(u, v) = (P_1(u, v), P_2(u, v)) = (u, v)$ it follows from Lemma 2.5 this fixed point is a solution of (9) and (1)–(4). □

4. ASYMPTOTIC STABILITY OF THE ZERO SOLUTION

Now we will show that the zero solution of (1)–(4) is stable at $t = 0$ is exactly like the one given for the next theorem. For this purpose, let us denote

$$X^0 := \{\varphi \in X \text{ such that } \varphi(t) \to 0 \text{ as } t \to 0\}.$$

Since X^0 is closed in X and $(X, \|\cdot\|_X)$ is complete, then the metric space $\left(X^0, \|\cdot\|_X\right)$ is also complete.

THEOREM 4.1. *Under the hypotheses of Theorem* 3.1 *we have:*
i) *The zero solution of the system* (1)–(4) *is stable at* $t = 0$.
ii) *If, in addition*

$$\lim_{t\to+\infty} \int_0^t A_1(t) = \lim_{t\to+\infty} \int_0^t A_2(t) = +\infty. \tag{16}$$

then the zero solution of the system (1-4) *is asymptotically stable.*

Proof. Step one: Stability

Firstly, the results of Theorem 3.1 holds also when the l is replaced by $\frac{\varepsilon}{4}$ for $\varepsilon > 0$. Choose $\psi = (\psi_1, \psi_2)$ satisfying $\|\psi_1\|_0 + \|\psi_2\|_0 \le \delta$ ($\delta \le \varepsilon/2$), with δ such that

$$\begin{aligned} \left(\delta + L_1\delta \int_0^{p_1(0)} a(s)\,\mathrm{d}s\right) &< (1-\alpha)\frac{\varepsilon}{2}, \\ \left(\delta + L_2\delta \int_0^{p_2(0)} b(s)\,\mathrm{d}s\right) &< (1-\alpha)\frac{\varepsilon}{2}. \end{aligned}$$

Notice that with such a $\|P\varphi(t)\| = \|\psi(t)\| \le \|\psi_1\|_0 + \|\psi_2\|_0 \le \delta$ on $[m(0), 0]$. We claim that $\|(u, v)\| \le \varepsilon$ for all $t \ge 0$. If (u, v) is a solution of (1)–(4), we have

$$\begin{aligned} \|(u,v)\|_X = \|P(u,v)\|_X &\le \|P_1(u,v)\| + \|P_1(u,v)\| \\ &\le \left(\delta + L_1\delta \int_0^{p_1(0)} a(s)\,\mathrm{d}s\right) \\ &\quad + \left(\delta + L_2\delta \int_0^{p_2(0)} b(s)\,\mathrm{d}s\right) + \alpha 2\,\|(u,v)\|_X \\ &< (1-\alpha)\frac{\varepsilon}{2} + (1-\alpha)\frac{\varepsilon}{2} + 2\alpha\left(\|u\|_0 + \|v\|_0\right) \\ &= \varepsilon - \alpha\varepsilon + 2\alpha\left(\|u\|_0 + \|v\|_0\right) \end{aligned}$$

$$\leq \varepsilon - \alpha\varepsilon + 2\alpha\frac{\varepsilon}{2} = \varepsilon.$$

Thus

$$\|(u,v)\|_X = \|P(u,v)\|_X \leq \|u\|_0 + \|v\|_0 \leq \varepsilon.$$

The zero solution of the system (1)–(4) is stable at $t = 0$.

Step two: Stability asymptotic

Now we will show that for every $\varphi = (\varphi_1, \varphi_2) \in X^0$ we have $P\varphi(t) \to 0$ as $t \to +\infty$ that is $(P_1\varphi(t), P_2\varphi(t)) \to (0,0)$ as $t \to +\infty$. We begin by proving that $P_1\varphi(t) \to 0$. So, recalling (10), and denote the four terms on the right hand side of (10) by I_1, I_2, I_3, I_4 respectively. for any functions $\varphi \in X^0$ be fixed, we prove that each term of $P\varphi(t)$ tends to 0 as $t \to 0$. Obviously, the first term I_1 tends to zero by condition (16) as $t \to +\infty$. For the terms I_2, I_3 and I_4. Let $\varepsilon > 0$ be any positive number be given and choose $T \geq J$ large enough so that for $t > J$, $|\varphi_1(t - \tau(t))| \leq \varepsilon$ and $|\varphi_2(t - \tau(t))| \leq \varepsilon$. by making use conditions (10) the term I_2 satisfies

$$|I_2| \leq \int_t^{p_1(t)} |a(s) f_1(\varphi_1(s - \tau(s)))| \, \mathrm{d}s \leq \varepsilon\alpha L_1.$$

Also, the term I_3 satisfies

$$\begin{aligned} |I_3| &\leq \int_0^t |A_1(s)| \, \mathrm{e}^{-\int_s^t A_1(\theta)\mathrm{d}\theta} \int_y^{p_1(y)} |a(y) f_1(\varphi_1(y - \tau(y)))| \, \mathrm{d}y \, \mathrm{d}s \\ &\leq \mathrm{e}^{-\int_J^t A_1(\theta)\mathrm{d}\theta} \left(L_1 \|\varphi_1\| \int_0^J |A_1(s)| \, \mathrm{e}^{-\int_s^J A_1(\theta)\mathrm{d}\theta} \right. \\ &\quad \left. \times \int_y^{p_1(y)} |a(y)| \, \mathrm{d}y \, \mathrm{d}s \right) \\ &\quad + \varepsilon \left(L_1 \int_J^t |A_1(s)| \, \mathrm{e}^{-\int_s^t A_1(\theta)\mathrm{d}\theta} \int_y^{p_1(y)} |a(y)| \, \mathrm{d}y \, \mathrm{d}s \right) \\ &\leq \mathrm{e}^{-\int_J^t A_1(\theta)\mathrm{d}\theta} \|\varphi_1\| \, \alpha + \varepsilon\alpha. \end{aligned} \tag{17}$$

By condition (16) the first factor on the r.h.s of (17) tends to 0, as $t \to +\infty$, while the second is arbitrarily small. Thus, I_2 tends to 0, as $t \to +\infty$. Nevertheless,

$$\begin{aligned} |I_4| &\leq \mathrm{e}^{-\int_J^t A_1(\theta)\mathrm{d}\theta} \left(\|\varphi_2\| \int_0^J \mathrm{e}^{-\int_s^J A_1(\theta)\mathrm{d}\theta} |R_1(s)| \, \mathrm{d}s \right) \\ &\quad + \varepsilon \left(\int_J^t \mathrm{e}^{-\int_s^t A_1(\theta)\mathrm{d}\theta} |R_1(s)| \, \mathrm{d}s \right) \\ &\leq \varepsilon\alpha + \mathrm{e}^{-\int_J^t A_1(\theta)\mathrm{d}\theta} M. \end{aligned}$$

Then,

$$P_1\varphi(t) \to 0, \text{ as } t \to +\infty.$$

One can take the same way to prove that $P_2\varphi(t) \to 0$, as $t \to +\infty$. thus $P\varphi(t) \in X^0$. Consequently, P maps X^0 into itself. That is $P : X^0 \to X^0$. However by Theorem 3.1 P has a unique fixed point in X^0. It is clear in view of Lemma 2.5 that the existence of solutions for (1)–(4) is equivalent to the existence of solutions for the operator equation $P(u,v) = (P_1(u,v), P_2(u,v)) = (u,v)$. Finally, there is a unique continuous function $(u,v) \in X^0$ satisfying $(u(t), v(t)) = (\psi_1(t), \psi_2(t))$ on $[m(0), 0]$, which is a solution of (1)–(4) on $[0,+\infty)$. Since we have obtained the stability of the zero solution in step one, it follows that the zero solution is asymptotically stable at 0. □

5. APPLICATION

Let us consider the following system with a delay of the form

$$x'(t) + \frac{1}{2}\frac{0.5}{1+t}\sin(x(t-0.5t)) + \frac{\delta}{1+t^2}\frac{y(t-0.5t)}{y^2(t)+1} = 0, \tag{18}$$

$$y'(t) + \frac{1}{2}\frac{t}{1+t^2}\tan(y(t-0.5t)) + \frac{\mu\cos(x(t-0.5t))}{5+t^2} = 0, \tag{19}$$

with the following assumptions $(\mu, \delta < 0,3)$. Then, the trivial solution of (18)–(19) is asymptotically stable.

Proof. It is not difficult to verify that the hypotheses (**H1**), (**H2**) and (**H3**) are possible.

H1) It is clear that for $x, y \in \mathbb{R}$ we have

$$\begin{aligned} |f_1(x) - f_1(y)| &= |\sin y - \sin x| \leq L_1 |y-x|,\ L_1 = 1, \\ f_1(0) &= g_1(t,0,0) = 0 \text{ for } t \in \mathbb{R}^+. \\ |g_1(t,x) - g_1(t,y)| &= \frac{\delta}{1+t^2}\frac{|yx-1|}{(x^2+1)(y^2+1)}|y-x| \\ &\leq \frac{\delta}{1+t^2}\|y-x\| = R_1(t)\|x-y\| \text{ for all } t \in \mathbb{R}, \end{aligned}$$

where

$$\sup_{x,y\in\mathbb{R}}\left|\frac{(yx-1)}{(x^2+1)(y^2+1)}\right| \leq 1.$$

H2) There is a constant $0 < \alpha_1 < 1$ satisfying

$$2L_1 \sup \int_t^{p_1(t)} a(s)\,\mathrm{d}s + \sup \int_0^t R_1(s)\,\mathrm{e}^{-\int_s^t A_1(\theta)\mathrm{d}\theta}\mathrm{d}s \leq \alpha_1.$$

Indeed, firstly we have

$$\begin{aligned} a(p_1(t)) &= a(2t) = \frac{1}{2}\frac{0.5}{1+2t} \text{ and } p_1'(t) = 2, \\ A_1(t) &= p_1'(t)\,a(p_1(t))\frac{f_1(u(t))}{u(t)} \end{aligned}$$

$$= \frac{1}{1+2t}\frac{f_1(u(t))}{u(t)} \geq \frac{1}{1+2t},$$

$$2L_1\int_t^{p_1(t)} a(s)\,\mathrm{d}s = \int_t^{2t}\frac{0.5\mathrm{d}s}{1+s} = 0.5\ln\frac{1+2t}{1+t} \leq 0.5\ln 2,$$

and

$$I = \int_0^t R_1(s)\,\mathrm{e}^{-\int_s^t A_1(\theta)\mathrm{d}\theta}\mathrm{d}s \leq \int_0^t \frac{\delta}{1+s^2}\mathrm{e}^{-\int_s^t \frac{\mathrm{d}\theta}{1+2\theta}}\mathrm{d}s.$$

Also, for $t \geq 2$ we have

$$\frac{1}{1+t^2} \leq \frac{1}{1+2t}.$$

Then

$$I \leq \int_0^2 \frac{\delta}{1+s^2}\mathrm{e}^{-\int_s^2 \frac{\mathrm{d}\theta}{1+2\theta}}\mathrm{d}s + \int_2^t \frac{\delta}{1+2s}\mathrm{e}^{-\int_s^t \frac{\mathrm{d}\theta}{1+2\theta}}\mathrm{d}s,$$

in fact that

$$\delta\int_0^2 \frac{1}{1+s^2}\mathrm{e}^{-\int_s^2 \frac{\mathrm{d}\theta}{1+2\theta}}\mathrm{d}s = \frac{\delta}{\sqrt{5}}\int_0^2 \frac{\sqrt{1+2s}}{1+s^2}\mathrm{d}s \leq 1.7,$$

$$\int_2^t \frac{\delta}{1+2s}\mathrm{e}^{-\int_s^t \frac{\mathrm{d}\theta}{1+2\theta}}\mathrm{d}s = 1-\mathrm{e}^{-\int_0^t \frac{\mathrm{d}\theta}{1+2\theta}} \leq 1.$$

Thus

$$0.5\ln 2 + I \leq \frac{\ln 2}{2} + \delta\left(\frac{1.7}{\sqrt{5}}+1\right) \leq 0.88 = \alpha_1 < 1.$$

H3) $\beta_1 = 1$,

$$\lim \frac{f_1(x)}{x} \geq \beta_1 = 1 > 0 \text{ as } x \to 0.$$

As before, a direct calculation shows that $R_2(t) = \frac{\mu}{1+t^2}$, $L_2 = 1.083$, $\beta_2 = 1$, $l \leq \frac{\pi}{8}$ and $\alpha_2 = 0.9$. We have from Theorem 4.1 that the zero solution is asymptotically stable. □

REMARK 5.1. We assume that f_1, f_2, g_1 and g_2 are linear functions and a_1, b_1, a_2, b_2, τ, r are positive constants satisfy the appropriate condition

$$\alpha = \max\left\{2a_1\tau + \frac{a_2}{a_1}, 2b_1 r + \frac{b_2}{b_1}\right\} < 1.$$

Then the trivial solution of (1)–(4) is asymptotically stable.

REFERENCES

[1] R.P. Agarwal, D. O'Regan and S.H. Saker, *Oscillation and Stability of Delay Models in Biology*, Springer, 2014.

[2] A. Ardjouni and A. Djoudi, *Stability in nonlinear neutral integro-differential equations with variable delay using fixed point theory*, J. Appl. Math. Comput., **44** (2014), 317–336.

[3] L.C. Becker and T.A. Burton, *Stability, fixed points and inverse of delays*, Proc. Roy. Soc. Edinburgh Sect. A, **136A** (2006), 245–275.

[4] D.R. David, *Ordinary and Delay Differential Equations, Mathematical Sciences*, Springer-Verlag, New York, 1977.
[5] H. Gabsi, A. Ardjouni and A. Djoudi, *New stability conditions for the delayed Liénard nonlinear equation via fixed point technique*, Azerb. J. Math., **8** (2018), 15–34.
[6] H. Gabsi, A. Ardjouni and A. Djoudi, *Existence of positive periodic solutions of nonlinear neutral differential systems with variable delays*, Ann. Univ. Ferrara Sez. VII Sci. Mat., **64** (2018), 83–97.
[7] H. Gabsi, A. Ardjouni and A. Djoudi, *Fixed points and stability of a class of nonlinear delay integro-differential equations with variable delays*, Facta Univ. Ser. Math. Inform., **32** (2017), 31–57.
[8] H. Gabsi, A. Ardjouni and A. Djoudi, *Existence of positive periodic solutions of a class of delay differential systems*, Rom. J. Math. Comput. Sci., **9** (2019), 1–14.
[9] H. Gabsi, A. Ardjouni and A. Djoudi, *Nonlinear neutral integro-differential equations, stability by fixed point and inverses of delays*, Novi Sad J. Math., **47** (2017), 89–113.
[10] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Springer, Dordrecht, 1992.
[11] J.K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.
[12] J.K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer, New York, 1993.
[13] Y. Kuang, *Delay Differential Equations Population Dynamics with Applications*, Academic Press, 1993.
[14] H. Smith, *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, Springer, New York, 2011.
[15] J.M. Smith, *Models in Ecology*, Cambridge University Press, 1974.

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