MATHEMATICA, 64 (87), N$^{\circ}$ 1, 2022, pp. 39–50

A COMMON PROPERTY TO GENERALIZED CONTRACTIONS AND EXISTENCE OF FIXED POINTS

SFYA BENAHMED

Abstract. The definitions of several types of generalized contractions for an application T from a complete metric space (X, d) in itself given in [14] are recalled and reviewed. A common property to all these concepts is put in light, namely: there exists $\alpha > 0$ such that, for all $x \in X$, $x \neq T(x)$, there exists $u \in X \setminus \{x\}$ satisfying:

$$[H] \qquad \mathrm{d}(u, T(u)) + \alpha \mathrm{d}(x, u) \leq \mathrm{d}(x, T(x)).$$

We observe that assumption $[H]$ is fulfilled in most cases treated in [14] and we prove that assumption $[H]$ and lower semi-continuity of the function $x \longmapsto \mathrm{d}(x, T(x))$ ensure existence of a fixed point along with a sharp estimate for the distance to the fixed-points set.

MSC 2010. 54H25, 47H10.

Key words. Ekeland's variational principle, fixed point, lower semicontinuty, generalized contractions.

1. INTRODUCTION

In his nice survey [14] on generalized contractions, Rhoades reviewed 25 generalizations of the notion of contraction mapping and precised relationships between these various generalizations. In the present work, we focus in some of them which we consider most representative. We show that these assumptions imply a simple assumption which guarantees existence of a fixed point and provides an estimate for the distance to fixed-points set (or to the fixed point if it is unique). For the sake of brevity, we restrict to 10 of these assumptions, but many of the others also enter within our framework. Furthermore, we improve some of these results by weakening some assumptions of the original articles. Our generalized assumption is the following: there exists $\alpha > 0$ such that, for all $x \in X$, $x \neq T(x)$, there exists $u \in X \setminus \{x\}$ satisfying:

$$[H] \qquad \mathrm{d}(u, T(u)) + \alpha \mathrm{d}(x, u) \leq \mathrm{d}(x, T(x)).$$

Observe that assumption $[H]$ also makes sense for multivalued maps with closed values by replacing $x \neq T(x)$ by $x \notin T(x)$. We show in Lemma 3.2

The author would like to express her sincere thanks to the anonymous referee for having contributed to improve the present paper.

DOI: 10.24193/mathcluj.2022.1.06

that assumptions used in [3, 4, 6, 7, 13, 14, 15] fulfil assumption $[H]$. The paper is organized as follows. In section 2 we give our basic lemma based on the Ekeland variational principle. In section 3 we list some definitions on generalized contractions and we show that our assumption $[H]$ is satisfied in all cases. At last, section 4 is devoted to fixed point results, some of them improving the quoted one and giving an estimate for the distance to the fixed-points set. The main references in this work are [1, 4, 6, 7, 13, 14, 15].

2. A BASIC LEMMA

DEFINITION 2.1. Let (X, d) be a metric space, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function. A point $x \in X$ is said to be a *d-point* of f if

$$f(x) < f(z) + \mathrm{d}(z, x) \qquad \text{for all } z \in X \setminus \{x\}.$$

Here is the well-known Ekeland's variational principle under its simpler form (see [8, 9, 12, 17]).

THEOREM 2.2. *The following are equivalent:*
(a) *The metric space (X, d) is complete.*
(b) *Every proper (not identically equal to $+\infty$), lower semicontinuous, and bounded from below function $f : X \to \mathbb{R} \cup \{+\infty\}$ has a d-point.*

Now, given $x \in X$, let us set $M(x) = \{z \in X : f(z) + \mathrm{d}(x, z) \leq f(x)\}$. It is an immediate consequence of the triangle inequality that a d-point of the restriction of f to the subset $M(x)$ is a d-point of f on the whole X. Thus we have,

COROLLARY 2.3. *Let (X, d) be a complete metric space. Assume that the function $f : X \to \mathbb{R} \cup \{+\infty\}$ is proper lower semicontinuous and bounded from below. Then, for all $x \in X$ there exists a d-point of f belonging to $M(x)$.*

For $\lambda \in \mathbb{R}$, we further denote by $[f{\leq}\lambda]$ the sublevel set $f^{-1}((-\infty, \lambda])$ and we define analogously $[f{<}\lambda]$, $[f{>}\lambda]$ etc. The following simple lemma in the lines of [1, 2, 16, 10] is our basic tool in the sequel.

LEMMA 2.4. *Let $f : X \to [0, +\infty]$ be a proper lower semicontinuous function defined on a complete metric space (X, d), and let $0 \leq \lambda < \mu \leq +\infty$ be such that $[f{<}\mu] \neq \emptyset$. Assume that:*

$$\textit{for all } x \in [\lambda{<}f{<}\mu] \textit{ there exists } y \neq x \textit{ such that } f(y) + \mathrm{d}(x, y) \leq f(x).$$

Then, $[f{\leq}\lambda] \neq \emptyset$, and, for all $x \in [f{<}\mu]$, we can find $y \in [f{\leq}\lambda]$ such that $\mathrm{d}(x, y) \leq f(x)$.

Proof. Given $x \in [f{<}\mu]$, one has $M(x) \subset [f{<}\mu]$. Then, a d-point y of f which belongs to $M(x)$, whose existence is guaranteed by Corollary 2.3, is in $[f{\leq}\lambda]$ since, from our assumption, an element of $[\lambda{<}f{<}\mu]$ is not a d-point, and y satisfies $\mathrm{d}(x, y) \leq f(y) + \mathrm{d}(x, y) \leq f(x)$. $\square$

3. GENERALIZED CONTRACTIONS

Let (X, d) be a metric space and T be a function from a set X into itself . We shall denote by $\mathcal{F}_T = \{x \in X : T(x) = x\}$ the set of fixed point of T.

We now give a list of properties mainly taken from [14].

(1) $[H_1]$ (Clarke [7]). Clarke's directional contraction: there exists $0 \leq k < 1$ such that, for all $x \in X$, $x \neq T(x)$, there exists $u \neq x$ such that
$$\begin{cases} \mathrm{d}(x,u) + \mathrm{d}(u,T(x)) = \mathrm{d}(x,T(x)) \\ \mathrm{d}(T(u),T(x)) \leq k\mathrm{d}(x,u); \end{cases}$$

(2) $[H_2]$ (Song [15]). Song's directional contraction: $T : D \to X$, where D is a closed subset of X, is a mapping such that there exists $0 \leq k < \gamma \leq 1$ such that, for all $x \in X$, $x \neq T(x)$, there exists $u \in D \setminus \{x\}$ such that
$$\begin{cases} \gamma\mathrm{d}(x,u) + \mathrm{d}(u,T(x)) \leq \mathrm{d}(x,T(x)) \\ \mathrm{d}(T(u),T(x)) \leq k\mathrm{d}(x,u); \end{cases}$$

(3) $[H_3]$ ($[H_1]$ with $u = T(x)$). There exists $0 \leq k < 1$ such that, for all $x \in X$, $x \neq T(x)$, we have
$$\mathrm{d}(T(x),T(T(x))) \leq k\mathrm{d}(x,T(x));$$

(4) $[H_4]$ There exist nonnegative constants a, b satisfying $a + b < 1$ such that, for all $x, u \in X$
$$\mathrm{d}(T(x),T(u)) \leq a\mathrm{d}(x,T(x)) + b\mathrm{d}(u,T(u));$$

(5) $[H_5]$ (Reich [13]). There exist nonnegative constants a, b, c satisfying $a + b + c < 1$ such that, for all $x, u \in X$
$$\mathrm{d}(T(x),T(u)) \leq a\mathrm{d}(x,T(x)) + b\mathrm{d}(u,T(u)) + c\mathrm{d}(x,u);$$

(6) $[H_6]$ (Bianchini [4]). There exists real number $0 \leq k < 1$ such that, for all $x, u \in X$
$$\mathrm{d}(T(x),T(u)) \leq k \max\left(\mathrm{d}(x,T(x)), \mathrm{d}(u,T(u))\right);$$

(7) $[H_7]$ (Chatterjea [5]). There exists real number $0 \leq k < \frac{1}{2}$ such that, for all x, $u \in X$
$$\mathrm{d}(T(x),T(u)) \leq k\left(\mathrm{d}(u,T(x)) + \mathrm{d}(x,T(u))\right);$$

(8) $[H_8]$ (Ciric [6]). There exist nonnegative functions q, r, s, t and $\lambda \in [0,1)$ satisfying
$$\sup_{x,u \in X} \left(q(x,u) + r(x,u) + s(x,u) + 2t(x,u)\right) \leq \lambda < 1$$
such that, for all x, $u \in X$:
$$\begin{aligned} \mathrm{d}(T(x),T(u)) \leq{}& q(x,u)\mathrm{d}(x,u) + r(x,u)\mathrm{d}(x,T(x)) \\ &+ s(x,u)\mathrm{d}(u,T(u)) + t(x,u)[\mathrm{d}(x,T(u)) + \mathrm{d}(u,T(x))]; \end{aligned}$$

(9) $[H_9]$ (Zamfirescu [18]). There exist real numbers a, b, c, $0 \leq a < 1$, $0 \leq b, c < \frac{1}{2}$, such that, for all $x, u \in X$, at least one of the following is true:
   (i) $\mathrm{d}(T(x), T(u)) \leq a\mathrm{d}(x, u)$,
   (ii) $\mathrm{d}(T(x), T(u)) \leq b[\mathrm{d}(x, T(x)) + \mathrm{d}(u, T(u))]$,
   (iii) $\mathrm{d}(T(x), T(u)) \leq c[\mathrm{d}(x, T(u)) + \mathrm{d}(u, T(x))]$,

(10) $[H_{10}]$ There exist nonnegative functions a, b, c satisfying

$$\limsup_{t \searrow s}(a(t) + b(t) + c(t)) < 1 \text{ for all } s > 0,$$

such that, for all $x, u \in X$,

$$\mathrm{d}(T(x), T(u)) \leq a(\mathrm{d}(x, u))\mathrm{d}(x, T(x)) + b(\mathrm{d}(x, u))\mathrm{d}(u, T(u)) + c(\mathrm{d}(x, u))\mathrm{d}(x, u).$$

Remark 3.1.

(1) Property $[H_4]$ coincides with the assumption of [11], in which $0 < a = b < \frac{1}{2}$, because by symmetry, we get $a = b < \frac{1}{2}$.
(2) In $[H_{10}]$ we can take $a = b$ and it is a weakening of the assumption of [13] in which a, b and c are positive decreasing functions satisfying:

$$a(t) + b(t) + c(t) < 1 \text{ for all } t > 0.$$

(3) It is easy to verify that all assumptions $[H_i]$, with $i \in \{1, \ldots, 7\}$, do imply Ciric's contractive assumption $[H_8]$.

In order to prove the existence of fixed points for applications, we establish a relationship between the hypotheses $[H_i]$ such that $i \in \{1, \ldots, 10\}$ and the following main hypothesis: there exists $\alpha > 0$ such that, for all $x \in X$, $x \neq T(x)$, there exists $u \in X \setminus \{x\}$ satisfying:

$$[H] \qquad \mathrm{d}(u, T(u)) + \alpha\mathrm{d}(x, u) \leq \mathrm{d}(x, T(x)).$$

These relationships are summarized in the following:

Lemma 3.2. *Let (X, d) be a metric space and $T : X \longrightarrow X$ be a mapping satisfying one of the hypotheses $[H_i]$, $i \in \{1, \ldots, 9\}$. Then $[H]$ is satisfied.*

Proof. Let $x \in X$ be such that $\mathrm{d}(x, T(x)) > 0$.
1) If $[H_1]$ is satisfied, then we can find $u \neq x$ such that

$$\mathrm{d}(x, u) + \mathrm{d}(u, T(x)) = \mathrm{d}(x, T(x))$$

from which

$$\mathrm{d}(x, u) + \mathrm{d}(u, T(u)) - \mathrm{d}(T(u), T(x)) \leq \mathrm{d}(x, T(x)),$$

and as $\mathrm{d}(T(x), T(u)) \leq k\mathrm{d}(x, u)$, we have:

$$\mathrm{d}(x, u) + \mathrm{d}(u, T(u)) - k\mathrm{d}(x, u) \leq \mathrm{d}(x, T(x)),$$

and therefore

$$\mathrm{d}(u, T(u)) + (1 - k)\mathrm{d}(x, u)) \leq \mathrm{d}(x, T(x)),$$

thus $[H]$ is satisfied with $\alpha = 1 - k > 0$.

2) If $[H_2]$ is satisfied, then in the same way, we can prove that $[H]$ is satisfied with $\alpha = \gamma - k > 0$.

3) Note that applications that check $[H_1]$, are a special case of Clarke's directional contractions ie they satisfy $[H_1]$, indeed, putting $u = T(x)$ in $[H_3]$, we obtain, for all $x \neq T(x)$, there exists $u = T(x)$ such that

$$\begin{cases} \mathrm{d}(x,u) + \mathrm{d}(u,T(x)) = \mathrm{d}(x,T(x)) \\ \mathrm{d}(T(u),T(x)) \leq k\mathrm{d}(x,u). \end{cases}$$

Thus, if $[H_3]$ is satisfied then $[H_1]$ is also and therefore $[H]$ is also with $\alpha = 1 - k$.

4) If $[H_4]$ is satisfied then by writing for $u = T(x)$, we obtain:

$$\mathrm{d}(u,T(u)) \leq a\mathrm{d}(x,u) + b\mathrm{d}(u,T(u))$$

and then

$$\mathrm{d}(u,T(u)) \leq a(1-b)^{-1}\mathrm{d}(x,u)$$

and therefore:

$$\begin{array}{rcl} \mathrm{d}(u,T(u)) - \mathrm{d}(x,T(x)) & \leq & a(1-b)^{-1}\mathrm{d}(x,u) - \mathrm{d}(x,u) \\ & \leq & (a+b-1)(1-b)^{-1}\mathrm{d}(x,u). \end{array}$$

Thus we obtain:

$$\mathrm{d}(u,T(u)) + (1-(a+b))(1-b)^{-1}\mathrm{d}(x,u) \leq \mathrm{d}(x,T(x)).$$

Then, there exists $\alpha = (1-(a+b))(1-b)^{-1} > 0$, such that $[H]$ is satisfied.

5) In the same way, if $[H_5]$ is satisfied then, $[H]$ holds with

$$\alpha = (1-(a+b+c))(1-b)^{-1} > 0.$$

6) Writing $[H_6]$ for $u = T(x)$, we obtain:

$$\mathrm{d}(u,T(u)) \leq k\max(\mathrm{d}(u,T(u)),\mathrm{d}(x,u))$$

as $k < 1$, obviously, we have $\mathrm{d}(u,T(u)) \leq k\mathrm{d}(x,u)$, and therefore

$$\mathrm{d}(u,T(u)) - \mathrm{d}(x,T(x)) \leq k\mathrm{d}(x,u) - \mathrm{d}(x,u),$$

hence

$$\mathrm{d}(u,T(u)) + (1-k)\mathrm{d}(x,u) \leq \mathrm{d}(x,T(x)).$$

Then $[H]$ is fulfilled with $\alpha = 1 - k > 0$.

7) For $[H_7]$, we get

$$\mathrm{d}(u,T(u)) \leq k(\mathrm{d}(x,T(u)) + \mathrm{d}(u,u)) = k\mathrm{d}(x,T(u)) \leq k(\mathrm{d}(x,u) + \mathrm{d}(u,T(u))$$

then

$$\mathrm{d}(u,T(u)) \leq k(1-k)^{-1}\mathrm{d}(u,T(u))$$

and then

$$\mathrm{d}(u,T(u)) - \mathrm{d}(x,T(x)) \leq k(1-k)^{-1}\mathrm{d}(x,u) - \mathrm{d}(x,u),$$

hence
$$\mathrm{d}(u,T(u)) + (1-2k)(1-k)^{-1}\mathrm{d}(x,u) \le \mathrm{d}(x,T(x)).$$
Then $[H]$ is satisfied with $\alpha = (1-2k)(1-k)^{-1} > 0$.

8) For $[H_8]$, we obtain
$$\mathrm{d}(u,T(u)) \le q(x,u)\mathrm{d}(x,u)+r(x,u)\mathrm{d}(x,u)+s(x,u)\mathrm{d}(u,T(u))+t(x,u)\mathrm{d}(x,T(u)),$$
then
$$\mathrm{d}(u,T(u)) \le (q(x,u)+r(x,u))\mathrm{d}(x,u)+s(x,u)\mathrm{d}(u,T(u))+ \\ t(x,u)(\mathrm{d}(x,u)+\mathrm{d}(u,T(u))),$$
and
$$\mathrm{d}(u,T(u)) \le (1-(s(x,u)+t(x,u))^{-1}(q(x,u)+r(x,u)+t(x,u))\mathrm{d}(x,u),$$
and
$$\mathrm{d}(u,T(u))-\mathrm{d}(x,T(x)) \le (1-(s(x,u)+t(x,u))^{-1}(q+r+t)(x,u)\mathrm{d}(x,u)-\mathrm{d}(x,u).$$
Then
$$\mathrm{d}(u,T(u))-\mathrm{d}(x,T(x)) \le ((q+r+s+2t)(x,u)-1)(1-(s(x,u)+t(x,u))^{-1}\mathrm{d}(x,u),$$
that is to say
$$\mathrm{d}(u,T(u)) - \mathrm{d}(x,T(x)) \le (\lambda-1)(1-\lambda)^{-1}\mathrm{d}(x,u) = \mathrm{d}(x,u),$$
and therefore $[H]$ is satisfied with $\alpha = 1$.

9) Finally, for $[H_9]$, we have for each $x \in X$ and for $u = T(x)$, at least one of the following is true:

(i) $\mathrm{d}(u,T(u)) \le a\mathrm{d}(x,u)$,

(ii) $\mathrm{d}(u,T(u)) \le b[\mathrm{d}(x,u)+\mathrm{d}(u,T(u))]$,

(iii) $\mathrm{d}(u,T(u)) \le c[\mathrm{d}(x,T(u))+\mathrm{d}(u,u)] \le c[\mathrm{d}(x,u)+\mathrm{d}(u,T(u))]$,

Thus we obtain:
$$\mathrm{d}(u,T(u)) \le \max\left\{a\mathrm{d}(x,u), \beta[\mathrm{d}(x,u)+\mathrm{d}(u,T(u))]\right\},$$
with $0 \le \beta = \max(b,c) < \frac{1}{2}$.

If we have $\mathrm{d}(u,T(u)) \le a\mathrm{d}(x,u)$, therefore
$$\mathrm{d}(u,T(u)) - \mathrm{d}(x,T(x)) \le a\mathrm{d}(x,u) - \mathrm{d}(x,u)$$
hence
$$\mathrm{d}(u,T(u)) + (1-a)\mathrm{d}(x,u) \le \mathrm{d}(x,T(x)), \tag{1}$$
or if we have
$$\mathrm{d}(u,T(u)) \le \beta[\mathrm{d}(x,u)+\mathrm{d}(u,T(u))],$$
then
$$\mathrm{d}(u,T(u)) \le \beta(1-\beta)^{-1}\mathrm{d}(x,u),$$
therefore
$$\mathrm{d}(u,T(u)) - \mathrm{d}(x,T(x)) \le \beta(1-\beta)^{-1}\mathrm{d}(x,u) - \mathrm{d}(x,u),$$

hence

$$\mathrm{d}(u, T(u)) + (1 - 2\beta)(1 - \beta)^{-1}\mathrm{d}(x, u) \leq \mathrm{d}(x, T(x)). \tag{2}$$

Then, from (1) and (2), $[H]$ is fulfilled with

$$\alpha = \min(1 - a, (1 - 2\beta)(1 - \beta)^{-1}) > 0.$$

□

4. FIXED POINTS RESULTS

PROPOSITION 4.1. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a mapping satisfying the hypothesis $[H_{10}]$, then,*

$$\inf_{x \in X} \mathrm{d}(x, T(x)) = 0.$$

Proof. Let $x_0 \in X$. We may assume that $\mathrm{d}(x_0, T(x_0)) > 0$ (otherwise there is nothing to prove). Assume that are known $x_0, \ldots, x_n \in X$ such that, for all $k \in \{0, \ldots, n\}$, one has $\mathrm{d}(x_k, T(x_k)) > 0$ and for all $k \in \{0, \ldots, n-1\}$, $x_k = T(x_{k-1})$ and thus, one has, setting $d_k = \mathrm{d}(x_k, T(x_k))$:

$$\begin{aligned} d_{k+1} &= \mathrm{d}(T(x_k), T(x_{k+1})) \\ &\leq a(d_k)\mathrm{d}(x_k, T(x_k)) + b(d_k)\mathrm{d}(x_{k+1}, T(x_{k+1})) + c(d_k)\mathrm{d}(x_{k+1}, T(x_{k+1})) \\ &\leq a(d_k)\mathrm{d}(x_k, x_{k+1}) + b(d_k)\mathrm{d}(x_{k+1}, x_{k+2}) + c(d_k)\mathrm{d}(x_{k+1}, (x_{k+2}), \qquad (3) \end{aligned}$$

and, as

$$d_{k+1} = \mathrm{d}(x_{k+1}, x_{k+2}) = \mathrm{d}(x_{k+1}, T(x_{k+1})),$$

we derive from (3) that

$$d_{k+1} \leq \frac{a(d_k)}{1 - (b(d_k) + c(d_k))} d_k < d_k. \tag{4}$$

By induction, either the process ends if $\mathrm{d}(x_k, T(x_k)) = 0$ for some $k \in \mathbb{N}$, either we obtain a sequence $(x_n)_{n \in \mathbb{N}} \subset X$ such that the sequence

$$(\mathrm{d}(x_n, T(x_n)))_{n \in \mathbb{N}} \subset \mathbb{R}$$

is decreasing. Denoting by $d \geq 0$ the limit of the decreasing sequence

$$(\mathrm{d}(x_n, x_{n+1}))_{n \in \mathbb{N}},$$

and assuming that $d > 0$, we then get using (4), the contradiction

$$d \leq \limsup_{t \downarrow s} \frac{a(t)}{1 - (b(t) + c(t))} d < d.$$

It follows that $\lim_{n \to \infty} \mathrm{d}(x_n, T(x_n)) = \lim_{n \to \infty} \mathrm{d}(x_n, x_{n+1}) = 0$ thus

$$\inf_{x \in X} \mathrm{d}(x, T(x)) = 0.$$

□

Lemma 4.2. *Let* (X, d) *be a complete metric space and* $T : X \longrightarrow X$ *be a mapping satisfying the hypothesis* $[H_{10}]$*, then we can find* $\delta > 0$ *and* $\lambda \in (0,1)$ *such that for each* $x \in X$ *with* $0 < \mathrm{d}(x, T(x)) < \delta$*, there exists* $u \neq x$ *such that*

$$\mathrm{d}(u, T(u)) + \mathrm{d}(x, u) \leq \mathrm{d}(x, T(x)).$$

Proof. Writing assumption $[H_{10}]$ with $u = T(x)$, we have:

$$\begin{aligned}\mathrm{d}(u, T(u)) &\leq a(\mathrm{d}(x,u))\mathrm{d}(x,u) + b(\mathrm{d}(x,u))\mathrm{d}(u,T(u)) + c(\mathrm{d}(x,u))\mathrm{d}(x,u) \\ &\leq (a(\mathrm{d}(x,u)) + c(\mathrm{d}(x,u))\mathrm{d}(x,u) + b(\mathrm{d}(x,u))\mathrm{d}(u,T(u)),\end{aligned}$$

and then

$$\mathrm{d}(u, T(u)) \leq (a(\mathrm{d}(x,u)) + c(\mathrm{d}(x,u))(1 - b(\mathrm{d}(x,u)))^{-1}\mathrm{d}(x,u),$$

leading to

$$\mathrm{d}(u,T(u)) - \mathrm{d}(x,T(x)) \leq ((a+b+c)(\mathrm{d}(x,u)) - 1)(1 - b(\mathrm{d}(x,u)))^{-1}\mathrm{d}(x,u). \tag{5}$$

As $\limsup_{t \downarrow s}(a(t) + b(t) + c(t)) < 1$ for all $s \geq 0$, then there exist $\delta > 0$, $\lambda > 0$ such that

$$\sup_{t \in (0,\delta)} (a(t) + b(t) + c(t)) < 1 - \lambda.$$

As a, b, c are nonegative functions, we also have

$$\sup_{t \in (0,\delta)} b(t) < 1 - \lambda,$$

then for the inequality (1), we obtain, as $0 < \mathrm{d}(x, T(x)) = \mathrm{d}(x, u) < \delta$,

$$\mathrm{d}(u,T(u)) - \mathrm{d}(x,T(x)) \leq ((a+b+c)(\mathrm{d}(x,u)) - 1)\lambda^{-1}\mathrm{d}(x,u),$$

yielding

$$\mathrm{d}(u,T(u)) + (1 - (a+b+c)(\mathrm{d}(x,u)))\lambda)^{-1}\mathrm{d}(x,u) \leq \mathrm{d}(x,T(x))$$

and then

$$\mathrm{d}(u,T(u)) + \lambda\lambda^{-1}\mathrm{d}(x,u) \leq \mathrm{d}(x,T(x)),$$

Thus, for all $x \in X$ such that $0 < \mathrm{d}(x, T(x)) < \delta$, we have

$$\mathrm{d}(u, T(u)) + \mathrm{d}(x, u) \leq \mathrm{d}(x, T(x)).$$

□

Theorem 4.3. *Let* (X, d) *be a complete metric space and* $T : X \longrightarrow X$ *be a mapping. Assume that:*

(1) $[H]$ *is satisfied*
(2) *the function* $x \longmapsto \mathrm{d}(x, T(x))$ *is lower semicontinuous.*

Then, $\mathcal{F}_T \neq \emptyset$ *and for all* $x \in X$*, and*

$$\mathrm{d}(x, \mathcal{F}_T) \leq \alpha^{-1}\mathrm{d}(x, T(x)).$$

Proof. Consider the lower semicontinuous function

$$f(x) = \mathrm{d}(x, T(x)).$$

From $[H]$ the assumptions of Lemma 2.4 are satisfied with $\mu = +\infty$ for the distance $\tilde{d} = \alpha d$. It then suffices to apply Lemma 2.4 and we obtain the conclusion of the theorem. □

COROLLARY 4.4. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a mapping. If T is continuous and if one of the assumptions $[H_1]$ or $[H_2]$ or $[H_3]$ is satisfied then $\mathcal{F}_T \neq \emptyset$ and for all $x \in X$*

$$\mathrm{d}(x, \mathcal{F}_T) \leq \alpha^{-1}\mathrm{d}(x, T(x))$$

with

$$\alpha = \begin{cases} 1-k & \text{in the case } [H_1] \text{ or } [H_3] \\ \gamma - k & \text{in the case } [H_2]. \end{cases}$$

Proof. Let us define $f : X \longrightarrow \mathbb{R}$ by $f(x) = \mathrm{d}(x, T(x))$, as T is continuous, f is lower semicontinuous. From Lemma 3.2, if $[H_1]$ or $[H_2]$ or $[H_3]$ is satisfied then $[H]$ is also and demonstration ends by applying Theorem 4.3. □

COROLLARY 4.5. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a mapping.*
If T is continuous and if one of assumptions $[H_i]$ with $i \in \{4, \dots, 9\}$ is satisfied then T has a unique fixed point $\bar{x}$ and for all $x \in X$

$$\mathrm{d}(x, \bar{x}) \leq \alpha^{-1}\mathrm{d}(x, T(x))$$

with

$$\alpha = \begin{cases} (1-(a+b))(1-b)^{-1} & \text{in the case } [H_4] \\ (1-(a+b+c))(1-b)^{-1} & \text{in the case } [H_5] \\ 1-k & \text{in the case } [H_6] \\ (1-2k)(1-k)^{-1} & \text{in the case } [H_7] \\ 1 & \text{in the case } [H_8] \\ \min(1-a, (1-2\beta)(1-\beta)^{-1}) \text{ with } \beta = \max(b,c) & \text{in the case } [H_9]. \end{cases}$$

Proof. As in Theorem 4.4, existence of a fixed point is ensured by Lemma 3.2. Now, assume that there are two points $x_1 \neq x_2$ such that $T(x_1) = x_1$ and $T(x_2) = x_2$.

1) If $[H_4]$ is satisfied, by writing $[H_4]$ for x_1 and x_2 we obtain

$$\mathrm{d}(x_1, x_2) \leq a\mathrm{d}(x_1, x_1) + b\mathrm{d}(x_2, x_2) = 0,$$

thus $x_1 = x_2$.

2) If $[H_5]$ is satisfied, by writing $[H_5]$ for x_1 and x_2 we obtain

$$\mathrm{d}(x_1, x_2) \leq a\mathrm{d}(x_1, x_1) + b\mathrm{d}(x_2, x_2) + c\mathrm{d}(x_1, x_2) = c\mathrm{d}(x_1, x_2)$$

leading to $x_1 = x_2$ since $c < 1$.

3) Similarly for $[H_6]$, we obtain

$$\mathrm{d}(x_1, x_2) \le k \max(\mathrm{d}(x_1, x_1), \mathrm{d}(x_2, x_2)) = 0,$$

and then $x_1 = x_2$.

4) Similarly for $[H_7]$, we obtain

$$\mathrm{d}(x_1, x_2) \le k(\mathrm{d}(x_1, x_2) + \mathrm{d}(x_1, x_2)) = k\mathrm{d}(x_1, x_2)),$$

yielding $x_1 = x_2$.

5) Similarly for $[H_8]$, we obtain

$$\mathrm{d}(x_1, x_2) \le (q(x_1, x_2) + 2t(x_1, x_2))\mathrm{d}(x_1, x_2) \le \lambda \mathrm{d}(x_1, x_2) < \mathrm{d}(x_1, x_2),$$

yielding $x_1 = x_2$.

6) If $[H_9]$ is satisfied, by writing (i), (ii) and (iii) for x_1 and x_2 we obtain

(i) $\mathrm{d}(x_1, x_2) \le a\mathrm{d}(x_1, x_2)$,
(ii) $\mathrm{d}(x_1, x_2)) \le 0$,
(iii) $\mathrm{d}(x_1, x_2) \le 2c\mathrm{d}(x_1, x_2)$,

leading to $x_1 = x_2$ since $a < 1$ and $c < \frac{1}{2}$. □

Corollary 4.6. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a mapping. If T is continuous and if assumption $[H_{10}]$ is satisfied then T has a unique fixed point $\bar{x}$ and there exists $\delta > 0$ such that for all $x \in X$ satisfying $\mathrm{d}(x, T(x)) < \delta$, we have*

$$\mathrm{d}(x, \bar{x}) \le \mathrm{d}(x, T(x)).$$

Proof. Let $f(x) = \mathrm{d}(x, T(x))$, which is lower semicontinuous. Under the assumption $[H_{10}]$, we get from Lemma 4.2, the existence of $\delta > 0$ such that for all $x \in X$, $0 < \mathrm{d}(x, T(x)) < \delta$, there exists $u \neq x$ such that:

$$\mathrm{d}(u, T(u)) + \mathrm{d}(x, u) \le \mathrm{d}(x, T(x))$$

and then $[f < \delta] \neq \emptyset$ and for all $x \in [0 < f < \delta]$ there exists $u \neq x$ such that $f(u) + \mathrm{d}(x, u) \le f(x)$, and thus the assumptions of Lemma 2.4 are satisfied with $\mu = \delta$, hence the existence of a fixed point $\bar{x}$ satisfying

$$\mathrm{d}(x, \bar{x}) \le \mathrm{d}(x, T(x)) \quad \text{for all } x \in [f < \delta].$$

Suppose now that there are two points $x_1 \neq x_2$ such that $T(x_1) = x_1$ and $T(x_2) = x_2$, by writing $[H_{10}]$ for x_1 and x_2, we obtain

$$\mathrm{d}(x_1, x_2) \le a(\mathrm{d}(x_1, x_2))\mathrm{d}(x_1, x_1) + b(\mathrm{d}(x_1, x_2))\mathrm{d}(x_2, x_2) + c(\mathrm{d}(x_1, x_2))\mathrm{d}(x_1, x_2)$$

then

$$\mathrm{d}(x_1, x_2) \le c(\mathrm{d}(x_1, x_2))\mathrm{d}(x_1, x_2),$$

so that $x_1 = x_2$. □

In order to prove the existence of fixed point, we used the lemma 2.4 and this requires that the function $x \longmapsto \mathrm{d}(x, T(x))$ is lower semicontinuous. Now we will see results removing this condition by assuming only that T has a closed graph.

COROLLARY 4.7. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping with closed graph. Assume that one of the assumptions $[H_i]$ with $i \in \{4, \dots, 9\}$ is satisfied then T has a unique fixed point $\bar{x}$ and for all $x \in X$*

$$\mathrm{d}(x, \bar{x}) \leq \alpha^{-1}\mathrm{d}(x, T(x))$$

with

$$\alpha = \begin{cases} (1-(a+b))(1-b)^{-1} & \textit{in the case } [H_4] \\ (1-(a+b+c))(1-b)^{-1} & \textit{in the case } [H_5] \\ 1-k & \textit{in the case } [H_6] \\ (1-2k)(1-k)^{-1} & \textit{in the case } [H_7] \\ 1 & \textit{in the case } [H_8] \\ \min(1-a, (1-2\beta)(1-\beta)^{-1}) \ \textit{with } \beta = \max(b, c) & \textit{in the case } [H_9]. \end{cases}$$

Proof. Under one of the assumptions $[H_i]$, $i \in \{4, \dots, 9\}$, and for $u = T(x)$, assumption $[H]$ is satisfied. Thus we can find $\alpha > 0$ such that for all $x \in X$, $x \neq T(x)$, we have

$$\mathrm{d}(u, T(u)) + \alpha\mathrm{d}(x, u) \leq \mathrm{d}(x, T(x)),$$

with $u = T(x)$. Let $x_0 \in X$ and let $x_k = T^k(x_0)$. If $T(x_k) = x_k$ for one k, the result is proved. Otherwise, we have

$$\mathrm{d}(x_{k+1}, x_{k+2}) + \alpha\mathrm{d}(x_k, x_{k+1}) \leq \mathrm{d}(x_k, x_{k+1}) \text{ for all } k \in \mathbb{N}.$$

Summing p inequalities and using the triangle inequality, we obtain:

$$\mathrm{d}(x_{k+p}, x_{k+p+1}) + \alpha\mathrm{d}(x_{k+p}, x_k) \leq \mathrm{d}(x_k, x_{k+1}).$$

The last inequality shows that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Indeed the sequence $(\mathrm{d}(x_k, x_{k+1}))_{k\in\mathbb{N}}$ is convergent because it is decreasing, and

$$\mathrm{d}(x_{k+p}, x_k) \leq \alpha^{-1}(\mathrm{d}(x_k, x_{k+1}) - \mathrm{d}(x_{k+1}, x_{k+2})) \text{ for all } p \in \mathbb{N}.$$

Let then $\bar{x}$ its limit, as $x_{k+1} = T(x_k)$ we deduce that $T(\bar{x}) = \bar{x}$ due to the closedness of the graph of T. ☐

REFERENCES

[1] D. Azé and J.-N. Corvellec, *A variational method in fixed point results with inwardness conditions*, Proc. Amer. Math. Soc., **134** (2006), 3577–3583.

[2] S. Benahmed and D. Azé, *On fixed points of generalized set-valued contractions*, Bull. Aust. Math. Soc., **81** (2010), 16–22,

[3] V. Berinde, *Iterative Approximation of Fixed points*, Springer, 2007.

[4] R.M.T. Bianchini, *Su un problema di S. Reich riguardante la teoria dei punti fissi*, Boll. Unione Mat. Ital., **5** (1976), 103–108.

[5] S.K. Chatterjea, *Fixed points theorems*, C.R. Acad. Bulgare Sci., **25** (1972), 727–730.

[6] L.B. Ciric, *Generalized contractions and fixed-point theorems*, Publ. Inst. Math. (Beograd) (N.S.), **12(26)** (1971), 19–26.

[7] F.H. Clarke, *Pointwise contraction criteria for the existence of fixed points*, Canad. Math. Bull. **21** (1978), 7–11.

[8] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl., **47** (1974), 324–353.

[9] I. Ekeland, *Nonconvex minimization problems*, Bull. Amer. Math. Soc., **1** (1979), 443–474.

[10] A. Hamel, *Remarks to an equivalent formulation of Ekeland's variational principle*, Optimization, **31** (1994), 233–238.

[11] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., **60** (1968), 71–76.

[12] J.-P. Penot, *The drop theorem, the petal theorem and Ekeland's variational principle*, Nonlinear Anal., **10** (1986), 813–822.

[13] S. Reich, *Fixed points of contractive functions*, Boll. Unione Mat. Ital., **5** (1972), 26–42.

[14] B.E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., **226** (1977), 257–290

[15] W. Song, *A generalization of Clarke's fixed point theorem*, Appl. Math. J. Chinese Univ. Ser. B, **10** (1995), 463–466.

[16] W. Takahashi, *Existence theorems generalizing fixed point theorems for multivalued mappings.* In: *Fixed point theory and applications*, Marseille, 1989, Pitman Res. Notes Math. Ser., Vol. 252, Longman Sci. Tech., Harlow, 1991, pp. 397–406.

[17] J. D. Weston, *A characterization of metric completeness*, Proc. Amer. Math. Soc., **64** (1977), 186–188.

[18] T. Zamfirescu, *Fix point theorems in metric spaces*, Arch. Math. (Basel), **23** (1972), 292–298.

Received June 3, 2020
Accepted March 16, 2021

National Polytechnic School of Oran-MA
Department of FPST
BP 1523 El Ménaouer
31000 Oran, Algeria
E-mail: `sfyabenahmed@yahoo.fr`
E-mail: `sfya.ghamnia@enp-oran.dz`
`https://orcid.org/0000-0002-6404-5959`