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OPERATIONS ON GREEDOIDS

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Abstract. Our goal in this paper is to explore the operations of deletion, contraction, direct sum and ordered sum of greedoids. Moreover, we introduce the notion of balanced greedoid and give a necessary and sufficient condition for the direct sum and ordered sum of balanced greedoids to be balanced.

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1. INTRODUCTION

We begin with some background material, which follows the terminology and notation in [17]. A **greedoid** G is a pair $(E, \mathfrak{F})$, where $\mathfrak{F} \subseteq 2^E$ is a set system satisfying the following conditions.

(**G1**) For every non-empty $X \in \mathfrak{F}$, there is an $x \in X$ such that $X - x \in \mathfrak{F}$.

(**G2**) For all $X, Y \in \mathfrak{F}$ such that the cardinality $|X|$ of X is greater than the cardinality $|Y|$ of Y, there is an $x \in X - Y$ such that $Y \cup x \in \mathfrak{F}$.

Thus every matroid is a greedoid and a greedoid is a matroid if and only if the following axiom is satisfied:

(**M1**) If $X \in \mathfrak{F}$ and $Y \subseteq X$, then $Y \in \mathfrak{F}$.

For an introduction on matroids the reader is referred to [16] and [18]. Observe that axioms **M1** and **G2** together define a matroid and axiom **G1** and the following axiom defines a greedoid:

(**G2′**) For $X, Y \in \mathfrak{F}$ such that $|X| = |Y| + 1$, there is an $x \in X - Y$ such that $Y \cup x \in \mathfrak{F}$.

The set E is called the **ground set** of G, the sets in $\mathfrak{F}$ are called **feasible sets** and r is the **rank** of G which we denote by $r(G)$. For $A \subseteq E$, the **rank** of A is $r(A) = \max\{|X| : X \subseteq A, X \in \mathfrak{F}\}$. Thus A is feasible if and only if $r(A) = |A|$ and it is called a **basis** if $r(A) = |A| = r(G)$. The collection of all basis of G is denoted by $\mathcal{B}(G)$. Axiom **G2** implies that bases elements have the same size r (or $r(G)$). For $A \subseteq E$, define

$$\mathfrak{F}\backslash A := \{X \subseteq E - A : X \in \mathfrak{F}\},$$

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and, if A is feasible, define

$$\mathfrak{F}/A := \{X \subseteq E - A \,:\, X \cup A \in \mathfrak{F}\}.$$

Then it is easy to see that the set systems obtained in both cases are greedoids on the ground set $E-A$. The greedoid $G\backslash A = (E-A, \mathfrak{F}\backslash A)$ is called G **delete** A or the **restriction** of G to $E - A$ and $G/A = (E - A, \mathfrak{F}/A)$ is called G **contract** A. For all $X \subseteq E - A$, it is easy to see that

$$r_{G\backslash A}(X) = r(X) \text{ and } r_{G/A}(X) = r(X \cup A) - r(A).$$

A greedoid $G = (E, \mathfrak{F})$ is called an **interval greedoid** if it satisfies the **interval property** which is if $A \subseteq B \subseteq C$, $A, B, C \in \mathfrak{F}$, $x \in E-C$, $A \cup x \in \mathfrak{F}$ and $C \cup x \in \mathfrak{F}$, then $B \cup x \in \mathfrak{F}$. Thus, matroids are interval greedoids.

Operations as basic as deletion and contraction are those of **direct sum** and **ordered sum**.

Definition 1.1. Let $G_1 = (E_1, \mathfrak{F}_1)$ and $G_2 = (E_2, \mathfrak{F}_2)$ be two greedoids on disjoint ground sets. Then their direct sum is the greedoid $G_1 \oplus G_2 = (E_1 \cup E_2, \mathfrak{F}_1 \oplus \mathfrak{F}_2)$, where

$$\mathfrak{F}_1 \oplus \mathfrak{F}_2 = \{X_1 \cup X_2 \,:\, X_1 \in \mathfrak{F}_1 \text{ and } X_2 \in \mathfrak{F}_2\},$$

and the ordered sum of G_1 and G_2 is the greedoid $G_1 \otimes G_2 = (E_1 \cup E_2, \mathfrak{F}_1 \otimes \mathfrak{F}_2)$, where

$$\mathfrak{F}_1 \otimes \mathfrak{F}_2 = \mathfrak{F}_1 \cup \{B \cup X \,:\, B \in \mathcal{B}(G_1),\, X \in \mathfrak{F}_2\}.$$

Observe that $\varnothing \in \mathfrak{F}_1 \cap \mathfrak{F}_2$ and $\mathfrak{F}_1 \otimes \mathfrak{F}_2 \subseteq \mathfrak{F}_1 \oplus \mathfrak{F}_2$, thus $G_1 \otimes G_2$ is a subgreedoid of $G_1 \oplus G_2$.

Remark 1.2. Although we will only consider greedoids on disjoint ground sets when talking about the operations of direct sum and ordered sum, these operations can easily be defined on the disjoint union of the ground sets of any greedoids.

Greedoids were invented in 1981 by Korte and Lovász [15]. Originally, the main motivation for proposing this generalization of the matroid concept (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]) came from combinatorial optimization. Korte and Lovász had observed that the optimality of a “greedy” algorithm could in several instances be traced back to an underlying combinatorial structure that was not a matroid and so they named it a greedoid. In 1991, Korte, Lovász and Schrader [14] introduced greedoid as a special kind of antimatroids. In 1992, Björner and Ziegler [17] explained the basic ideas and gave a few glimpses of more specialized topics related to greedoids. In 1992, Broesma and Li [11] extended the “connectivity” concept from matroids to greedoids and in 1997, Gordon [13] extended Crapo’s β invariant from matroids to greedoids. In this paper, we extend the density concept from matroids and graphs to greedoids. We also study some greedoid preserving operations.

2. DELETION AND CONTRACTION GREEDOIDS

In this section, we study properties of greedoid deletion and contraction operations and show that these operations commute. We start by proving the following.

Proposition 2.1. *If B_A is a basis for the restriction $G|A$ of G to A, then*

$$\begin{aligned}\mathfrak{F}(G/A) &= \{X \subseteq E - A : G|A \text{ has a basis } B \text{ such that } X \cup B \in \mathfrak{F}(G)\}\\ &= \{X \subseteq E - A : X \cup B_A \in \mathfrak{F}(G)\}.\end{aligned}$$

Proof. Clearly $\{X \subseteq E - A : G|A$ has a basis B such that $X \cup B \in \mathfrak{F}(G)\}$ contains the set $\{X \subseteq E - A : X \cup B_A \in \mathfrak{F}(G)\}$. Suppose $X \cup B \in \mathfrak{F}(G)$ for some basis B of $G|A$. We shall show that $X \in \mathfrak{F}(G/A)$. Clearly $X \cup B$ is a basis of $X \cup A$, so $r(X \cup B) = r(X \cup A)$. Therefore,

$$r_{G/A}(X) = r(X \cup A) - r(B) = r(X \cup B) - r(B) = |X \cup B| - |B| = |X|,$$

that is, $X \in \mathfrak{F}(G/A)$. Hence,

$$\{X \subseteq E - A : G|A \text{ has a basis } B \text{ such that } X \cup B \in \mathfrak{F}(G)\} \subseteq \mathfrak{F}(G/A).$$

Finally we show $\{X \subseteq E - A : X \cup B_A \in \mathfrak{F}(G)\}$ contains $\mathfrak{F}(G/A)$. If $X \in \mathfrak{F}(G/A)$, then $|X| = r_{G/A}(X) = r(X \cup A) - r(A) = r(X \cup B_A) - |B_A|$. Hence $|X \cup B_A| = r(X \cup B_A)$, so $X \cup B_A \in \mathfrak{F}(G)$. □

Corollary 2.2. *If B_A is a basis for $G|A$, then a bases of G/A is*

$$\begin{aligned}\mathcal{B}(G/A) &= \{B \subseteq E - A : G|A \text{ has a basis } B' \text{ such that } B \cup B' \in \mathcal{B}(G)\}\\ &= \{B \subseteq E - A : B \cup B_A \in \mathcal{B}(G)\}.\end{aligned}$$

Observe that $\mathcal{B}(G\backslash A)$ is the set of maximal members of $\{B - A : B \in \mathcal{B}(G)\}$ and $\mathfrak{F}(G/A) \subseteq \mathfrak{F}(G\backslash A)$ for every feasible set A in G. Next, we give a necessary and sufficient condition for the contraction of a feasible set to be the same as the deletion of that set.

Proposition 2.3. *If A is a feasible set in G, then*

$$G/A = G\backslash A \text{ if and only if } r(G\backslash A) = r(G) - r(A).$$

Proof. Suppose $G/A = G\backslash A$ and let B be a basis of $G\backslash A$. Then B is a basis of G/A and hence by Corollary 2.2, $B \dot\cup B_A$ is a basis of G for some basis B_A of $G|A$. Thus $r(G) = |B \dot\cup B_A| = |B| + |B_A| = r(A) + r(G\backslash A)$. Suppose $r(G\backslash A) = r(G) - r(A)$. Since $\mathfrak{F}(G/A) \subseteq \mathfrak{F}(G\backslash A)$, to show $G/A = G\backslash A$, we need only show $\mathfrak{F}(G\backslash A) \subseteq \mathfrak{F}(G/A)$. But if $X \in \mathfrak{F}(G\backslash A)$, then X is a subset of a basis B of $G\backslash A$ and B is contained in a basis $B \dot\cup B'$ of G. Evidently,

$$r(G) = |B' \dot\cup B| = |B| + |B'| = r(G\backslash A) + |B'|.$$

Since $r(G\backslash A) = r(G) - r(A)$, we have $r(A) = |B'|$, that is, B' is a basis of $G|A$. Hence $B \in \mathcal{B}(G/A)$, so $X \in \mathfrak{F}(G/A)$ and $G/A = G\backslash A$. □

Corollary 2.4. *For all* $A \in \mathfrak{F}$, $G/A = G\backslash A$ *if and only if* $r(G\backslash A) \leq r(G/A)$.

Proof. If $G/A = G\backslash A$, then clearly $r(G\backslash A) \leq r(G/A)$. If $r(G\backslash A) \leq r(G/A)$, then as $\mathfrak{F}(G/A)$ is a subset of $\mathfrak{F}(G\backslash A)$ we must have $r(G\backslash A) \geq r(G/A)$. Thus $G/A = G\backslash A$. □

In the next proposition, we show that the operations of deletion and contraction commute.

Proposition 2.5. *Let* $G = (E, \mathfrak{F})$ *be a greedoid. Then*

$$(G\backslash A')/A = (G/A)\backslash A' = \{X \subseteq E - (A' \cup A) \,:\, X \cup A \in \mathfrak{F}\},$$

for $A \cap A' = \varnothing$, $A \in \mathfrak{F}$ *and* $A' \subseteq E$.

Proof. We need only show $(G\backslash A')/A$ and $(G/A)\backslash A'$ have the same collections of feasible sets. If $X \in \mathfrak{F}_{(G\backslash A')/A}$, then $X \subseteq (E - A') - A$ and $X \cup A \in \mathfrak{F}$. That is, $X \subseteq (E - A) - A'$ and $X \in \mathfrak{F}_{G/A}$ and hence $X \in \mathfrak{F}_{(G/A)\backslash A'}$. Conversely, if $X \in \mathfrak{F}_{(G/A)\backslash A'}$, then $X \subseteq (E - A) - A'$ and $X \in \mathfrak{F}_{G/A}$. That is, $X \subseteq (E - A') - A$ and $X \cup A \in \mathfrak{F}$ and hence $X \in \mathfrak{F}_{(G\backslash A')/A}$. Therefore, $\mathfrak{F}_{(G\backslash A')/A} = \mathfrak{F}_{(G/A)\backslash A'}$. □

The straightforward proof of the following proposition is omitted.

Proposition 2.6. $\{B_1 \cup B_2 \,:\, B_1 \in \mathcal{B}(G_1)$ *and* $B_2 \in \mathcal{B}(G_2)\} = \mathcal{B}(G_1 \otimes G_2)$ *which is equal to* $\mathcal{B}(G_1 \oplus G_2)$.

Corollary 2.7. *Let* $G_1 = (E_1, \mathfrak{F}_1)$ *and* $G_2 = (E_2, \mathfrak{F}_2)$ *be greedoids on disjoint ground sets. If* $X \subseteq E_1 \cup E_2$, *then*

$$r_{G_1 \otimes G_2}(X) = r_{G_1 \oplus G_2}(X) = r_{G_1}(X \cap E_1) + r_{G_2}(X \cap E_2).$$

3. ON GREEDOID PRESERVING OPERATIONS

In this section, we prove the operations of direct sum and ordered sum take interval greedoids to interval greedoids. In fact, we show that the direct sum and ordered sum of greedoids G_1 and G_2 is an interval greedoid if and only if G_1 and G_2 are both interval greedoids. We also introduce balanced greedoids and give a condition for the direct sum and ordered sum of balanced greedoids to be balanced.

Theorem 3.1. *Let* $G_1 = (E_1, \mathfrak{F}_1)$ *and* $G_2 = (E_2, \mathfrak{F}_2)$ *be greedoids on disjoint ground sets. Then* G_1 *and* G_2 *are interval greedoids if and only if* $G_1 \oplus G_2$ *is an interval greedoid.*

Proof. Suppose G_1 and G_2 are interval greedoids. If $A \subseteq B \subseteq C$, $A, B, C \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$, $x \in E_1 \cup E_2 - C$, $A \cup x \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$, and $C \cup x \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$, then $A = A_1 \cup A_2$, $B = B_1 \cup B_2$, $C = C_1 \cup C_2$ where A_i, B_i, C_i are feasible sets in G_i for $i = 1, 2$, $A_i \cup x \in \mathfrak{F}_i$ (as $A_i \cup x = (A_1 \cup A \cup x) \cap E_i$). Similarly, $C_i \cup x \in \mathfrak{F}_i$.

Moreover, $x \in (E_1 \cup E_2 - C_1) \cap (E_1 \cup E_2 - C_2)$. Hence suppose $x \in E_i - C_i$ for $i = 1$ or $i = 2$ and as $A_i \subseteq B_i \subseteq C_i$, $B_i \cup x \in \mathfrak{F}_i$. But

$$B \cup x = B_1 \cup B_2 \cup x = (B_1 \cup x) \cup B_2 \in \mathfrak{F}_1 \oplus \mathfrak{F}_2.$$

Therefore, $G_1 \oplus G_2$ is an interval greedoid.

Suppose $G_1 \oplus G_2$ is an interval greedoid. If $A \subseteq B \subseteq C$, $A, B, C \in \mathfrak{F}_1$, x an element in $E_1 - C$, $A \cup x \in \mathfrak{F}_1$, and $C \cup x \in \mathfrak{F}_1$, then as $\varnothing \in \mathfrak{F}_2$, $A \cup \varnothing \subseteq B \cup \varnothing \subseteq C \cup \varnothing$, $A \cup \varnothing, B \cup \varnothing, C \cup \varnothing \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$, $x \in E_1 \cup E_2 - C$, $(A \cup x) \cup \varnothing, (B \cup x) \cup \varnothing \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$ and as $G_1 \oplus G_2$ is an interval greedoid, $B \cup x = (B \cup \varnothing) \cup x \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$. But $B \cup x = (B \cup x) \cap E_1 \in \mathfrak{F}_1$ and hence G_1 is an interval greedoid. Similarly, G_2 is an interval greedoid. □

THEOREM 3.2. *Let $G_1 = (E_1, \mathfrak{F}_1)$ and $G_2 = (E_2, \mathfrak{F}_2)$ be greedoids on disjoint ground sets. Then G_1 and G_2 are interval greedoids if and only if $G_1 \otimes G_2$ is an interval greedoid.*

Proof. The proof of the necessary condition is similar to that of the direct sum one in the preceding theorem and is left to the reader. Suppose $G_1 \otimes G_2$ is an interval greedoid. If $A \subseteq B \subseteq C$, $A, B, C \in \mathfrak{F}_1$, $x \in E_1 - C$, $A \cup x \in \mathfrak{F}_1$, and $C \cup x \in \mathfrak{F}_1$, then $A, B, C \in \mathfrak{F}_1 \otimes \mathfrak{F}_2$, $x \in E_1 \cup E_2 - C$, $A \cup x, C \cup x \in \mathfrak{F}_1 \otimes \mathfrak{F}_2$ and as $G_1 \otimes G_2$ is an interval greedoid, $B \cup x \in \mathfrak{F}_1 \otimes \mathfrak{F}_2$. But as $\mathfrak{F}_1 \otimes \mathfrak{F}_2 \subseteq \mathfrak{F}_1 \oplus \mathfrak{F}_2$, $B \cup x \in \mathfrak{F}_1 \oplus \mathfrak{F}_2$. Thus $B \cup x = (B \cup x) \cap E_1 \in \mathfrak{F}_1$ and hence G_1 is an interval greedoid. Similarly, G_2 is an interval greedoid. □

In [12], a condition for the direct sum of balanced matroids to be balanced was given. Next, we prove a similar result for loopless greedoids.

DEFINITION 3.3. The **density** of a loopless greedoid (i.e. has no elements of rank zero) $G = (E, \mathfrak{F})$ is given by $d(G) := \frac{|G|}{r(G)}$. A greedoid G is **balanced** if $d(K) \leq d(G)$ for all non-empty subgreedoids K of G.

THEOREM 3.4. *The direct sum (respectively, the ordered sum) of balanced loopless greedoids G_1 and G_2, on disjoint ground sets, is balanced if and only if*

$$d(G_1) = d(G_2) = d(G_1 \oplus G_2)(\textit{respectively}, d(G_1) = d(G_2) = d(G_1 \otimes G_2)).$$

Proof. We only prove the direct sum part since the order sum one is similar. Let $G_1 = (E_1, \mathfrak{F}_1)$ and $G_2 = (E_2, \mathfrak{F}_2)$ be balanced greedoids on disjoint ground sets. Suppose that $G_1 \oplus G_2$ is balanced. Evidently $d(G_i) \leq d(G_1 \oplus G_2)$ for $i = 1, 2$ and thus

$$\begin{aligned} |E_1|r(G_1) + |E_1|r(G_2) &\leq |E_1|r(G_1) + |E_2|r(G_2) \text{ and} \\ |E_2|r(G_1) + |E_2|r(G_2) &\leq |E_1|r(G_2) + |E_2|r(G_2). \end{aligned}$$

So, $|E_1|r(G_2) \leq |E_2|r(G_1) \leq |E_1|r(G_2)$ which implies $|E_2|r(G_1) = |E_1|r(G_2)$ or

$$d(G_2) = \frac{|E_2|}{r(G_2)} = \frac{|E_1|}{r(G_1)} = d(G_1) = d(G_1 \oplus G_2).$$

Conversely, suppose that $d(G_1) = d(G_2) = d(G_1 \oplus G_2)$. If N is a subgreedoid of $G_1 \oplus G_2$, then $N = N_1 \oplus N_2$ where each $N_i = N \cap E_i$. Thus $d(N_i) = \frac{|E(N_i)|}{r(N_i)} \leq \frac{|E_1|}{r(G_1)}$ and hence

$$|E(N_1)|r(G_1) + |E(N_2)|r(G_1) \leq |E_1|r(N_1) + |E_1|r(N_2),$$

and

$$d(N) = \frac{|E(N_1)| + |E(N_2)|}{r(N_1) + r(N_2)} \leq \frac{|E_1|}{r(G_1)} = d(G_1 \oplus G_2).$$

Therefore, $G_1 \oplus G_2$ is balanced. □

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