MATHEMATICA, 63 (86), N° 2, 2021, pp. 297–302

STRONG REGULARITY OF SMASH PRODUCTS ASSOCIATED WITH G-SET GRADINGS

GABRIELA OLTEANU

Abstract. For a group G, a G-graded ring R and a finite left G-set A, we study the strong regularity of the smash product $R\#A$.

MSC 2010. 16E50, 16D90, 16W50.

Key words. Strongly regular ring, smash product, G-set grading.

1. INTRODUCTION

One of the most important classical concepts in ring theory has been that of regular ring in the sense of J. von Neumann [9]. A ring R is called *(von Neumann) regular* if for every $a \in R$, there exists $b \in R$ such that $a = aba$. Strongly regular rings have been introduced by R.F. Arens and I. Kaplansky [1] as a relevant specialization of (von Neumann) regular rings. A ring R is called *strongly regular* if for every $a \in R$, there exists $b \in R$ such that $a = a^2b$ (equivalently, $a = ba^2$). It is well known that a ring R is strongly regular if and only if R is (von Neumann) regular and abelian (i.e., every idempotent of R is central).

Both notions have been generalized to modules by G. Lee. S.T. Rizvi and C. Roman [6], but also to abelian categories by S. Dăscălescu, C. Năstăsescu, A. Tudorache and L. Dăuş [4] and S. Crivei, A. Kör and G. Olteanu [2, 3] respectively. In particular, such results are applicable to (graded) module categories. For instance, L. Dăuş, C. Năstăsescu and M. Năstăsescu have studied von Neumann regularity of smash products associated with G-set gradings [5], and we have studied strong regularity of modules under excellent extensions of rings [8]. In this paper we investigate strong regularity of smash products associated with G-set gradings.

2. MODULES GRADED BY G-SETS

For a multiplicative group G, recall that a ring R with identity is called *G-graded* if there is a direct sum decomposition $R = \bigoplus_{\sigma\in G} R_\sigma$ of additive subgroups such that $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ for every $\sigma, \tau \in G$. For every $\sigma \in G$, R_σ is called the *σ-homogeneous component* of R. For a G-graded ring R and a non-empty subset X of G, denote $R_X = \bigoplus_{x\in X} R_x$. For a subgroup H of G, R_H is a subring of R, which is an H-graded ring.

DOI: 10.24193/mathcluj.2021.2.14

For a group G with identity e, a non-empty set A is called a *left G-set* if there is a left action of G on A, say $G \times A \to A$ given by $(\sigma, x) \mapsto \sigma x$ such that $ex = x$ for every $x \in A$ and $(\sigma\tau)x = \sigma(\tau x)$ for every $\sigma, \tau \in G$ and $x \in A$.

Let $R = \bigoplus_{\sigma\in G} R_\sigma$ be a G-graded ring and let A be a finite left G-set. Following [7], the *smash product* $R\#A$ is the free left R-module with basis $\{p_x \mid x \in A\}$ and multiplication given by $(r_\sigma p_x)(s_\tau p_y) = (r_\sigma s_\tau)p_y$ if $\tau y = x$ and zero otherwise, for every $r_\sigma \in R_\sigma$, $s_\tau \in R_\tau$, $x, y \in A$. Thus $R\#A$ is a ring with identity $\sum_{x\in A} p_x$, and the ring R embeds into the smash product $R\#A$. Moreover, $R\#A$ is a G-graded ring with its σ-homogeneous component $(R\#A)_\sigma = \sum_{x\in A} R_\sigma p_x$.

Let R be a G-graded ring and let A be a left G-set. Following [7], a *graded left R-module of type A* is a left R-module M which has a direct-sum decomposition $M = \bigoplus_{x\in A} M_x$ as additive subgroups such that $R_\sigma M_x \subseteq M_{\sigma x}$ for every $\sigma \in G$ and $x \in A$. A *morphism of graded left R-modules of type A* is a morphism $f : M \to N$ of left R-modules such that $f(M_x) \subseteq N_x$ for every $x \in A$. We denote by (G, A, R)-gr the category of graded left R-modules of type A and corresponding morphisms.

For every $x \in A$, the *x-suspension* $R(x)$ of R is the object of (G, A, R)-gr which equals R as an R-module and whose grading is given by

$$R(x)_y = \bigoplus_{\sigma\in G} \{R_\sigma \mid \sigma \in G, \sigma x = y\}$$

for every $y \in A$. Since A is a left G-set, $(R(x))_{x\in A}$ is a family of finitely generated generators, and $V = \bigoplus_{x\in A} R(x)$ is a projective generator of the abelian category (G, A, R)-gr [7].

For every $x, y \in A$, denote $G_{x,y} = \{\sigma \in G \mid \sigma x = y\}$. Note that $G_x = G_{x,x}$ is the stabilizer of x. According to [5, Lemma 4.5], for every $x, y \in A$, there exist a group isomorphism

$$\mathrm{Hom}_{(G,A,R)\text{-}gr}(R(x), R(y)) \cong R_{G_{y,x}}$$

and a ring isomorphism

$$\mathrm{End}_{(G,A,R)\text{-}gr}(R(x)) \cong R_{G_x}.$$

3. STRONGLY RELATIVE REGULAR MODULES

The concept of strongly regular ring was generalized to strongly regular objects in abelian categories by using fully (co)invariant (co)kernels. Recall that a kernel $k : K \to M$ in an abelian category $\mathcal{A}$ is called *fully invariant* if for every $h \in \mathrm{End}_{\mathcal{A}}(M)$, there is a morphism $\alpha \in \mathrm{End}_{\mathcal{A}}(K)$ such that $hk = k\alpha$ [3]. The notion of *fully coinvariant cokernel* is defined dually.

Definition 3.1 ([3]). Let M and N be objects of an abelian category $\mathcal{A}$. Then N is called *strongly M-regular* if for every morphism $f : M \to N$, $\ker(f)$ is a fully invariant section and $\mathrm{coker}(f)$ is a fully coinvariant retraction. Also, N is called *strongly self-regular* if N is strongly N-regular.

Proposition 3.2 ([3]). *Let M be an object of an abelian category $\mathcal{A}$. Then M is strongly self-regular if and only if its endomorphism ring* $\mathrm{End}_{\mathcal{A}}(M)$ *is strongly regular.*

We now give a general key result on strongly relative regular objects in abelian categories, which will be useful throughout the paper.

Theorem 3.3. *Let $(M_i)_{i\in I}$ be a family of objects of an abelian category $\mathcal{A}$. Then the following are equivalent:*

(1) *For every $i, j \in I$, M_j is strongly M_i-regular.*
(2) *For every $i, j \in I$, M_j is M_i-regular and the ring* $\mathrm{End}_{\mathcal{A}}(M_i)$ *is abelian.*

Proof. (1)$\Longrightarrow$(2) Let $i, j \in I$. By hypothesis, M_j is clearly M_i-regular and $\mathrm{End}_{\mathcal{A}}(M_i)$ is abelian by [3, Propositions 2.4 and 2.5].

(2)$\Longrightarrow$(1) Let $i, j \in I$, and let $f : M_i \to M_j$ be a morphism in $\mathcal{A}$ with kernel $k : K \to M_i$ and cokernel $c : M_j \to C$. Since M_j is M_i-regular, k is a section and c is a retraction, and thus there are morphisms $p : M_i \to K$ and $q : C \to M_j$ such that $pk = 1_K$ and $cq = 1_C$. We need to show that k is a fully invariant kernel, and c is a fully coinvariant cokernel. We only prove the former, because the latter follows by duality. Let $h \in \mathrm{End}_{\mathcal{A}}(M_i)$. By hypothesis, the ring $\mathrm{End}_{\mathcal{A}}(M_i)$ is abelian, which implies that the idempotent $e = kp : M_i \to M_i$ is central. Then $hkp = he = eh = kph$ implies that $hk = kphk$, which shows that k is a fully invariant kernel. Finally, since c is a fully coinvariant cokernel, it follows that M_j is strongly M_i-regular. □

4. STRONG REGULARITY OF SMASH PRODUCTS

The next result relates regularity and strong regularity of smash products.

Theorem 4.1. *Let $R = \bigoplus_{\sigma\in G} R_\sigma$ be a G-graded ring and let A be a finite left G-set. Then the following are equivalent:*

(1) *$R\#A$ is a strongly regular ring.*
(2) *For every $x, y \in A$, $R(x)$ is a strongly $R(y)$-regular module.*
(3) *$R\#A$ is a regular ring and, for every $x \in A$, the ring R_{G_x} is abelian.*

Proof. (1)$\Longrightarrow$(2) Using the ring isomorphism

$$(R\#A)^{\mathrm{op}} \cong \mathrm{End}_{(G,A,R)\text{-}gr}(V)$$

[7, Theorem 2.14], where $V = \bigoplus_{x\in A} R(x)$, we deduce that $\mathrm{End}_{(G,A,R)\text{-}gr}(V)$ is a strongly regular ring. Then V is a strongly self-regular module [3, Proposition 2.5], and thus $R(x)$ is a strongly $R(y)$-regular module for every $x, y \in A$ by [3, Theorem 3.1].

(2)$\Longrightarrow$(1) By hypothesis and [3, Theorem 3.1], $V = \bigoplus_{x\in A} R(x)$ is a strongly $R(y)$-regular module for every $y \in A$. Again by [3, Theorem 3.1], it follows that V is strongly self-regular. Then $\mathrm{End}_{(G,A,R)\text{-}gr}(V)$ is a strongly regular ring by [3, Proposition 2.5], and thus $R\#A$ is a strongly regular ring by the ring isomorphism from the above implication.

(2)$\Longrightarrow$(3) Let $x, y \in A$. Since $R(x)$ is an $R(y)$-regular module, $R\#A$ is a regular ring by [5, Theorem 4.7]. Since $R(x)$ is a strongly self-regular module, $\mathrm{End}_{(G,A,R)\text{-}gr}(R(x))$ is a strongly regular ring by [3, Proposition 2.5], and thus it is abelian. By [5, Remark 4.6], there is a ring isomorphism $R_{G_x} \cong \mathrm{End}_{(G,A,R)\text{-}gr}(R(x))$. Hence the ring R_{G_x} is abelian.

(3)$\Longrightarrow$(2) Let $x, y \in A$. Then $R(x)$ is an $R(y)$-regular module by [5, Theorem 4.7]. Using again the ring isomorphism $R_{G_x} \cong \mathrm{End}_{(G,A,R)\text{-}gr}(R(x))$, it follows that the ring $\mathrm{End}_{(G,A,R)\text{-}gr}(R(x))$ is abelian. Now $R(x)$ is a strongly $R(y)$-regular module by Theorem 3.3. □

For a group G and a left G-set A, denote by $\mathrm{Fix}_G(A)$ the union of all trivial G-orbits, that is,

$$\mathrm{Fix}_G(A) = \{x \in A \mid \{\sigma x \mid \sigma \in G\} = \{x\}\}.$$

Theorem 4.2. *Let G be a group, let A be a finite left G-set and let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a G-graded ring.*

(1) *If R is a strongly regular ring, then so is $R\#A$.*
(2) *If $R\#A$ is a strongly regular ring, then so is R_{G_x} for every $x \in A$.*
(3) *If $\mathrm{Fix}_G(A) \neq \emptyset$ and $R\#A$ is a strongly regular ring, then so is R.*
(4) *If G is a p-group, $(|A|, p) = 1$ and $R\#A$ is a strongly regular ring, then so is R.*

Proof. (1) If R is a strongly regular ring, then $R\#A$ is a regular ring by [5, Corollary 4.8]. Also, since R is abelian, so is its subring R_{G_x} for every $x \in A$. Then $R\#A$ is a strongly regular ring by Theorem 4.1.

(2) Let $x \in A$. If $R\#A$ is a strongly regular ring, then the ring R_{G_x} is regular by [5, Theorem 4.7] and abelian by Theorem 4.1. Hence R_{G_x} is a strongly regular ring.

(3) If $a \in \mathrm{Fix}_G(A)$, then $\{a\}$ is a left G-set and the inclusion $i : \{a\} \to A$ is a morphism of G-sets. But the induced morphism $R\#i : R\#A \to R\#\{a\} \cong R$ is an epimorphism by [5, Corollary 3.3]. Since the ring $R\#A$ is strongly regular, clearly so is its homomorphic image R.

(4) If G is a p-group, $(|A|, p) = 1$, then $\mathrm{Fix}_G(A) \neq \emptyset$ [5, Remark 2.4]. Hence R is a strongly regular ring by (3). □

Theorem 4.3. *Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a G-graded ring, and let A, B be finite left G-sets. If either $R\#A$ or $R\#B$ is a strongly regular ring, then so is $(R\#A)\#B \cong R\#(A \times B)$.*

Proof. We assume that $R\#A$ is a strongly regular ring. Then $R\#(A \times B)$ is a regular ring by [5, Proposition 4.11]. For every $(x, y) \in A \times B$, $R_{G_{(x,y)}}$ is a strongly regular ring by Theorem 4.2 (2), and so it is abelian. Then $R\#(A \times B)$ is a regular ring by Theorem 4.1. For the isomorphism, see [5, Corollary 3.2]. □

5. SOME APPLICATIONS TO SUBGROUPS

PROPOSITION 5.1. *Let H be a subgroup of finite index of a group G and let $R = \bigoplus_{\sigma\in G} R_\sigma$ be a G-graded ring. If $R\#G/H$ is a strongly regular ring, then so is $R_{\sigma H\sigma^{-1}}$ for every $\sigma \in G$.*

Proof. As noted in [5, p. 52], if H is a subgroup of a group G, then $G/H = \{\sigma H \mid \sigma \in G\}$ is a left G-set, where the G-action is defined by $\tau(\sigma H) = \tau\sigma H$ for every $\tau, \sigma \in G$. Also, the stabilizer of the element $\sigma H \in G/H$ is $G_{\sigma H} = \sigma H\sigma^{-1}$. Now use Theorem 4.2 (2). □

PROPOSITION 5.2. *Let H be a subgroup of a group G and let $R = \bigoplus_{\sigma\in G} R_\sigma$ be a G-graded ring.*

(1) *If R is a strongly regular ring, then so is R_H.*
(2) *If A is a finite left G-set and $R\#A$ is a strongly regular ring, then so is $R_H\#A$.*

Proof. (1) If R is a strongly regular ring, then R_H is a regular ring [5, Lemma 4.1]. Also, since R is abelian, so is its subring R_H. Hence R_H is a strongly regular ring.

(2) Note that $R_H\#A = (R\#A)_H$ and use (1). □

Let R be a ring with automorphism group $\mathrm{Aut}(R)$, and let G be a finite group that acts as automorphisms on R, in the sense that there exists a group morphism $\psi : G \to \mathrm{Aut}(R)$. We denote by

$$R^G = \{r \in R \mid \psi(g)(r) = r \text{ for every } g \in G\}$$

the fixed subring of R under G.

LEMMA 5.3. *Let G be a finite group and let R be a ring such that G acts as automorphisms on R and $|G|^{-1} \in R$. If R is a strongly regular ring, then so is R^G.*

Proof. If the ring R is strongly regular, then the ring R^G is regular by [5, Lemma 4.16]. But R is also abelian, which implies that so is its subring R^G. Hence the ring R^G is strongly regular. □

PROPOSITION 5.4. *Let G be a finite group, let R be a G-graded ring and let H be a subgroup of G such that $|H|^{-1} \in R$. If $R\#G$ is a strongly regular ring, then so is $R\#G/H$.*

Proof. This follows by using the isomorphism $(R\#G)^H \cong R\#G/H$ [5, p. 56] and Lemma 5.3. □

REFERENCES

[1] R.F. Arens and I. Kaplansky, *Topological representations of algebras*, Trans. Amer. Math. Soc., **63** (1948), 457–481.

[2] S. Crivei and A. Kör, *Rickart and dual Rickart objects in abelian categories*, Appl. Categor. Struct., **24** (2016), 797–824.

[3] S. Crivei and G. Olteanu, *Strongly Rickart objects in abelian categories. Applications to strongly regular and strongly Baer objects*, Comm. Algebra, **46** (2018), 4426–4447.
[4] S. Dăscălescu, C. Năstăsescu, A. Tudorache and L. Dăuş, *Relative regular objects in categories*, Appl. Categ. Struct., **14** (2006), 567–577.
[5] L. Dăuş, C. Năstăsescu and M. Năstăsescu, *Von Neumann regularity of smash products associated with G-set gradings*, J. Algebra, **331** (2011), 46–57.
[6] G. Lee, S.T. Rizvi and C. Roman, *Modules whose endomorphism rings are von Neumann regular*, Comm. Algebra, **41** (2013), 4066–4088.
[7] C. Năstăsescu, Ş. Raianu and F. Van Oystaeyen, *Modules graded by G-sets*, Math. Z., **203** (1990), 605–627.
[8] G. Olteanu, *Strongly relative regular modules and excellent extensions of rings*, Mathematica, **59 (82)** (2017), 76-80.
[9] J. von Neumann, *On regular rings*, Proc. Natl. Acad. Sci. USA, **22** (1936), 707–712.

Received February 8, 2020
Accepted June 10, 2020

Babeş-Bolyai University
Department of Statistics-Forecasts-Mathematics
Str. T. Mihali 58-60
400591 Cluj-Napoca, Romania
E-mail: gabriela.olteanu@econ.ubbcluj.ro