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ON A SECOND-ORDER DIFFERENTIAL INCLUSION WITH CERTAIN INTEGRAL AND MULTI-STRIP BOUNDARY CONDITIONS

AURELIAN CERNEA

Abstract. We study a second-order differential inclusion with integral and multi-strip boundary conditions defined by a set-valued map with nonconvex values. We obtain an existence result and we prove the arcwise connectedness of the solution set of the considered problem.

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1. INTRODUCTION

This paper is concerned with the following problem

$$x''(t) \in F(t, x(t)) \quad \text{a.e. } ([0,1]), \tag{1}$$

$$\begin{aligned} \int_0^1 x(s)\mathrm{d}s = \sum_{j=1}^m \gamma_j \int_{\xi_j}^{\eta_j} x(s)\mathrm{d}s + c_1,\ \int_0^1 x'(s)\mathrm{d}s \\ = \sum_{j=1}^m \rho_j \int_{\xi_j}^{\eta_j} x'(s)\mathrm{d}s + c_2, \end{aligned} \tag{2}$$

where $F : [0,1] \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map, $0 < \xi_1 < \eta_1 < \xi_2 < \eta_2 < ... < \xi_m < \eta_m < 1$, $\gamma_j, \rho_j \geq 0$, $i = \overline{1,m}$ and $c_1, c_2 \in \mathbf{R}$.

In a recent paper [1], it is studied the problem (1)-(2) and several existence results are provided for this problem, when the right-hand side of (1) is single-valued and multi-valued. In the case of differential inclusions, the results in [1] are obtained using a nonlinear alternative of Leray Schauder type and some suitable theorems of fixed point theory.

The aim of our paper is to continue the study [1] in the case when the set-valued map $F(.,.)$ has nonconvex values. The main hypothesis in our approach is that $F(.,.)$ is Lipschitz in the second variable. Our goal is twofold. On one hand, we show that Filippov's ideas ([5]) can be suitably adapted in order to obtain the existence of solutions for problem (1)-(1). We recall that for a differential inclusion defined by a Lipschitzian set-valued map with nonconvex values, Filippov's theorem ([5]) consists in proving the existence of a solution

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starting from a given almost solution. Moreover, the result provides an estimate between the starting almost solution and the solution of the differential inclusion.

On the other hand, following the approach in [8], we prove the arcwise connectedness of the solution set of problem (1)-(2). The proof is based on a result (see [7, 8]) concerning the arcwise connectedness of the fixed point set of a class of set-valued contractions.

Motivation and examples for problem (1)-(2) may be found in [1] and the references therein. We also note that such kind of results exist in the literature (see e.g. [3, 4] etc.), but their presentation in the framework of problem (1)-(2) is new.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel, Section 3 is devoted to the existence theorem and in Section 4 we obtain the arcwise connectedness of the solution set.

2. PRELIMINARIES

In what follows we denote by I the interval $[0,1]$, $C(I,\mathbf{R})$ is the Banach space of all continuous functions from I to $\mathbf{R}$ with the norm $||x||_C = \sup_{t\in I}|x(t)|$ and $L^1(I,\mathbf{R})$ is the Banach space of integrable functions $u(.) : I \to \mathbf{R}$ endowed with the norm $||u||_1 = \int_0^T |u(t)|\mathrm{d}t$. The characteristic function of the set C it is denoted by $\chi_C(.)$ and if $a = (a_1, a_2) \in \mathbf{R}^2$ we put $||a|| = |a_1| + |a_2|$.

Let (X, d) be a metric space. We recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$D(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$.

Consider a set-valued map T on X with nonempty values in X. T is said to be a λ-contraction if there exists $0 < \lambda < 1$ such that

$$d_H(T(x), T(y)) \le \lambda d(x,y) \quad \forall x, y \in X.$$

A function $x \in C^2(I,\mathbf{R})$ is called a solution of problem (1)-(2) if there exists a function $f \in L^1(I,\mathbf{R})$ with $f(t) \in F(t, x(t))$, a.e. (I) such that $x''(t) = f(t)$ a.e. (I) and conditions (2) are satisfied.

In what follows we need the following technical lemma proved in [1].

LEMMA 2.1. *Assume that* $[1 - \sum_{j=1}^m \gamma_j(\eta_j - \xi_j)][1 - \sum_{j=1}^m \rho_j(\eta_j - \xi_j)] \neq 0$. *For a given integrable function* $f(.) : [0,T] \to \mathbf{R}$, *the unique solution of the differential equation* $x''(t) = f(t)$ *with boundary conditions* (2) *is given by*

$$\begin{aligned} x(t) = &\int_0^t (t-s)f(s)\mathrm{d}s + \frac{1}{a_1a_2}[\frac{1}{2}\int_0^1(2a(t) + a_1(t-s))(1-s)f(s)\mathrm{d}s + \\ &a_1c_1 + a_2c_2 + \sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_0^s [\rho_j a(t) + \gamma_j a_1(s-u)]f(u)\mathrm{d}u\mathrm{d}s], \end{aligned} \tag{3}$$

where $a(t) = a_2t - a_3$, $a_1 = 1 - \sum_{j=1}^m \gamma_j(\eta_j - \xi_j)$, $a_2 = 1 - \sum_{j=1}^m \rho_j(\eta_j - \xi_j)$ *and* $a_3 = \frac{1}{2} - \frac{1}{2}\sum_{j=1}^m \gamma_j((\eta_j)^2 - (\xi_j)^2)$.

Remark 2.2. For $c=(c_1,c_2)\in\mathbf{R}^2$ we set $P_c(t)=\frac{c_2}{a_1}t+\frac{c_1}{c_2}-\frac{c_2a_3}{a_1a_2}-\frac{ac_2}{c_1}$ and we denote $G(t,s)=G_1(t,s)+G(t,s)+G(t,s)$, where $G_1(t,s)=(t-s)\chi_{[0,t]}(s)$, $G_2(t,s)=-\frac{1}{2a_1a_2}(2a(t)+a_1(t-s))(1-s)$ and $G_3(t,s)=\frac{1}{a_1a_2}\sum_{j=1}^m[a(t)\rho_j((\eta_j-s)\chi_{[0,\eta_j]}(s)-(\xi_j-s)\chi_{[0,\xi_j]}(s))+a_1\gamma_j(\frac{(\eta_j-s)^2}{2}\chi_{[0,\eta_j]}(s)-\frac{(\xi_j-s)^2}{2}\chi_{[0,\xi_j]}(s))]$ then the solution in (3) may be written as

$$x(t)=P_c(t)+\int_0^1 G(t,s)f(s)\mathrm{d}s.$$

Moreover, $|G_1(t,s)|\le t\le 1\ \forall t,s\in I$, $|G_2(t,s)|\le\frac{1}{2|a_1a_2|}(2(|a_2|+|a_3|)+|a_1|)=:M_2\ \forall t,s\in I$, $|G_3(t,s)|\le\frac{1}{|a_1a_2|}\sum_{j=1}^m[\rho_j(|a_2|+|a_3|)(|\eta_j|+|\xi_j|)+\gamma_j|a_1|(\frac{\eta_j^2}{2}+\frac{\xi_j^2}{2})]=:M_3\ \forall t,s\in I$, and therefore,

$$|G(t,s)|\le 1+M_2+M_3=:M\quad\forall t,s\in I.$$

3. A FILIPPOV TYPE EXISTENCE RESULT

First we recall a selection result ([2]) which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem ([6]).

Lemma 3.1. *Consider X a separable Banach space, B is the closed unit ball in X, $H:I\to\mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g:I\to X, L:I\to\mathbf{R}_+$ are measurable functions. If*

$$H(t)\cap(g(t)+L(t)B)\ne\emptyset\quad a.e.(I),$$

then the set-valued map $t\to H(t)\cap(g(t)+L(t)B)$ has a measurable selection.

Hypothesis H1. i) $F:I\times\mathbf{R}\to\mathcal{P}(\mathbf{R})$ has nonempty closed values and for every $x\in\mathbf{R}$ $F(.,x)$ is measurable.

ii) There exists $L\in L^1(I,\mathbf{R})$ such that for almost all $t\in I, F(t,.)$ is $L(t)$-Lipschitz in the sense that

$$D(F(t,x),F(t,y))\le L(t)|x-y|\quad\forall\ x,y\in\mathbf{R}.$$

Theorem 3.2. *Assume that* Hypothesis H1 *is satisfied, assume that $M||L||_1<1$ and let $y\in C^2(I,\mathbf{R})$ be such that there exists $q(.)\in L^1(I,\mathbf{R})$ with $d(y''(t),F(t,y(t)))\le q(t)$ a.e. (I). Denote $\tilde{c}_1=\int_0^1 y(s)\mathrm{d}s-\sum_{j=1}^m\gamma_j\int_{\xi_j}^{\eta_j}y(s)\mathrm{d}s$, $\tilde{c}_2=\int_0^1 y'(s)\mathrm{d}s-\sum_{j=1}^m\rho_j\int_{\xi_j}^{\eta_j}y'(s)\mathrm{d}s$.*

Then there exists $x(.):I\to\mathbf{R}$ a solution of problem (1)-(2) *satisfying for all $t\in I$*

$$|x(t)-y(t)|\le\frac{1}{1-M||L||_1}\sup_{t\in I}|P_c(t)-P_{\tilde{c}}(t)|+\frac{M}{1-M||L||_1}||q||_1.$$

Proof. The set-valued map $t\to F(t,y(t))$ is measurable with closed values and the hypothesis that $d(y''(t),F(t,y(t)))\le q(t)$ a.e. (I) is equivalent to

$$F(t,y(t))\cap\{y''(t)+q(t)[-1,1]\}\ne\emptyset\quad a.e.\ (I).$$

Therefore, we can apply Lemma 2 in order to deduce that there exists a measurable selection $f_1(t) \in F(t, y(t))$ a.e. (I) such that

$$|f_1(t) - y''(t)| \le q(t) \quad \text{a.e. } (I) \tag{4}$$

Define $x_1(t) = P_c(t) + \int_0^1 G(t,s) f_1(s) \mathrm{d}s$ and one has

$$|x_1(t) - y(t)| = |P_c(t) - P_{\tilde{c}}(t) + \int_0^1 G(t,s)(f_1(s) - y''(s))\mathrm{d}s| \le$$
$$|P_c(t) - P_{\tilde{c}}(t)| + \int_0^1 |G(t,s)| q(s) \mathrm{d}s \le |P_c(t) - P_{\tilde{c}}(t)| + M||q||_1.$$

Our statement is that it is enough to construct the sequences $x_n(.) \in C(I, \mathbf{R})$, $f_n(.) \in L^1(I, \mathbf{R})$, $n \ge 1$ with the properties

$$x_n(t) = P_c(t) + \int_0^1 G(t,s) f_n(s) \mathrm{d}s, \quad t \in I, \tag{5}$$

$$f_n(t) \in F(t, x_{n-1}(t)) \quad \text{a.e.}\,(I),\ n \ge 1, \tag{6}$$

$$|f_{n+1}(t) - f_n(t)| \le L(t)|x_n(t) - x_{n-1}(t)| \quad \text{a.e.}\,(I), n \ge 1. \tag{7}$$

If this procedure is done, then from (4)-(7) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \le \int_0^1 |G(t,t_1)|.|f_{n+1}(t_1) - f_n(t_1)|\mathrm{d}t_1$$
$$\le M \int_0^1 L(t_1)|x_n(t_1) - x_{n-1}(t_1)|\mathrm{d}t_1 \le M \int_0^1 L(t_1) \int_0^1 |G(t_1,t_2)|.$$
$$|f_n(t_2) - f_{n-1}(t_2)|\mathrm{d}t_2 \le M^2 \int_0^1 L(t_1) \int_0^1 L(t_2)|x_{n-1}(t_2) - x_{n-2}(t_2)|\mathrm{d}t_2\mathrm{d}t_1$$
$$\le M^n \int_0^1 L(t_1) \int_0^1 L(t_2) ... \int_0^1 L(t_n)|x_1(t_n) - y(t_n)|\mathrm{d}t_n \ldots \mathrm{d}t_1$$
$$\le (M||L||_1)^n (\sup_{t\in I} |P_c(t) - P_{\tilde{c}}(t)| + M||q||_1).$$

Thus, $\{x_n(.)\}_{n\in\mathbf{N}}$ is a Cauchy sequence in the Banach space $C(I, \mathbf{R})$, hence converging uniformly to some $x(.) \in C(I, \mathbf{R})$. Therefore, by (7), for almost all $t \in I$, the sequence $\{f_n(t)\}_{n\in\mathbf{N}}$ is Cauchy in $\mathbf{R}$. Denote by f be the pointwise limit of f_n.

At the same time, we have

$$\begin{aligned} |x_n(t) - y(t)| &\le |x_1(t) - y(t)| + \sum_{i=1}^{n-1} |x_{i+1}(t) - x_i(t)| \\ &\le \sup_{t\in I} |P_c(t) - P_{\tilde{c}}(t)| + M||q||_1 \\ &+ \sum_{i=1}^{n-1} (\sup_{t\in I} |P_c(t) - P_{\tilde{c}}(t)| + M||q||_1)(M||L||_1)^i \\ &= \frac{\sup_{t\in I} |P_c(t) - P_{\tilde{c}}(t)| + M||q||_1}{1 - M||L||_1}. \end{aligned} \tag{8}$$

On the other hand, from (4), (7) and (8) we obtain for almost all $t \in I$

$$|f_n(t) - y''(t)| \leq \sum_{i=1}^{n-1} |f_{i+1}(t) - f_i(t)| + |f_1(t) - y''(t)|$$
$$\leq L(t) \frac{\sup_{t\in I} |P_c(t) - P_{\tilde{c}}(t)| + M||q||_1}{1 - M||L||_1} + q(t).$$

Hence the sequence f_n is integrably bounded and therefore $f \in L^1(I, \mathbf{R})$.

With Lebesque's dominated convergence theorem we may take the limit in (5), (6) and we find that $x(.)$ is a solution of (1). Finally, passing to the limit in (8) we obtained the desired estimate on $x(.)$.

In order to finish the proof it remains to construct the sequences $x_n(.)$, $f_n(.)$ with the properties in (5)-(7). The construction will be done by reccurence.

Since the first step is already realized, assume that for some $N \geq 1$ we already constructed $x_n(.) \in C(I, \mathbf{R})$ and $f_n(.) \in L^1(I, \mathbf{R}), n = 1, 2, ...N$ satisfying (5),(7) for $n = 1, 2, ...N$ and (6) for $n = 1, 2, ...N-1$. The set-valued map $t \to F(t, x_N(t))$ is measurable. Moreover, the map $t \to L(t)|x_N(t) - x_{N-1}(t)|$ is measurable. By the lipschitzianity of $F(t, .)$ we have that for almost all $t \in I$

$$F(t, x_N(t)) \cap \{f_N(t) + L(t)|x_N(t) - x_{N-1}(t)|[-1, 1]\} \neq \emptyset.$$

From Lemma 2 there exists a measurable selection $f_{N+1}(.)$ of $F(., x_N(.))$ such that

$$|f_{N+1}(t) - f_N(t)| \leq L(t)|x_N(t) - x_{N-1}(t)| \quad \text{a.e. } (I).$$

We define $x_{N+1}(.)$ as in (5) with $n = N + 1$. Thus $f_{N+1}(.)$ satisfies (6) and (7) and the proof is complete. □

If in Theorem 1 we take $y(.) = 0$ and $q(.) = L(.)$ we obtain the following consequence of Theorem 1.

COROLLARY 3.3. *Assume that Hypothesis H1 is satisfied,* $M||L||_1 < 1$ *and* $d(0, F(t, 0)) \leq L(t)$ *a.e.* (I). *Then there exists* $x(.) : I \to \mathbf{R}$ *a solution of problem* (1)-(2) *satisfying for all* $t \in I$

$$|x(t)| \leq \frac{1}{1 - M||L||_1} \sup_{t\in I} |P_c(t)| + \frac{M}{1 - M||L||_1} ||L||_1. \tag{9}$$

REMARK 3.4. A similar result to the one in Corollary 1 may be found in [1], namely, Theorem 4; this result does not contain a priori bounds as in (9).

4. ARCWISE CONNECTEDNESS OF THE SOLUTION SET

In this section we are concerned with the more general problem

$$x''(t) \in F(t, x(t), H(t, x(t))) \quad \text{a.e. } ([0, 1]), \tag{10}$$

$$\int_0^1 x(s)\mathrm{d}s = \sum_{j=1}^m \gamma_j \int_{\xi_j}^{\eta_j} x(s)\mathrm{d}s + c_1,$$
$$(11) \qquad \int_0^1 x'(s)\mathrm{d}s = \sum_{j=1}^m \rho_j \int_{\xi_j}^{\eta_j} x'(s)\mathrm{d}s + c_2,$$

where $F : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ and $H : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$.

We assume that F and H are closed-valued Lipschitzian set-valued maps with respect to the second variable and F is contractive in the third variable. Obviously, the right-hand side of the differential inclusion in (10) is in general neither convex nor closed. We prove the arcwise connectedness of the solution set to (10)-(11). When F does not depend on the last variable (10) reduces to (1) and the result remains valid for problem (1)-(2).

Let Z be a metric space with the distance d_Z. In what follows, when the product $Z = Z_1 \times Z_2$ of metric spaces $Z_i, i = 1, 2$, is considered, it is assumed that Z is equipped with the distance $d_Z((z_1, z_2), (z_1', z_2')) = \sum_{i=1}^2 d_{Z_i}(z_i, z_i')$.

Let X be a nonempty set and let $F : X \to \mathcal{P}(Z)$ be a set-valued map with nonempty closed values. The range of F is the set $F(X) = \cup_{x\in X} F(x)$. The multifunction F is called Hausdorff continuous if for any $x_0 \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $x \in X, d_X(x, x_0) < \delta$ implies $D_Z(F(x), F(x_0)) < \epsilon$.

Let $(T, \mathcal{F}, \mu)$ be a finite, positive, nonatomic measure space and let $(X, |.|_X)$ be a Banach space. We recall that a set $A \in \mathcal{F}$ is called atom of μ if $\mu(A) \neq 0$ and for any $B \in \mathcal{F}$, $B \subset A$ one has $\mu(B) = 0$ or $\mu(B) = \mu(A)$. μ is called nonatomic measure if $\mathcal{F}$ does not contains atoms of μ. For example, Lebesgue's measure on a given interval in $\mathbf{R}^n$ is a nonatomic measure.

We denote by $L^1(T, X)$ the Banach space of all (equivalence classes of) Bochner integrable functions $u : T \to X$ endowed with the norm

$$|u|_{L^1(T,X)} = \int_T |u(t)|_X \mathrm{d}\mu$$

A nonempty set $K \subset L^1(T, X)$ is called decomposable if, for every $u, v \in K$ and every $A \in \mathcal{F}$, one has

$$\chi_A.u + \chi_{T\setminus A}.v \in K$$

where $\chi_B, B \in \mathcal{F}$ indicates the characteristic function of B.

Next we recall some preliminary results ([7]) that are the main tools in the proof of our result. To simplify the notation we write E in place of $L^1(T, X)$.

LEMMA 4.1. *Assume that $\phi : S \times E \to \mathcal{P}(E)$ and $\psi : S \times E \times E \to \mathcal{P}(E)$ are Hausdorff continuous set-valued maps with nonempty, closed, decomposable values, satisfying the following conditions*

a) *There exists $L \in [0, 1)$ such that, for every $s \in S$ and every $u, u' \in E$,*

$$D_E(\phi(s, u), \phi(s, u')) \leq L|u - u'|_E.$$

b) *There exists* $\mathcal{L} \in [0,1)$ *such that* $L+\mathcal{L} < 1$ *and for every* $s \in S$ *and every* $(u,v),(u',v') \in E \times E$,

$$D_E(\psi(s,u,v),\psi(s,u',v')) \leq \mathcal{L}(|u-u'|_E + |v-v'|_E).$$

Set $Fix(\Gamma(s,.)) = \{u \in E; u \in \Gamma(s,u)\}$, *where* $\Gamma(s,u) = \psi(s,u,\phi(s,u))$, $(s,u) \in S \times E$. *Then*

1) *For every* $s \in S$ *the set* $Fix(\Gamma(s,.))$ *is nonempty and arcwise connected.*

2) *For any* $s_i \in S$, *and any* $u_i \in Fix(\Gamma(s,.)), i = 1,\dots,p$ *there exists a continuous function* $\gamma : S \to E$ *such that* $\gamma(s) \in Fix(\Gamma(s,.))$ *for all* $s \in S$ *and* $\gamma(s_i) = u_i, i = 1,\dots,p$.

Lemma 4.2. *Let* $U : T \to \mathcal{P}(X)$ *and* $V : T \times X \to \mathcal{P}(X)$ *be two set-valued maps with nonempty closed values satisfying the following conditions*

a) *U is measurable and there exists* $r \in L^1(T)$ *such that* $D_X(U(t),\{0\}) \leq r(t)$ *for almost all* $t \in T$.

b) *The set-valued map* $t \to V(t,x)$ *is measurable for every* $x \in X$.

c) *The set-valued map* $x \to V(t,x)$ *is Hausdorff continuous for all* $t \in T$.

Let $v : T \to X$ *be a measurable selection from* $t \to V(t,U(t))$. *Then there exists a selection* $u \in L^1(T,X)$ *of* $U(.)$ *such that* $v(t) \in V(t,u(t))$, $t \in T$.

Hypothesis H2. Let $F : I \times \mathbf{R}^2 \to \mathcal{P}(\mathbf{R})$ and $H : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ be two set-valued maps with nonempty closed values, satisfying the following assumptions

i) The set-valued maps $t \to F(t,u,v)$ and $t \to H(t,u)$ are measurable for all $u,v \in \mathbf{R}$.

ii) There exists $l \in L^1(I,\mathbf{R}_+)$ such that, for every $u,u' \in \mathbf{R}$,

$$D(H(t,u),H(t,u')) \leq l(t)|u-u'| \quad a.e.\ (I).$$

iii) There exist $m \in L^1(I,\mathbf{R}_+)$ and $\theta \in [0,1)$ such that, for every u,v,u', $v' \in \mathbf{R}$,

$$D(F(t,u,v),F(t,u',v')) \leq m(t)|u-u'| + \theta|v-v'| \quad a.e.\ (I).$$

iv) There exist $f,g \in L^1(I,\mathbf{R}_+)$ such that

$$d(0,F(t,0,0)) \leq f(t), \quad d(0,H(t,0)) \leq g(t) \quad a.e.\ (I).$$

For $c = (c_1,c_2) \in \mathbf{R}^2$ we denote by $S(c)$ the solution set of (10)-(11).

In what follows $N(t) := \max\{l(t),m(t)\}$, $t \in I$.

Theorem 4.3. *Assume that* Hypothesis H2 *is satisfied and* $2M\int_0^T N(s)ds + \theta < 1$. *Then*

1) *For every* $c \in \mathbf{R}^2$, *the solution set* $S(c)$ *of (10)-(11) is nonempty and arcwise connected in the space* $C(I,\mathbf{R})$.

2) *For any* $c_i \in \mathbf{R}^2$ *and any* $u_i \in S(c_i)$, $i = 1,\dots,p$, *there exists a continuous function* $s : \mathbf{R}^2 \to C(I,\mathbf{R})$ *such that* $s(c) \in S(c)$ *for any* $c \in \mathbf{R}^2$ *and* $s(c_i) = u_i, i = 1,\dots,p$.

3) *The set* $S = \cup_{c\in\mathbf{R}^2} S(c)$ *is arcwise connected in* $C(I,\mathbf{R})$.

Proof. 1) For $c \in \mathbf{R}^2$ and $u \in L^1(I, \mathbf{R})$, we define

$$u_c(t) = P_c(t) + \int_0^1 G(t,s)u(s)\mathrm{d}s, \quad t \in I.$$

First, we prove that the set-valued maps $\phi : \mathbf{R}^2 \times L^1(I, \mathbf{R}) \to \mathcal{P}(L^1(I, \mathbf{R}))$ and $\psi : \mathbf{R}^2 \times L^1(I, \mathbf{R}) \times L^1(I, \mathbf{R}) \to \mathcal{P}(L^1(I, \mathbf{R}))$ given by

$$\phi(c,u) = \{v \in L^1(I, \mathbf{R}); \quad v(t) \in H(t, u_c(t)) \quad \text{a.e. } (I)\},$$

$$\psi(c,u,v) = \{w \in L^1(I, \mathbf{R}); \quad w(t) \in F(t, u_c(t), v(t)) \quad \text{a.e. } (I)\},$$

$c \in \mathbf{R}^2$, $u, v \in L^1(I, \mathbf{R})$ verify the assumptions in Lemma 3.

Since u_c is measurable and H satisfies Hypothesis H2 i) and ii), the set-valued $t \to H(t, u_c(t))$ is measurable and nonempty closed valued, thus it has a measurable selection. Hence taking into account Hypothesis H2 iv), the set $\phi(c,u)$ is nonempty. The fact that the set $\phi(c,u)$ is closed and decomposable follows by simple computation. Similarly, we get that $\psi(c,u,v)$ is a nonempty closed decomposable set.

Pick $(c,u), (\tilde{c}, u_1) \in \mathbf{R}^2 \times L^1(I, \mathbf{R})$ and choose $v \in \phi(c,u)$. For each $\varepsilon > 0$ there exists $v_1 \in \phi(\tilde{c}, u_1)$ such that, for every $t \in I$, one has

$$|v(t) - v_1(t)| \leq D(H(t, u_c(t)), H(t, u_{\tilde{c}}(t))) + \varepsilon \leq N(t)[|P_c(t) - P_{\tilde{c}}(t)| + \int_0^1 |G(t,s)|.|u(s) - u_1(s)|\mathrm{d}s] + \varepsilon.$$

Thus there exists $M_0 \geq 0$ such that

$$||v - v_1||_1 \leq M_0||c - \tilde{c}||. \int_0^1 N(t)\mathrm{d}t + M\int_0^1 N(t)\mathrm{d}t||u - u_1||_1 + \varepsilon$$

for any $\varepsilon > 0$.

This implies

$$d_{L^1(I,\mathbf{R})}(v, \phi(\tilde{c}, u_1)) \leq M_0||c - \tilde{c}||. \int_0^1 N(t)\mathrm{d}t + M\int_0^1 N(t)\mathrm{d}t||u - u_1||_1$$

for all $v \in \phi(c,u)$. Consequently,

$$D_{L^1(I,\mathbf{R})}(\phi(c,u), \phi(\tilde{c}, u_1)) \leq M_0||c - \tilde{c}||. \int_0^1 N(t)\mathrm{d}t + M\int_0^1 N(t)\mathrm{d}t||u - u_1||_1$$

which shows that ϕ is Hausdorff continuous and satisfies the assumptions of Lemma 3.

Pick $(c,u,v), (\tilde{c}, u_1, v_1) \in \mathbf{R}^2 \times L^1(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ and choose $w \in \psi(c,u,v)$. Then, as before, for each $\varepsilon > 0$ there exists $w_1 \in \psi(\tilde{c}, u_1, v_1)$ such that for every $t \in I$

$$|w(t) - w_1(t)| \leq D(F(t, u_c(t), v(t)), F(t, u_{\tilde{c}}(t), v_1(t))) + \varepsilon \leq N(t)|u_c(t) - u_{\tilde{c}}(t)| + \theta|v(t) - v_1(t)| + \varepsilon \leq N(t)[|P_c(t) - P_{\tilde{c}}(t)| + \int_0^1 |G(t,s)|.|u(s) - u_1(s)|\mathrm{d}s] + \theta|v(t) - v_1(t)| + \varepsilon \leq N(t)[M_0||c - \tilde{c}|| + M||u - u_1||_1] + \theta|v(t) - v_1(t)| + \varepsilon.$$

Hence

$$||w - w_1||_1 \leq M_0||c - \tilde{c}||.\int_0^1 N(t)\mathrm{d}t + M\int_0^1 N(t)\mathrm{d}t||u - u_1||_1$$

$$+ \theta||v - v_1||_1 + \varepsilon \leq M_0||c - \tilde{c}||.\int_0^1 N(t)\mathrm{d}t +$$

$$(M\int_0^1 N(t)\mathrm{d}t + \theta)d_{L^1(I,\mathbf{R})\times L^1(I,\mathbf{R})}((u,v),(u_1,v_1)) + \varepsilon.$$

As above, we deduce that

$$D_{L^1(I,\mathbf{R})}(\psi(c,u,v),\psi(\tilde{c},u_1,v_1)) \leq M_0||c - \tilde{c}||.\int_0^1 N(t)\mathrm{d}t$$

$$+ (M\int_0^1 N(t)\mathrm{d}t + \theta)d_{L^1(I,\mathbf{R})\times L^1(I,\mathbf{R})}((u,v),(u_1,v_1)),$$

namely, the set-valued map ψ is Hausdorff continuous and verifies the hypothesis of Lemma 3.

Define $\Gamma(c,u) = \psi(c,u,\phi(c,u)), (c,u) \in \mathbf{R}^2 \times L^1(I,\mathbf{R})$. With Lemma 3, the set $Fix(\Gamma(c,.)) = \{u \in L^1(I,\mathbf{R}); u \in \Gamma(c,u)\}$ is nonempty and arcwise connected in $L^1(I,\mathbf{R})$. Moreover, for fixed $c_i \in \mathbf{R}^2$ and $v_i \in Fix(\Gamma(c_i,.)), i = 1,\ldots,p$, there exists a continuous function $\gamma : \mathbf{R}^2 \to L^1(I,\mathbf{R})$ such that

$$\gamma(c) \in Fix(\Gamma(c,.)), \quad \forall c \in \mathbf{R}^2, \tag{12}$$

$$\gamma(c_i) = v_i, \quad i = 1,\ldots,p. \tag{13}$$

Next, we prove that

$$Fix(\Gamma(c,.)) = \{u \in L^1(I,\mathbf{R});\ u(t) \in F(t,u_c(t),H(t,u_c(t)))\ \text{a.e.}\ (I)\}. \tag{14}$$

Denote by $A(c)$ the right-hand side of (14). If $u \in Fix(\Gamma(c,.))$ then there is $v \in \phi(c,v)$ such that $u \in \psi(c,u,v)$. Therefore, $v(t) \in H(t,u_c(t))$ and

$$u(t) \in F(t,u_c(t),v(t)) \subset F(t,u_c(t),H(t,u_c(t))) \quad \text{a.e.}\ (I),$$

so that $Fix(\Gamma(c,.)) \subset A(c)$.

Let now $u \in A(c)$. By Lemma 4, there exists a selection $v \in L^1(I,\mathbf{R})$ of the set-valued map $t \to H(t,u_c(t)))$ satisfying

$$u(t) \in F(t,u_c(t),v(t)) \quad \text{a.e.}\ (I).$$

Hence, $v \in \phi(c,v)$, $u \in \psi(c,u,v)$ and thus $u \in \Gamma(c,u)$, which completes the proof of (14).

Finally, we note that the function $\mathcal{T} : L^1(I,\mathbf{R}) \to C(I,\mathbf{R})$,

$$\mathcal{T}(u)(t) := \int_0^1 G(t,s)u(s)\mathrm{d}s, \quad t \in I$$

is continuous and one has

$$S(c) = P_c(.) + \mathcal{T}(Fix(\Gamma(c,.))), \quad c \in \mathbf{R}^2. \tag{15}$$

Since $Fix(\Gamma(c,.))$ is nonempty and arcwise connected in $L^1(I,\mathbf{R})$, the set $S(c)$ has the same properties in $C(I,\mathbf{R})$.

2) Let $c_i \in \mathbf{R}^2$ and let $u_i \in S(c_i), i = 1,\dots,p$ be fixed. By (15) there exists $v_i \in Fix(\Gamma(c_i,.))$ such that

$$u_i = P_{c_i}(.) + \mathcal{T}(v_i), \quad i = 1,\dots,p.$$

If $\gamma : \mathbf{R}^2 \to L^1(I,\mathbf{R})$ is a continuous function satisfying (12) and (13) we define, for every $c \in \mathbf{R}$,

$$s(c) = P_c(.) + \mathcal{T}(\gamma(c)).$$

Obviously, the function $s : \mathbf{R}^2 \to C(I,\mathbf{R})$ is continuous, $s(c) \in S(c)$ for all $c \in \mathbf{R}^2$, and

$$s(c_i) = P_{c_i}(.) + \mathcal{T}(\gamma(c_i)) = P_{c_i}(.) + \mathcal{T}(v_i) = u_i, \quad i = 1,\dots,p.$$

3) Let $u_1, u_2 \in S = \cup_{c\in\mathbf{R}^2} S(c)$ and choose $\hat{c}, \tilde{c} \in \mathbf{R}^2$, such that $u_1 \in S(\hat{c})$ and $u_2 \in S(\tilde{c})$. From the conclusion of 2) we deduce the existence of a continuous function $s : \mathbf{R}^2 \to C(I,\mathbf{R})$ satisfying $s(\hat{c}) = u_1$, $s(\tilde{c}) = u_2$ and $s(c) \in S(c)$, $c \in \mathbf{R}^2$. Let $h : [0,1] \to \mathbf{R}$ be a continuous mapping such that $h(0) = \hat{c}$ and $h(1) = \tilde{c}$. Then the function $s \circ h : [0,1] \to C(I,\mathbf{R})$ is continuous and verifies

$$s \circ h(0) = u_1, \quad s \circ h(1) = u_2, \quad s \circ h(\tau) \in S(h(\tau)) \subset S, \quad \tau \in [0,1].$$

□

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University of Bucharest
Faculty of Mathematics and Computer Science
010014 Bucharest, Romania
E-mail: acernea@fmi.unibuc.ro
https://orcid.org/0000-0002-9174-9855