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DIFFERENTIAL IDENTITIES IN PRIME RINGS

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Abstract. Let $\mathcal{R}$ be a prime ring with center $Z(\mathcal{R})$ and $\alpha, \beta : \mathcal{R} \to \mathcal{R}$ be automorphisms. This paper is divided into two parts. The first tackles the notions of (generalized) skew derivations on $\mathcal{R}$, as the subject of the present study, several characterization theorems concerning commutativity of prime rings are obtained and an example proving the necessity of the primeness hypothesis of $\mathcal{R}$ is given. The second part of the paper tackles the notions of symmetric Jordan bi (α, β)-derivations. In addition, the researchers illustrated that for a prime ring with char$(\mathcal{R}) \neq 2$, every symmetric Jordan bi (α, α)-derivation D of $\mathcal{R}$ is a symmetric bi (α, α)-derivation.

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1. INTRODUCTION

It is worth mentioning that the notion of derivation was extended to the notion of skew derivation as follows: an additive map $D : \mathcal{R} \to \mathcal{R}$ is called skew derivation (or skew derivation associated with α) if $D(xy) = D(x)y + \alpha(x)D(y)$ for all $x, y \in \mathcal{R}$ where $\alpha : \mathcal{R} \to \mathcal{R}$ is an automorphism. Also, the concepts of derivation, generalized derivation and skew derivation were extended to the concept of a generalized skew derivation as follows: an additive map $F : \mathcal{R} \to$ is called a generalized skew derivation (or generalized skew derivation associated with D and α) if $F(xy) = F(x)y + \alpha(x)D(y)$ for all $x, y \in \mathcal{R}$, where D is a skew derivation and $\alpha : \mathcal{R} \to \mathcal{R}$ is an automorphism. There has been a continuous enthusiasm concerning the connection between the commutativity of a prime ring $\mathcal{R}$ and the behavior of a derivation or generalized derivation on $\mathcal{R}$ (see [1], [2], where further references can be found).

Herstein, as in [7], defined the Jordan derivation on associative rings and proved that for prime rings of characteristic different from 2, a Jordan derivation is an ordinary derivation. Also, 1988, Bresar and Vukman gave an alternative proof of the result in [1]. As in [5], Bresar and Vukman defined left derivation from a ring to a left $\mathcal{R}$ module Y and they showed the existence of a nonzero Jordan left derivation of $\mathcal{R}$ into Y implies $\mathcal{R}$ is commutative. The

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concept of a symmetric bi-derivation was presented and studied by Maksa as in [8].

In the following, $\mathcal{R}$ denotes an associative ring with center $Z(\mathcal{R})$. Additionally, we will write for all $x, y \in \mathcal{R}$, the symbol $[x, y]$ sign to the commutator $xy - yx$ and the symbol $x \circ y$ sign to the anti-commutator $xy + yx$. Our objective in this paper is to extend the result [3, Theorem 2.6] to generalized skew derivation, and also to generalize some results which exist in [6] in the case where D is a symmetric Jordan bi-(α, β)-derivation.

2. GENERALIZED SKEW DERIVATIONS IN PRIME RINGS

Let $\alpha : \mathcal{R} \to \mathcal{R}$ be an automorphism of $\mathcal{R}$. In this section, we suppose that d is a nonzero skew derivation associated with α and G is a generalized skew derivation associated with d and α. By abbreviation, we note that d is a nonzero skew derivation and G is a generalized skew derivation. Also we shall make use of the following identities without any specific mention:
(i) $[xy, t] = x[y, t] + [x, t]y$ for all $x, y, t \in \mathcal{R}$.
(ii) $[x, yt] = y[x, t] + [x, y]t$ for all $x, y, t \in \mathcal{R}$.

The following lemma is a generalization of a result of E. Posner for more details see [9, Lemma 3].

Lemma 2.1. *Let $\mathcal{R}$ be a prime ring. If $d : \mathcal{R} \to \mathcal{R}$ is a nonzero skew derivation of $\mathcal{R}$ such that $[d(x), x] = 0$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Proof. Suppose that

$$[d(x), x] = 0 \text{ for all } x \in \mathcal{R}. \tag{1}$$

Linearizing (1), we obtain

$$[d(x), y] + [d(y), x] = 0 \text{ for all } x, y \in \mathcal{R}. \tag{2}$$

Taking yx instead of y in (2), we get

$$[d(x), yx] + [d(y)x + \alpha(y)d(x), x] = 0 \text{ for all } x, y \in \mathcal{R}. \tag{3}$$

This can be rewritten as

$$[d(x), y]x + [d(y), x]x + [\alpha(y)d(x), x] = 0 \text{ for all } x, y \in \mathcal{R}. \tag{4}$$

Using (2), (4) becomes $[\alpha(y)d(x), x] = 0$ for all $x, y \in \mathcal{R}$ which implies that $\alpha(y)d(x)x = x\alpha(y)d(x)$ for all $x, y \in \mathcal{R}$. Replacing y by $\alpha^{-1}(t)y$ where $t \in \mathcal{R}$ in the last relation and use it to get $tx\alpha(y)d(x) = xt\alpha(y)d(x)$ for all $x, y, t \in \mathcal{R}$. So $[x, t]\alpha(y)d(x) = 0$ for all $x, y, t \in \mathcal{R}$. Since α is an automorphism of $\mathcal{R}$, we obtain $[x, t]\mathcal{R}d(x) = \{0\}$ for all $x, t \in \mathcal{R}$ and the primeness of $\mathcal{R}$ yields that $x \in Z(\mathcal{R})$ or $d(x) = 0$ for all $x \in \mathcal{R}$. Therefore, $\mathcal{R}$ is the union of its additive subgroups $Z(\mathcal{R})$ and $H = \{x \in \mathcal{R} \mid d(x) = 0\}$. But a group cannot be the union of two of its proper subgroups. Hence, either $R = Z(\mathcal{R})$ or $\mathcal{R} = H$. Since $d \neq 0$, we conclude that $\mathcal{R}$ is commutative. Hence the proof is complete. $\square$

THEOREM 2.2. *Let* $\mathcal{R}$ *be a prime ring with* $\text{char}(\mathcal{R}) \neq 2$. *If* G *is a generalized skew derivation on* $\mathcal{R}$, *then the following assertions are equivalent:*

(i) $G(x \circ y) \in Z(\mathcal{R})$ *for all* $x, y \in \mathcal{R}$,

(ii) $\mathcal{R}$ *is commutative.*

Proof. It is obvious that (ii) $\Rightarrow$ (i). Now we prove

(i) $\Rightarrow$ (ii) Suppose that $\mathcal{R}$ satisfies the following property

$$G(x \circ y) \in Z(\mathcal{R}) \text{ for all } x, y \in \mathcal{R}. \tag{5}$$

First if $Z(\mathcal{R}) = \{0\}$, so we get $G(x \circ y) = 0$ for all $x, y \in \mathcal{R}$. Replacing y by yx and using the fact that $(x \circ yx) = (x \circ y)x$ we get $0 = \alpha(x \circ y)d(x)$ for all $x, y \in \mathcal{R}$. But α is an automorphism of $\mathcal{R}$, so we have $-xy\alpha^{-1}(d(x)) = yx\alpha^{-1}(d(x))$ for all $x, y \in \mathcal{R}$. Again substituting y with ty in the last expression and use it to obtain $[x, t]y\alpha^{-1}(d(x)) = 0$ for all $x, y, t \in \mathcal{R}$ i.e. $[x, t]\mathcal{R}\alpha^{-1}(d(x)) = \{0\}$ for all $x, t \in \mathcal{R}$. Since $\mathcal{R}$ is prime, we can easily arrive at $x \in Z(\mathcal{R})$ or $d(x) = 0$ for all $x \in \mathcal{R}$. From the above we find that $\mathcal{R}$ is commutative.

Second if $Z(\mathcal{R}) \neq \{0\}$, then there exists an element $z \in Z(\mathcal{R}) - \{0\}$. Now from our hypotheses, we get $G(z^2 \circ x) \in Z(\mathcal{R})$, and by using with $\text{char}(\mathcal{R}) \neq 2$, we have

$$G(z^2)x + \alpha(z^2)d(x) \in Z(\mathcal{R}) \text{ for all } x \in \mathcal{R}. \tag{6}$$

Thus $[G(z^2)x + \alpha(z^2)d(x), x] = 0$ for all $x \in \mathcal{R}$, but $G(z^2) \in Z(\mathcal{R})$ from (5), so

$$G(z^2)[x, x] + \alpha(z^2)[d(x), x] = 0 \text{ for all } x \in \mathcal{R}. \tag{7}$$

Since $\alpha(z^2) \in Z(\mathcal{R})$, (7) becomes

$$\alpha(z^2)\mathcal{R}[d(x), x] = \{0\} \text{ for all } x \in \mathcal{R}. \tag{8}$$

The fact that α is an automorphism of $\mathcal{R}$ forces that $[d(x), x] = 0$ for all $x \in \mathcal{R}$, and $\mathcal{R}$ is commutative by Lemma 2.1. □

It is clear that if G is a generalized skew derivation on $\mathcal{R}$ associated with a nonzero skew derivation d, then $G \pm id_R$ is also a generalized skew derivation on $\mathcal{R}$ associated with d and α. In this case, when G is replaced by $G \pm id_R$, we obtain the following result:

THEOREM 2.3. *Let* $\mathcal{R}$ *be a prime ring with* $\text{char}(\mathcal{R}) \neq 2$. *If* G *is a generalized skew derivation on* $\mathcal{R}$, *then the following assertions are equivalent:*

(i) $G(x \circ y) - (x \circ y) \in Z(\mathcal{R})$ *for all* $x, y \in \mathcal{R}$.

(ii) $G(x \circ y) + (x \circ y) \in Z(\mathcal{R})$ *for all* $x, y \in \mathcal{R}$.

(iii) $\mathcal{R}$ *is commutative.*

Notice that if we put $\alpha = id_{\mathcal{R}}$ in the previous theorem we obtain:

COROLLARY 2.4 ([3, Theorem 2.6]). *Let* $\mathcal{R}$ *be a prime ring with* $\text{char}(\mathcal{R}) \neq 2$ *and* F *is a generalized derivation on* $\mathcal{R}$ *associated with a nonzero derivation* d. *Then the following assertions are equivalent:*

(i) $F(x \circ y) - (x \circ y) \in Z(\mathcal{R})$ *for all* $x, y \in \mathcal{R}$.
(ii) $F(x \circ y) + (x \circ y) \in Z(\mathcal{R})$ *for all* $x, y \in \mathcal{R}$.
(iii) $\mathcal{R}$ *is commutative.*

In the next example, we show that the condition "$\mathcal{R}$ is prime" is necessary in Theorem 2.2 and Theorem 2.3.

Example 2.5. Let us defined $\mathcal{R}$ and $G, d, \alpha : \mathcal{R} \to \mathcal{R}$ as follows:

$$\mathcal{R} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}, \; G \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$d \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \alpha = -id_{\mathcal{R}}.$$

It is clear that $\mathcal{R}$ is not prime with $\mathrm{char}(\mathcal{R}) \neq 2$. Moreover, d is a nonzero skew derivation of $\mathcal{R}$ and G is a generalized skew derivation of $\mathcal{R}$ associated with d and α such that

(i) $G(x \circ y) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$,
(ii) $G(x \circ y) - x \circ y \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$,
(iii) $G(x \circ y) + x \circ y \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$,

but $\mathcal{R}$ is not commutative.

3. SOME RESULTS INVOLVING SYMMETRIC JORDAN BI-(α, β)-DERIVATION

In this section, we investigated some properties of symmetric bi (α, β)-derivation and symmetric Jordan bi (α, β)-derivation for associative rings. We showed that for an associative prime ring with $\mathrm{char}(\mathcal{R}) \neq 2$ if D is a symmetric Jordan bi (α, α)-derivation, then D is symmetric bi (α, α)-derivation.

Definition 3.1. Let $\mathcal{R}$ be an associative ring, $\alpha, \beta : \mathcal{R} \to \mathcal{R}$ are automorphisms and $D : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ be a symmetric bi-additive mapping. If

$$D(xy, z) = \alpha(x)D(y, z) + D(x, z)\beta(y) \text{ for all } x, y, z \in \mathcal{R},$$

then D is called a symmetric bi-(α, β)-derivation.

Definition 3.2. Let $\mathcal{R}$ be an associative ring, $\alpha, \beta : \mathcal{R} \to \mathcal{R}$ are automorphisms and $D : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ be a symmetric bi-additive mapping. If

$$D(x^2, z) = \alpha(x)D(x, z) + D(x, z)\beta(x) \text{ for all } x, z \in \mathcal{R},$$

then D is called a symmetric Jordan bi-(α, β)-derivation.

Clearly, every symmetric bi-(α, β)-derivation is a symmetric Jordan bi-(α, β)-derivation. But the converse is not true in general, see the following example:

Example 3.3. Let S be a zero-square ring, a ring which verifies $a^2 = 0$ for all $a \in S$. It is obvious that every zero-square ring is anti-commutative, that is, $xy + yx = 0$ for all $x, y \in S$.
Let

$$\mathcal{R} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in S \right\}$$

and define the maps $D : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ and $\alpha : \mathcal{R} \to \mathcal{R}$ as follows:

$$D(\begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ax & ay + bx \\ 0 & 0 & ax \\ 0 & 0 & 0 \end{pmatrix}$$

$$\alpha \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & b \\ 0 & 0 & -a \\ 0 & 0 & 0 \end{pmatrix}, \quad \beta = \alpha.$$

It is easy to see that D is a symmetric Jordan bi-(α, β)-derivation which is not a symmetric bi-(α, β)-derivation.

Theorem 3.4. *Let $\mathcal{R}$ be a ring with* $\mathrm{char}(\mathcal{R}) \neq 2$, $\alpha, \beta : \mathcal{R} \to \mathcal{R}$ *are automorphisms and* $D : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ *be a symmetric Jordan bi-(α, β)-derivation. Then the following statements are hold for all* $a, x, y, z \in \mathcal{R}$

(i) $D(xy+yx, z) = \alpha(x)D(y, z)+\alpha(y)D(x, z)+D(x, z)\beta(y)+D(y, z)\beta(x)$,
(ii) $D(xyx, z) = \alpha(xy)D(x, z) + \alpha(x)D(y, z)\beta(x) + D(x, z)\beta(yx)$,
(iii) $D(xay + yax, z) = \alpha(xa)D(y, z) + \alpha(ya)D(x, z) + \alpha(x)D(a, z)\beta(y) + \alpha(y)D(a, z)\beta(x) + D(y, z)\beta(ax) + D(x, z)\beta(ay)$
(iv) $(D(ay, z) - \alpha(a)D(y, z) - D(a, z)\beta(y))(\beta(ay - ya)) = 0$

Proof. (i) Since $D(x^2, z) = \alpha(x)D(x, z) + D(x, z)\beta(x)$, we have

$$\begin{aligned} D((x + y)^2, z) &= \alpha(x + y)D(x + y, z) + D(x + y, z)\beta(x + y) \\ &= \alpha(x)D(x + y, z) + \alpha(y)D(x + y, z) \\ &+ D(x + y, z)\beta(x) + D(x + y, z)\beta(y) \\ &= \alpha(x)D(x, z) + \alpha(x)D(y, z) + \alpha(y)D(x, z) + \alpha(y)D(y, z) \\ &+ D(x, z)\beta(x) + D(y, z)\beta(x) + D(x, z)\beta(y) + D(y, z)\beta(y). \end{aligned}$$

On the other hand

$$\begin{aligned} D((x + y)^2, z) &= D(x^2 + xy + yx + y^2, z) \\ &= D(x^2, z) + D(xy + yx, z) + D(y^2, z) \\ &= \alpha(x)D(x, z) + D(x, z)\beta(x) + D(xy + yx, z) \\ &+ \alpha(y)D(y, z) + D(y, z)\beta(y). \end{aligned}$$

Comparing the above two expressions, we get the first equality.
(ii) Putting $xy + yx$ instead of y in (i), we have

$$D((x(xy + yx) + (xy + yx)x), z) = \alpha(x)D(xy + yx, z) + \alpha(xy + yx)D(x, z)$$

$$\begin{aligned}
&+ D(x,z)\beta(xy+yx) + D(xy+yx,z)\beta(x)\\
&= \alpha(x^2)D(y,z) + 2\alpha(xy)D(x,z)\\
&+ \alpha(x)D(x,z)\beta(y) + 2\alpha(x)D(y,z)\beta(x)\\
&+ \alpha(yx)D(x,z) + D(x,z)\beta(xy)\\
&+ \alpha(y)D(x,z)\beta(x) + 2D(x,z)\beta(yx) + D(y,z)\beta(x^2).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
D((x(xy+yx)+(xy+yx)x),z) &= D(x^2y+xyx+xyx+yx^2,z)\\
&= D(x^2y+yx^2,z) + 2D(xyx,z)\\
&= \alpha(x^2)D(y,z) + \beta(y)D(x^2,z)\\
&+ D(x^2,z)\beta(y) + D(y,z)\beta(x^2)\\
&= \alpha(x^2)D(y,z) + \beta(y)\alpha(x)D(x,z)\\
&+ \beta(y)D(x,z)\beta(x) + \alpha(x)D(x,z)\beta(y)\\
&+ D(x,z)\beta(xy) + D(y,z)\beta(x^2)\\
&+ 2D(xyx,z).
\end{aligned}$$

So by comparing the previous two expressions and using the 2-torsion freeness of $\mathcal{R}$, we obtain the (ii).

(iii) Putting $a+b$ instead of x in (ii), we get

$$\begin{aligned}
D(((a+b)y(a+b),z) &= \alpha((a+b)y)D(a+b,z) + \alpha(a+b)D(y,z)\beta(a+b)\\
&+ D(a+b,z)\beta(y(a+b))\\
&= \alpha(ay)D(a,z) + \alpha(by)D(a,z) + \alpha(ay)D(b,z)\\
&+ \alpha(by)D(b,z)) + \alpha(a)D(y,z)\beta(a)\\
&+ \alpha(a)D(y,z)\beta(b) + \alpha(b)D(y,z)\beta(a)\\
&+ \alpha(b)D(y,z)\beta(b) + D(a,z)\beta(ya)\\
&+ D(a,z)\beta(yb) + D(b,z)\beta(ya) + D(y,z)\beta(yb).
\end{aligned}$$

In another way,

$$\begin{aligned}
D(((a+b)y(a+b),z) &= D(aya+ayb+bay+byb,z)\\
&= D(aya,z) + D(ayb+bay,z) + D(byb,z)\\
&= \alpha(ay)D(a,z) + \alpha(a)D(y,z)\beta(a) + D(a,z)\beta(ya)\\
&+ D(ayb+bay,z) + \alpha(by)D(b,z) + \alpha(b)D(y,z)\beta(b)\\
&+ D(b,z)\beta(yb).
\end{aligned}$$

Now by comparing the two equations, we get (iii).

(iv) Let us take ay instead of b in (iii) and get:

$$\begin{aligned}
D(((ay)(ay)+ay^2a,z) &= \alpha(ay)D(ay,z) + \alpha(ay^2)D(a,z)\\
&+ \alpha(a)D(y,z)\beta(ay) + \alpha(ay)D(y,z)\beta(a)
\end{aligned}$$

$$+ \quad D(ay,z)\beta(ya) + D(a,z)\beta(yay).$$

But

$$\begin{aligned} D(((ay)(ay)+(ay^2a),z) &= D((ay)^2,z) + D(ay^2a,z),\\ &= \alpha(ay)D(ay,z) + D(ay,z)\beta(ay)\\ &+ \alpha(ay^2)D(a,z) + \alpha(a)D(y^2,z)\beta(b)\\ &+ D(a,z)\beta(y^2a). \end{aligned}$$

By comparing the previous two equations, we get

$$(D(ay,z) - \alpha(a)D(y,z) - D(a,z)\beta(y))(\beta(ay-ya)) = 0,$$

which finishes the proof. □

For accommodation of utilization in an associative ring $\mathcal{R}$ with $\mathrm{char}(\mathcal{R}) \neq 2$, we utilize the image for all $x, y, z \in \mathcal{R}$

$$x^y_{\alpha,\beta} := D(x.y,z) - \alpha(x)D(y,z) - D(x,z)\beta(y).$$

LEMMA 3.5. *Let $\mathcal{R}$ be a ring with* $\mathrm{char}(\mathcal{R}) \neq 2$, *$\alpha, \beta : \mathcal{R} \to \mathcal{R}$ are automorphisms and $D : \mathcal{R}\times\mathcal{R} \to \mathcal{R}$ be a symmetric Jordan bi-(α,β)-derivation. Then the following statements are hold for all $a, b, x, y, z \in \mathcal{R}$*

(i) $x^y_{\alpha,\beta} + y^x_{\alpha,\beta} = 0$,
(ii) $x^{a+b}_{\alpha,\beta} = x^a_{\alpha,\beta} + x^b_{\alpha,\beta}$,
(iii) $\alpha(yxz - xyz)x_{\alpha,\beta} + x^y_{\alpha,\beta}\beta(zyx - zxy) = 0$.

Proof. (i) Let $x, y, z \in \mathcal{R}$. We have

$$\begin{aligned} x^y_{\alpha,\beta} + y^x_{\alpha,\beta} &= D(xy,z) - \alpha(x)D(y,z) - D(x,z)\beta(y) + D(yx,z)\\ &- \alpha(y)D(x,z) - D(y,z)\beta(x)\\ &= D(xy+yx,z) - \alpha(x)D(y,z) - D(x,z)\beta(y)\\ &- \alpha(y)D(x,z) - D(y,z)\beta(x)\\ &= \alpha(x)D(y,z) + \alpha(y)D(x,z) + D(x,z)\beta(y) + D(y,z)\beta(x)\\ &- \alpha(x)D(y,z) - D(x,z)\beta(y) - \alpha(y)D(x,z) - D(y,z)\beta(x)\\ &= 0. \end{aligned}$$

(ii) Let $x, z, a, b \in \mathcal{R}$. We have

$$\begin{aligned} x^{a+b}_{\alpha,\beta} &= D(x(a+b),z) - \alpha(x)D(a+b,z) - D(x,z)\beta(a+b)\\ &= D(xa,z) + D(xb,z)\\ &- \alpha(x)D(a,z) - \alpha(x)D(b,z) - D(x,z)\beta(a) - D(x,z)\beta(b). \end{aligned}$$

Using the definition of $x^y_{\alpha,\beta}$, we can easily arrive at $x^{a+b}_{\alpha,\beta} = x^a_{\alpha,\beta} + x^b_{\alpha,\beta}$.

(iii) Putting $w := x(yzy)x + y(xzx)y$, $A := yzy$ and $B := xzx$ for every $x, y, z \in \mathcal{R}$, and using additivity and Theorem 3.4 (ii), we get

$$D(w,t) \quad = \quad D(x(yzy)x + y(xzx)y, t),$$

$$
\begin{aligned}
&= D(x(yzy)x,t) + D(y(xzx)y,t),\\
&= D(xAx,t) + D(yBy,t),\\
&= \alpha(xA)D(x,t) + \alpha(x)D(A,t)\beta(x) + D(x,t)\beta(Ax)\\
&+ \alpha(yB)D(y,t) + \alpha(y)D(B,t)\beta(y) + D(y,t)\beta(By),\\
&= \alpha(xyzy)D(x,t) + \alpha(xyz)D(y,t)\beta(x) + \alpha(xy)D(z,t)\beta(yx)\\
&+ \alpha(x)D(y,t)\beta(zyx) + D(x,t)\beta(yzyx) + \alpha(yxzx)D(y,t)\\
&= \alpha(yxz)D(x,t)\beta(y) + \alpha(yx)D(z,t)\beta(xy)\\
&+ \alpha(y)D(x,t)\beta(zxy) + D(y,t)\beta(xzxy).
\end{aligned}
$$

Conversely, since $w := (xy)z(yx) + (yx)z(xy)$, so by using Theorem 3.4 (iii), we get

$$
\begin{aligned}
D(w,t) &= D((xy)z(yx) + (yx)z(xy),t)\\
&= \alpha(xyz)D(yx,t) + \alpha(yxz)D(xy,t) + \alpha(xy)D(z,t)\beta(yx)\\
&+ \alpha(yx)D(z,t)\beta(xy) + D(yx,t)\beta(zxy) + D(xy,t)\beta(zyx).
\end{aligned}
$$

By comparing the previous two equations, we get

$$\alpha(yxz)x^y_{\alpha,\beta} + \alpha(xyz)y^x_{\alpha,\beta} + x^y_{\alpha,\beta}\beta(zyx) + y^x_{\alpha,\beta}\beta(zxy) = 0.$$

Now by using $x^y_{\alpha,\beta} = -y^x_{\alpha,\beta}$, we obtain

$$\alpha(yxz - xyz)x_{\alpha,\beta} + x^y_{\alpha,\beta}\beta(zyx - zxy) = 0.$$

Thus the proof of this lemma is completed. □

THEOREM 3.6. *Let $\mathcal{R}$ be an associative prime ring with* char$(\mathcal{R}) \neq 2$ *and $D : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ be a symmetric Jordan bi-(α,α)-derivation. Then D is a symmetric bi-(α,α)-derivation.*

Proof. Let x and y be fixed elements of $\mathcal{R}$. For the proof of this result, it is good to treat two cases $xy = yx$ and $xy \neq yx$.

For $xy = yx$, then $x, y \in Z(\mathcal{R})$. Using Theorem 3.1, we get

$$
\begin{aligned}
2D(xy,z) &= D(xy + yx, z)\\
&= \alpha(x)D(y,z) + \alpha(y)D(x,z) + D(x,z)\alpha(y) + D(y,z)\alpha(x)\\
&= \alpha(x)D(y,z) + D(x,z)\alpha(y) + D(x,z)\alpha(y) + \alpha(x)D(y,z)\\
&= 2\big(\alpha(x)D(y,z) + D(x,z)\alpha(y)\big).
\end{aligned}
$$

Since char$(\mathcal{R}) \neq 2$, it is obvious that $D(xy,z) = \alpha(x)D(y,z) + D(x,z)\alpha(y)$. So that D is symmetric bi-derivation.

If $xy \neq yx$, then using Lemma 1, we can easily arrive at $x^y_{\alpha,\alpha} = 0$, i.e., $D(xy,z) = \alpha(x)D(y,z) + D(x,z)\alpha(y)$. So that D is a bi-(α,α)-derivation. □

COROLLARY 3.7. *Let $\mathcal{R}$ be an associative prime ring with* char$(\mathcal{R}) \neq 2$ *and $D : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ be a symmetric Jordan bi-derivation. Then D is a symmetric bi-derivation.*

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