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A STUDY OF SOME FORMS OF CONTINUITY FOR MULTIFUNCTIONS IN IDEAL TOPOLOGICAL SPACES

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Abstract. The main goal of this article is to introduce the concepts of $\star(\alpha)$-continuous multifunctions and almost $\star(\alpha)$-continuous multifunctions. Some characterizations of $\star(\alpha)$-continuous multifunctions and almost $\star(\alpha)$-continuous multifunctions are established. Furthermore, the relationships between $\star(\alpha)$-continuity and almost $\star(\alpha)$-continuity are discussed.

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1. INTRODUCTION

Continuity in topological spaces, as significant and fundamental subject in the study of topology, has been researched by several mathematicians. In 1965, Njåstad [14] introduced a weak form of open sets called α-sets. Mashhour et al. [13] defined a function to be α-continuous if the inverse image of each open set is an α-set and obtained several characterizations of such functions. Noiri [16] investigated the relationships between α-continuous functions and several known functions, for example, almost continuous functions, η-continuous functions, δ-continuous functions or irresolute functions. In [17], the present author introduced the concept of almost α-continuity in topological spaces as a generalization of α-continuity and almost continuity. Neubrunn [15] introduced the notion of upper (resp. lower) α-continuous multifunctions. These multifunctions are further investigated by the present authors [19]. In 1996, Popa and Noiri [18] introduced the notion of upper (resp. lower) almost α-continuous multifunctions and investigated several characterizations and some basic properties concerning upper (resp. lower) almost α-continuous multifunctions. Topological ideals have played an important role in topology. Kuratowski [12] and Vaidyanathswamy [20] introduced and studied the concept of ideals in topological spaces. Every topological space is an ideal topological space and all the results of ideal topological spaces are generalizations of the results established in topological spaces. Janković and Hamlett [11] introduced

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the concept of $\mathscr{I}$-open sets in ideal topological spaces. Abd El-Monsef et al. [1] further investigated $\mathscr{I}$-open sets and $\mathscr{I}$-continuous functions. Later, several authors studied ideal topological spaces giving several convenient definitions. Some authors obtained decompositions of continuity. For instance, Açikgöz et al. [3] studied the concepts of α-$\mathscr{I}$-continuity and α-$\mathscr{I}$-openness in ideal topological spaces and obtained several characterizations of these functions. Hatir and Noiri [10] introduced the notions of semi-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets and β-$\mathscr{I}$-open sets via idealization and using these sets obtained new decompositions of continuity. In [2], the present authors introduced and investigated the notions of weakly-$\mathscr{I}$-continuous and weak*-$\mathscr{I}$-continuous functions in ideal topological spaces. In 2005, Hatir and Noiri [9] investigated further properties of semi-$\mathscr{I}$-open sets and semi-$\mathscr{I}$-continuity. Moreover, the present authors [8] introduced and investigated the notions of strong β-$\mathscr{I}$-open sets and strongly β-$\mathscr{I}$-continuous functions. The purpose of the present article is to introduce the notions of $\star(\alpha)$-continuous multifunctions and almost $\star(\alpha)$-continuous multifunctions. Moreover, several interesting characterizations of $\star(\alpha)$-continuous multifunctions and almost $\star(\alpha)$-continuous multifunctions are investigated. In particular, the relationships between $\star(\alpha)$-continuity and almost $\star(\alpha)$-continuity are discussed.

2. PRELIMINARIES

Throughout the present paper, spaces (X,τ) and (Y,σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. In a topological space (X,τ), the closure and the interior of any subset A of X will denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. An ideal $\mathscr{I}$ on a topological space (X,τ) is a non-empty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X,τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X,\tau,\mathscr{I})$. For an ideal topological space $(X,\tau,\mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows: $A^\star(\mathscr{I}) = \{x \in X \mid U \cap A \notin \mathscr{I}$ for every open neighbourhood U of $x\}$. In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [12], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ. Observe additionally that $\mathrm{Cl}^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ, generated by the base $\mathscr{B}(\mathscr{I},\tau) = \{U - I_0 \mid U \in \tau$ and $I_0 \in \mathscr{I}\}$. However, $\mathscr{B}(\mathscr{I},\tau)$ is not always a topology [20]. A subset A is said to be $\star$-closed [11] if $A^\star \subseteq A$. The interior of a subset A in $(X,\tau^\star(\mathscr{I}))$ is denoted by $\mathrm{Int}^\star(A)$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [4], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^{-}(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x\in A}F(x)$. Then F is said to be surjection if $F(X) = Y$, or equivalent, if for each $y \in Y$ there exists $x \in X$ such that $y \in F(x)$ and F is called injection if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$. Let $\mathscr{P}(X)$ be the collection of all non-empty subsets of X. For any $\star$-open set V of an ideal topological space $(X, \tau, \mathscr{I})$, we denote $V^{+} = \{A \in \mathscr{P}(X) \mid A \subseteq V\}$ and $V^{-} = \{A \in \mathscr{P}(X) \mid A \cap V \neq \emptyset\}$.

Definition 2.1 ([7]). A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be:

(i) R-$\mathscr{I}^{\star}$-open if $A = \mathrm{Int}^{\star}(\mathrm{Cl}^{\star}(A))$;
(ii) R-$\mathscr{I}^{\star}$-closed if its complement is R-$\mathscr{I}^{\star}$-open;
(iii) $\mathscr{I}^{\star}$-preopen if $A \subseteq \mathrm{Int}^{\star}(\mathrm{Cl}^{\star}(A))$;
(iv) $\mathscr{I}^{\star}$-preclosed if its complement is $\mathscr{I}^{\star}$-preopen.

Definition 2.2 ([6]). A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be:

(i) semi-$\mathscr{I}^{\star}$-open if $A \subseteq \mathrm{Cl}^{\star}(\mathrm{Int}^{\star}(A))$;
(ii) semi-$\mathscr{I}^{\star}$-closed if its complement is semi-$\mathscr{I}^{\star}$-open;
(iii) semi-$\mathscr{I}^{\star}$-preopen if $A \subseteq \mathrm{Cl}^{\star}(\mathrm{Int}^{\star}(\mathrm{Cl}^{\star}(A)))$;
(iv) semi-$\mathscr{I}^{\star}$-preclosed if its complement is semi-$\mathscr{I}^{\star}$- preopen.

For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all semi-$\mathscr{I}^{\star}$-closed sets containing A is called the *semi-$\mathscr{I}^{\star}$-closure* [6] of A and is denoted by $s\mathrm{Cl}_{\mathscr{I}^{\star}}(A)$. The union of all semi-$\mathscr{I}^{\star}$-open sets contained in A is called the *semi-$\mathscr{I}^{\star}$-interior* [6] of A and is denoted by $s\mathrm{Int}_{\mathscr{I}^{\star}}(A)$.

Proposition 2.3. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A, B \subseteq X$. If A is $\star$-open and B is semi-$\mathscr{I}^{\star}$-open, then $A \cap B$ is semi-$\mathscr{I}^{\star}$-open.*

Lemma 2.4 ([6]). *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, $x \in s\mathrm{Cl}_{\mathscr{I}^{\star}}(A)$ if and only if $U \cap A \neq \emptyset$ for every semi-$\mathscr{I}^{\star}$-open set U containing x.*

Lemma 2.5 ([6]). *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) $s\mathrm{Cl}_{\mathscr{I}^{\star}}(A) = A \cup \mathrm{Int}^{\star}(\mathrm{Cl}^{\star}(A))$.
(2) $s\mathrm{Int}_{\mathscr{I}^{\star}}(A) = A \cap \mathrm{Cl}^{\star}(\mathrm{Int}^{\star}(A))$.

Proposition 2.6. *Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. If A is $\star$-open, then $s\mathrm{Cl}_{\mathscr{I}^{\star}}(A) = \mathrm{Int}^{\star}(\mathrm{Cl}^{\star}(A))$.*

Definition 2.7. A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be:

(i) α-$\mathscr{I}^{\star}$-open if $A \subseteq \mathrm{Int}^{\star}(\mathrm{Cl}^{\star}(\mathrm{Int}^{\star}(A)))$;
(ii) α-$\mathscr{I}^{\star}$-closed if its complement is α-$\mathscr{I}^{\star}$-open.

PROPOSITION 2.8. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties are equivalent:*

(1) *A is α-$\mathscr{I}^\star$-open in X.*
(2) *$G \subseteq A \subseteq \mathrm{Int}^\star(\mathrm{Cl}^\star(G))$ for some $\star$-open set G.*
(3) *$G \subseteq A \subseteq s\mathrm{Cl}_{\mathscr{I}^\star}(G)$ for some $\star$-open set G.*
(4) *$A \subseteq s\mathrm{Cl}_{\mathscr{I}^\star}(\mathrm{Int}^\star(A))$.*

For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all α-$\mathscr{I}^\star$-closed sets containing A is called the *α-$\mathscr{I}^\star$-closure* of A and is denoted by $\alpha\mathrm{Cl}_{\mathscr{I}^\star}(A)$. The *$\alpha$-$\mathscr{I}^\star$-interior* of A is defined by the union of all α-$\mathscr{I}^\star$-open sets contained in A and is denoted by $\alpha\mathrm{Int}_{\mathscr{I}^\star}(A)$.

PROPOSITION 2.9. *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) *A is α-$\mathscr{I}^\star$-closed in X if and only if $s\mathrm{Int}_{\mathscr{I}^\star}(\mathrm{Cl}^\star(A)) \subseteq A$.*
(2) *$s\mathrm{Int}_{\mathscr{I}^\star}(\mathrm{Cl}^\star(A)) = \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(A)))$.*
(3) *$\alpha\mathrm{Cl}_{\mathscr{I}^\star}(A) = A \cup \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(A)))$.*
(4) *$\alpha\mathrm{Int}_{\mathscr{I}^\star}(A) = A \cap \mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(A)))$.*

3. SOME CHARACTERIZATIONS

In this section, we introduce the notions of $\star(\alpha)$-continuous multifunctions and almost $\star(\alpha)$-continuous multifunctions. Some characterizations of $\star(\alpha)$-continuous multifunctions and almost $\star(\alpha)$-continuous multifunctions are investigated. Moreover, the relationships between $\star(\alpha)$-continuity and almost $\star(\alpha)$-continuity are discussed.

DEFINITION 3.1. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be *$\star(\alpha)$-continuous* at $x \in X$ if for every $\star$-open sets V_1 and V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists an α-$\mathscr{I}^\star$-open set U of X containing x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be *$\star(\alpha)$-continuous* if F has this property at each point of X.

THEOREM 3.2. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is $\star(\alpha)$-continuous at $x \in X$;*
(2) *for every $\star$-open sets V_1, V_2 of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$, $x \in s\mathrm{Cl}_{\mathscr{I}^\star}(\mathrm{Int}^\star(F^+(V_1) \cap F^-(V_2)))$;*
(3) *for every $\star$-open sets V_1, V_2 of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$ and every semi-$\mathscr{I}^\star$-open set U of X containing x, there exists a nonempty $\star$-open set G of X such that $G \subseteq U$, $F(G) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in G$.*

Proof. (1) $\Rightarrow$ (2) Let V_1, V_2 be any $\star$-open sets of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$. By (1), there exists an α-$\mathscr{I}^\star$-open set U of X containing x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$. Therefore, we have $x \in U \subseteq F^{+}(V_1) \cap F^{-}(V_2)$. Since U is α-$\mathscr{I}^{\star}$-open, by Proposition 2.8, we obtain $x \in U \subseteq s\mathrm{Cl}_{\mathscr{I}^{\star}}(\mathrm{Int}^{\star}(U)) \subseteq s\mathrm{Cl}_{\mathscr{I}^{\star}}(\mathrm{Int}^{\star}(F^{+}(V_1) \cap F^{-}(V_2)))$.

(2) $\Rightarrow$ (3) Let V_1, V_2 be any $\star$-open sets of Y such that $F(x) \subseteq V_1$ and

$$F(x) \cap V_2 \neq \emptyset.$$

By (2), we have $x \in s\mathrm{Cl}_{\mathscr{I}^{\star}}(\mathrm{Int}^{\star}(F^{+}(V_1) \cap F^{-}(V_2)))$. Let U be any semi-$\mathscr{I}^{\star}$-open set of X containing x. By Lemma 2.4, $U \cap \mathrm{Int}^{\star}(F^{+}(V_1) \cap F^{-}(V_2)) \neq \emptyset$ and Proposition 2.3, we have $U \cap \mathrm{Int}^{\star}(F^{+}(V_1) \cap F^{-}(V_2))$ is semi-$\mathscr{I}^{\star}$-open. Put $G = \mathrm{Int}^{\star}(U \cap \mathrm{Int}^{\star}(F^{+}(V_1) \cap F^{-}(V_2)))$. Then G is a non-empty $\star$-open set of X such that $G \subseteq U$, $G \subseteq F^{+}(V_1)$ and $G \subseteq F^{-}(V_2)$. Consequently, we obtain $F(G) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in G$.

(3) $\Rightarrow$ (1) Let $S_{\mathscr{I}^{\star}}O(X, x)$ be the family of all semi-$\mathscr{I}^{\star}$-open sets of X containing x. Let V_1, V_2 be any $\star$-open sets of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$. For each $U \in S_{\mathscr{I}^{\star}}O(X, x)$, there exists a non-empty $\star$-open set G_U such that $G_U \subseteq U$, $F(G_U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in G_U$. Let $W = \cup\{G_U \mid U \in S_{\mathscr{I}^{\star}}O(X, x)\}$. Then W is $\star$-open in X, $x \in s\mathrm{Cl}_{\mathscr{I}^{\star}}(W)$, $F(W) \subseteq V_1$ and $F(w) \cap V_2 \neq \emptyset$ for every $w \in W$. Put $H = W \cup \{x\}$, then $W \subseteq H \subseteq s\mathrm{Cl}_{\mathscr{I}^{\star}}(W) = \mathrm{Int}^{\star}(\mathrm{Cl}^{\star}(W))$. Therefore, H is an α-$\mathscr{I}^{\star}$-open set containing x, $F(H) \subseteq V_1$ and $F(h) \cap V_2 \neq \emptyset$ for every $h \in H$. This shows that F is $\star(\alpha)$-continuous at x. $\square$

DEFINITION 3.3. A subset N of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be a *$\star$-neighbourhood* [5] (resp. *$\star(\alpha)$-neighbourhood*) of $x \in X$ if there exists a $\star$-open (resp. α-$\mathscr{I}^{\star}$-open) set V of X such that $x \in V \subseteq N$.

DEFINITION 3.4. Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. A subset B of X is called a *$\star$-neighbourhood which intersects A* if there exists a $\star$-open set V of X such that $V \subseteq B$ and $V \cap A \neq \emptyset$.

THEOREM 3.5. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is $\star(\alpha)$-continuous;*
(2) *$F^{+}(V_1) \cap F^{-}(V_2)$ is α-$\mathscr{I}^{\star}$-open in X for every $\star$-open sets V_1, V_2 of Y;*
(3) *$F^{-}(K_1) \cup F^{+}(K_2)$ is α-$\mathscr{I}^{\star}$-closed in X for every $\star$-closed sets K_1, K_2 of Y;*
(4) *$s\mathrm{Int}_{\mathscr{I}^{\star}}(\mathrm{Cl}^{\star}(F^{-}(B_1) \cup F^{+}(B_2))) \subseteq F^{-}(\mathrm{Cl}^{\star}(B_1)) \cup F^{+}(\mathrm{Cl}^{\star}(B_2))$ for every subsets B_1, B_2 of Y;*
(5) *$\alpha\mathrm{Cl}_{\mathscr{I}^{\star}}(F^{-}(B_1) \cup F^{+}(B_2)) \subseteq F^{-}(\mathrm{Cl}^{\star}(B_1)) \cup F^{+}(\mathrm{Cl}^{\star}(B_2))$ for every subsets B_1, B_2 of Y;*
(6) *for each $x \in X$, for each $\star$-neighbourhood V_1 of $F(x)$ and for each $\star$-neighbourhood V_2 which intersects $F(x)$, $F^{+}(V_1) \cap F^{-}(V_2)$ is a $\star(\alpha)$-neighbourhood of x;*
(7) *for each $x \in X$, for each $\star$-neighbourhood V_1 of $F(x)$ and for each $\star$-neighbourhood V_2 which intersects $F(x)$, there exists a $\star(\alpha)$-neighbourhood U of x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$.*

Proof. (1) ⇒ (2) Let V_1, V_2 be any $\star$-open sets of Y and

$$x \in F^+(V_1) \cap F^-(V_2).$$

Then $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$. By Theorem 3.2,

$$x \in s\mathrm{Cl}_{\mathscr{I}^\star}(\mathrm{Int}^\star(F^+(V_1) \cap F^-(V_2)))$$

and so $F^+(V_1) \cap F^-(V_2) \subseteq s\mathrm{Cl}_{\mathscr{I}^\star}(\mathrm{Int}^\star(F^+(V_1) \cap F^-(V_2)))$. It follows from Proposition 2.8 that $F^+(V_1) \cap F^-(V_2)$ is α-$\mathscr{I}^\star$-open in X.

(2) ⇔ (3) This follows from the relations: $F^-(Y - B) = X - F^+(B)$ and $F^+(Y - B) = X - F^-(B)$ for any subset B of Y.

(3) ⇒ (4) Let B_1, B_2 be any subsets of Y. By (3), we have

$$F^-(\mathrm{Cl}^\star(B_1)) \cup F^+(\mathrm{Cl}^\star(B_2))$$

is α-$\mathscr{I}^\star$-closed in X and by Proposition 2.9,

$$\begin{aligned} s\mathrm{Int}_{\mathscr{I}^\star}(\mathrm{Cl}^\star(F^-(B_1) \cup F^+(B_2))) &\subseteq s\mathrm{Int}_{\mathscr{I}^\star}(\mathrm{Cl}^\star(F^-(\mathrm{Cl}^\star(B_1)) \cup F^+(\mathrm{Cl}^\star(B_2)))) \\ &\subseteq F^-(\mathrm{Cl}^\star(B_1)) \cup F^+(\mathrm{Cl}^\star(B_2)). \end{aligned}$$

(4) ⇒ (5) Let B_1, B_2 be any subsets of Y. By (4) and Proposition 2.9,

$$\begin{aligned} &\alpha\mathrm{Cl}_{\mathscr{I}^\star}(F^-(B_1) \cup F^+(B_2)) \\ &= (F^-(B_1) \cup F^+(B_2)) \cup s\mathrm{Int}_{\mathscr{I}^\star}(\mathrm{Cl}^\star(F^-(B_1) \cup F^+(B_2))) \\ &\subseteq (F^-(B_1) \cup F^+(B_2)) \cup (F^-(\mathrm{Cl}^\star(B_1) \cup F^+(\mathrm{Cl}^\star(B_2)))) \\ &= F^-(\mathrm{Cl}^\star(B_1) \cup F^+(\mathrm{Cl}^\star(B_2))). \end{aligned}$$

(5) ⇒ (3) Let K_1, K_2 be any $\star$-closed sets of Y. By (5), we have

$$\begin{aligned} \alpha\mathrm{Cl}_{\mathscr{I}^\star}(F^-(K_1) \cup F^+(K_2)) &\subseteq F^-(\mathrm{Cl}^\star(K_1)) \cup F^+(\mathrm{Cl}^\star(K_2)) \\ &= F^-(K_1) \cup F^+(K_2) \end{aligned}$$

and hence $F^-(K_1) \cup F^+(K_2)$ is α-$\mathscr{I}^\star$-closed in X.

(2) ⇒ (6) Let $x \in X$, V_1 be a $\star$-neighbourhood of $F(x)$ and V_2 a $\star$-neighbourhood which intersects $F(x)$. There exist $\star$-open sets U_1, U_2 of X such that $F(x) \subseteq U_1 \subseteq V_1$, $U_2 \subseteq V_2$ and $F(x) \cap U_2 \neq \emptyset$. Therefore, we have $x \in F^+(U_1) \cap F^-(U_2) \subseteq F^+(V_1) \cap F^-(V_2)$. Since $F^+(U_1) \cap F^-(U_2)$ is α-$\mathscr{I}^\star$-open in X, $F^+(V_1) \cap F^-(V_2)$ is a $\star(\alpha)$-neighbourhood of x.

(6) ⇒ (7) Let $x \in X$, V_1 be a $\star$-neighbourhood of $F(x)$ and V_2 a $\star$-neighbourhood which intersects $F(x)$. Put $U = F^+(V_1) \cap F^-(V_2)$, then U is a $\star(\alpha)$-neighbourhood of x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$.

(7) ⇒ (1) Let $x \in X$ and V_1, V_2 be any $\star$-open sets of Y such that $F(x) \subseteq V_1$ and $F(x) \cap V_2 \neq \emptyset$. Then V_1 is a $\star$-neighbourhood of $F(x)$ and V_2 is a $\star$-neighbourhood which intersects $F(x)$. There exists a $\star(\alpha)$-neighbourhood U of x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$. Therefore, there exists an α-$\mathscr{I}^\star$-open set W of X such that $x \in W \subseteq U$; hence $F(W) \subseteq V_1$ and $F(w) \cap V_2 \neq \emptyset$ for every $w \in W$. This shows that F is $\star(\alpha)$-continuous. □

Definition 3.6. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be $\star(\alpha)$-*continuous* if $f^{-1}(V)$ is α-$\mathscr{I}^\star$-open in X for every $\star$-open set V of Y.

Corollary 3.7. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is $\star(\alpha)$-continuous;*
(2) *$f^{-1}(V)$ is α-$\mathscr{I}^\star$-open in X for every $\star$-open set V of Y;*
(3) *$s\mathrm{Int}_{\mathscr{I}^\star}(\mathrm{Cl}^\star(f^{-1}(B))) \subseteq f^{-1}(\mathrm{Cl}^\star(B))$ for every subset B of Y;*
(4) *$\alpha\mathrm{Cl}_{\mathscr{I}^\star}(f^{-1}(B)) \subseteq f^{-1}(\mathrm{Cl}^\star(B))$ for every subset B of Y;*
(5) *for each $x \in X$, for each $\star$-neighbourhood V of $f(x)$, $f^{-1}(V)$ is a $\star(\alpha)$-neighbourhood of x;*
(6) *for each $x \in X$, for each $\star$-neighbourhood V of $f(x)$, there exists a $\star(\alpha)$-neighbourhood U of x such that $f(U) \subseteq V$.*

Definition 3.8. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be *almost $\star(\alpha)$-continuous* at a point x of X if for every $\star$-open sets V_1 and V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$ and each semi-$\mathscr{I}^\star$-open set U of X containing x, there exists a non-empty $\star$-open set G of X such that $G \subseteq U$, $F(G) \subseteq s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)$ and $F(z) \cap s\mathrm{Cl}_{\mathscr{J}^\star}(V_2) \neq \emptyset$ for every $z \in G$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be *almost $\star(\alpha)$-continuous* if F has this property at each point of X.

Theorem 3.9. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is almost $\star(\alpha)$-continuous at $x \in X$;*
(2) *for every $\star$-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists an α-$\mathscr{I}^\star$-open set U of X containing x such that $F(U) \subseteq s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)$ and $F(z) \cap s\mathrm{Cl}_{\mathscr{J}^\star}(V_2) \neq \emptyset$ for every $z \in U$;*
(3) *$x \in \alpha\mathrm{Int}_{\mathscr{I}^\star}[F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))]$ for every $\star$-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$;*
(4) *$x \in \mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star[F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))]))$ for every $\star$-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$.*

Proof. (1) $\Rightarrow$ (2) Let $S^\star_{\mathscr{I}}O(X, x)$ be the family of all semi-$\mathscr{I}^\star$-open sets of X containing x. Let V_1, V_2 be any $\star$-open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. For each $H \in S_{\mathscr{I}^\star}O(X, x)$, there exists a non-empty $\star$-open set $G_H \subseteq H$ such that $F(G_H) \subseteq s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)$ and $F(z) \cap s\mathrm{Cl}_{\mathscr{J}^\star}(V_2) \neq \emptyset$ for every $z \in G_H$. Let $W = \cup\{G_H \mid H \in S_{\mathscr{I}^\star}O(X, x)\}$. Then W is $\star$-open in X, $x \in s\mathrm{Cl}_{\mathscr{I}^\star}(W)$, $F(W) \subseteq s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)$ and $F(w) \cap s\mathrm{Cl}_{\mathscr{J}^\star}(V_2) \neq \emptyset$ for every $w \in W$. Put $U = W \cup \{x\}$, then $W \subseteq U \subseteq s\mathrm{Cl}_{\mathscr{I}^\star}(W) = \mathrm{Int}^\star(\mathrm{Cl}^\star(W))$. Therefore, we obtain U is an α-$\mathscr{I}^\star$-open set of X containing x such that $F(U) \subseteq s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)$ and $F(z) \cap s\mathrm{Cl}_{\mathscr{J}^\star}(V_2) \neq \emptyset$ for every $z \in U$.

(2) $\Rightarrow$ (3) Let V_1, V_2 be any $\star$-open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. Then, there exists an α-$\mathscr{I}^\star$-open set U of X containing x such that $F(U) \subseteq$

$s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)$ and $F(z) \cap s\mathrm{Cl}_{\mathscr{J}^\star}(V_2) \neq \emptyset$ for every $z \in U$. Thus,

$$x \in U \subseteq F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2)).$$

Since U is α-$\mathscr{I}^\star$-open, $x \in U \subseteq \alpha\mathrm{Int}_{\mathscr{I}^\star}(F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2)))$.

(3) $\Rightarrow$ (4) Let V_1, V_2 be any $\star$-open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. Now put $U = \alpha\mathrm{Int}_{\mathscr{I}^\star}(F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2)))$. Then U is α-$\mathscr{I}^\star$-open in X and $x \in U \subseteq F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))$. Thus, $x \in U \subseteq \mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(U))) \subseteq \mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2)))))$.

(4) $\Rightarrow$ (1) Let $U \in S_{\mathscr{I}^\star}O(X,x)$ and V_1, V_2 be any $\star$-open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. Then, we have

$$\begin{aligned} x &\in \mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))))) \\ &= s\mathrm{Cl}_{\mathscr{I}^\star}(\mathrm{Int}^\star(F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2)))). \end{aligned}$$

It follows from Proposition 2.3 and Lemma 2.4 that

$$\emptyset \neq U \cap \mathrm{Int}^\star(F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))) \in S_{\mathscr{I}^\star}O(X,x).$$

Put $G = \mathrm{Int}^\star(U \cap \mathrm{Int}^\star(F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))))$, then G is a nonempty $\star$-open set of X such that $G \subseteq U$, $F(G) \subseteq s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)$ and

$$F(z) \cap s\mathrm{Cl}_{\mathscr{J}^\star}(V_2) \neq \emptyset$$

for every $z \in G$. This shows that F is almost $\star(\alpha)$-continuous at x. $\square$

Remark 3.10. For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following implication holds:

$$\star(\alpha)\text{-continuity} \Rightarrow \text{almost } \star(\alpha)\text{-continuity}.$$

The converse of the implication is not true in general. We give example for the implication as follows.

Example 3.11. Let $X = \{1, 2, 3\}$ with a topology $\tau = \{\emptyset, \{1\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset, \{2\}\}$. Let $Y = \{a, b, c\}$ with a topology $\sigma = \{\emptyset, \{a\}, Y\}$ and an ideal $\mathscr{J} = \{\emptyset\}$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is defined as follows: $F(1) = \{c\}$ and $F(2) = F(3) = \{a, b\}$. Then F is almost $\star(\alpha)$-continuous but F is not $\star(\alpha)$-continuous.

Theorem 3.12. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is almost $\star(\alpha)$-continuous;*
(2) *for each $x \in X$ and every $\star$-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists an α-$\mathscr{I}^\star$-open set U of X containing x such that $F(U) \subseteq s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)$ and $F(z) \cap s\mathrm{Cl}_{\mathscr{J}^\star}(V_2) \neq \emptyset$ for every $z \in U$;*
(3) *for each $x \in X$ and every R-$\mathscr{J}^\star$-open sets V_1, V_2 of Y such that $F(x) \in V_1^+ \cap V_2^-$, there exists an α-$\mathscr{I}^\star$-open set U of X containing x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$;*
(4) *$F^+(V_1) \cap F^-(V_2)$ is α-$\mathscr{I}^\star$-open in X for every R-$\mathscr{J}^\star$-open sets V_1, V_2 of Y;*

(5) $F^+(K_1) \cup F^-(K_2)$ *is* α-$\mathscr{I}^\star$-*closed in* X *for every* R-$\mathscr{J}^\star$-*closed sets* K_1, K_2 *of* Y*;*
(6) $F^+(V_1) \cup F^-(V_2) \subseteq \alpha\mathrm{Int}_{\mathscr{I}^\star}(F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2)))$ *for every* $\star$-*open sets* V_1, V_2 *of* Y*;*
(7) $\alpha\mathrm{Cl}_{\mathscr{I}^\star}[F^-(s\mathrm{Int}_{\mathscr{J}^\star}(K_1)) \cup F^+(s\mathrm{Int}_{\mathscr{J}^\star}(K_2))] \subseteq F^-(K_1) \cup F^+(K_2)$ *for every* $\star$-*closed sets* K_1, K_2 *of* Y*;*
(8) $\alpha\mathrm{Cl}_{\mathscr{I}^\star}[F^-(\mathrm{Cl}^\star(\mathrm{Int}^\star(K_1))) \cup F^+(\mathrm{Cl}^\star(\mathrm{Int}^\star(K_2)))] \subseteq F^-(K_1) \cup F^+(K_2)$ *for every* $\star$-*closed sets* K_1, K_2 *of* Y*;*
(9) $\alpha\mathrm{Cl}_{\mathscr{I}^\star}[F^-(\mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(B_1)))) \cup F^+(\mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(B_2))))]$ $\subseteq F^-(\mathrm{Cl}^\star(B_1)) \cup F^+(\mathrm{Cl}(B_2))$ *for every subsets* B_1, B_2 *of* Y*;*
(10) $\mathrm{Cl}^\star[\mathrm{Int}^\star[\mathrm{Cl}^\star[F^-(\mathrm{Cl}^\star(\mathrm{Int}^\star(K_1))) \cup F^+(\mathrm{Cl}^\star(\mathrm{Int}^\star(K_2)))]]]$ $\subseteq F^-(K_1) \cup F^+(K_2)$ *for every* $\star$-*closed sets* K_1, K_2 *of* Y*;*
(11) $\mathrm{Cl}^\star[\mathrm{Int}^\star[\mathrm{Cl}^\star[F^-(s\mathrm{Int}_{\mathscr{J}^\star}(K_1)) \cup F^+(s\mathrm{Int}_{\mathscr{J}^\star}(K_2))]]] \subseteq F^-(K_1) \cup F^+(K_2)$ *for every* $\star$-*closed sets* K_1, K_2 *of* Y*;*
(12) $F^+(V_1) \cap F^-(V_2) \subseteq \mathrm{Int}^\star[\mathrm{Cl}^\star[\mathrm{Int}^\star[F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))]]]$ *for every* $\star$-*open sets* V_1, V_2 *of* Y.

Proof. (1) $\Rightarrow$ (2) The proof follows from Theorem 3.9.

(2) $\Rightarrow$ (3) The proof is obvious.

(3) $\Rightarrow$ (4) Let V_1, V_2 be any R-$\mathscr{J}^\star$-open sets of Y and $x \in F^+(V_1) \cap F^-(V_2)$. Then $F(x) \in V_1^+ \cap V_2^-$ and there exists an α-$\mathscr{I}^\star$-open set U of X containing x such that $F(U) \subseteq V_1$ and $F(z) \cap V_2 \neq \emptyset$ for every $z \in U$. Thus, $x \in U \subseteq F^+(V_1) \cap F^-(V_2)$ and hence $F^+(V_1) \cap F^-(V_2) \subseteq \alpha\mathrm{Int}_{\mathscr{I}^\star}[F^+(V_1) \cap F^-(V_2)]$. This shows that $F^+(V_1) \cap F^-(V_2)$ is α-$\mathscr{I}^\star$-open in X.

(4) $\Rightarrow$ (5) This follows from the fact that $F^+(Y - B) = X - F^-(B)$ and $F^-(Y - B) = X - F^+(B)$ for every subset B of Y.

(5) $\Rightarrow$ (6) Let V_1, V_2 be any $\star$-open sets of Y and $x \in F^+(V_1) \cap F^-(V_2)$. Then $F(x) \subseteq V_1 \subseteq s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)$ and $\emptyset \neq F(x) \cap V_2 \subseteq F(x) \cap s\mathrm{Cl}_{\mathscr{J}^\star}(V_2)$. Thus, $x \in F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) = X - F^-(Y - s\mathrm{Cl}_{\mathscr{J}^\star}(V_1))$ and

$$x \in F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2)) = X - F^+(Y - s\mathrm{Cl}_{\mathscr{J}^\star}(V_2)).$$

Since $Y - s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)$ and $Y - s\mathrm{Cl}_{\mathscr{J}^\star}(V_2)$ are R-$\mathscr{J}^\star$-closed, by (5),

$$F^-(Y - s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cup F^+(Y - s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))$$

is α-$\mathscr{I}^\star$-closed in X. Since

$$\begin{aligned} &F^-(Y - s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cup F^+(Y - s\mathrm{Cl}_{\mathscr{J}^\star}(V_2)) \\ &= (X - F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1))) \cup (X - F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))) \\ &= X - (F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))), \end{aligned}$$

we have $F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))$ is α-$\mathscr{I}^\star$-open in X and hence

$$x \in \alpha\mathrm{Int}_{\mathscr{I}^\star}(F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))).$$

Consequently, we obtain

$$F^+(V_1) \cap F^-(V_2) \subseteq \alpha\mathrm{Int}_{\mathscr{I}^\star}(F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))).$$

(6) $\Rightarrow$ (7) Let K_1, K_2 be any $\star$-closed sets of Y. Then $Y - K_1$ and $Y - K_2$ are $\star$-open sets. By (6), we have

$$\begin{aligned}
&X - (F^-(K_1) \cup F^+(K_2)) \\
&= (X - F^-(K_1)) \cap (X - F^+(K_2)) \\
&= F^+(Y - K_1) \cap F^-(Y - K_2) \\
&\subseteq \alpha\mathrm{Int}_{\mathscr{I}^\star}(F^+(s\mathrm{Cl}_{\mathscr{I}^\star}(Y - K_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{I}^\star}(Y - K_2))) \\
&= \alpha\mathrm{Cl}_{\mathscr{I}^\star}(F^+(Y - s\mathrm{Int}_{\mathscr{I}^\star}(K_1)) \cap F^-(Y - s\mathrm{Int}_{\mathscr{I}^\star}(K_2))) \\
&= \alpha\mathrm{Int}_{\mathscr{I}^\star}[(X - F^-(s\mathrm{Int}_{\mathscr{I}^\star}(K_1))) \cap (X - F^+(s\mathrm{Int}_{\mathscr{I}^\star}(K_2)))] \\
&= X - \alpha\mathrm{Cl}_{\mathscr{I}^\star}[F^-(s\mathrm{Int}_{\mathscr{I}^\star}(K_1)) \cup F^+(s\mathrm{Int}_{\mathscr{I}^\star}(K_2))].
\end{aligned}$$

Thus, $\alpha\mathrm{Cl}_{\mathscr{I}^\star}(F^-(s\mathrm{Int}_{\mathscr{I}^\star}(K_1)) \cup F^+(s\mathrm{Int}_{\mathscr{I}^\star}(K_2))) \subseteq F^-(K_1) \cup F^+(K_2)$.

(7) $\Rightarrow$ (8) The proof is obvious since $s\mathrm{Int}_{\mathscr{I}^\star}(K) = \mathrm{Cl}^\star(\mathrm{Int}^\star(K))$ for every $\star$-closed set K of Y.

(8) $\Rightarrow$ (9) The proof is obvious.

(9) $\Rightarrow$ (10) Let K_1, K_2 be any $\star$-closed sets of Y. By (9) and Proposition 2.9, we have

$$\begin{aligned}
&\mathrm{Cl}^\star[\mathrm{Int}^\star[\mathrm{Cl}^\star[F^-(\mathrm{Cl}^\star(\mathrm{Int}^\star(K_1))) \cup F^+(\mathrm{Cl}^\star(\mathrm{Int}^\star(K_2)))]]] \\
&\subseteq \alpha\mathrm{Cl}_{\mathscr{I}^\star}[F^-(\mathrm{Cl}^\star(\mathrm{Int}^\star(K_1))) \cup F^+(\mathrm{Cl}^\star(\mathrm{Int}^\star(K_2)))] \\
&= \alpha\mathrm{Cl}_{\mathscr{I}^\star}[F^-(\mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(K_1)))) \cup F^+(\mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(K_2))))] \\
&\subseteq F^-(K_1) \cup F^+(K_2).
\end{aligned}$$

(10) $\Rightarrow$ (11) The proof is obvious since $s\mathrm{Int}_{\mathscr{I}^\star}(K) = \mathrm{Cl}^\star(\mathrm{Int}^\star(K))$ for every $\star$-closed set K of Y.

(11) $\Rightarrow$ (12) Let V_1, V_2 be any $\star$-open sets of Y. Then $Y - V_1$ and $Y - V_2$ are $\star$-closed sets of Y. By (11),

$$\begin{aligned}
&\mathrm{Cl}^\star[\mathrm{Int}^\star[\mathrm{Cl}^\star[F^-(s\mathrm{Int}_{\mathscr{I}^\star}(Y - V_1)) \cup F^+(s\mathrm{Int}_{\mathscr{I}^\star}(Y - V_2))]]] \\
&\subseteq F^-(Y - V_1) \cup F^+(Y - V_2) \\
&= (X - F^+(V_1)) \cup (X - F^-(V_2)) \\
&= X - (F^+(V_1) \cap F^-(V_2)).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\mathrm{Cl}^\star[\mathrm{Int}^\star[\mathrm{Cl}^\star[F^-(s\mathrm{Int}_{\mathscr{I}^\star}(Y - V_1)) \cup F^+(s\mathrm{Int}_{\mathscr{I}^\star}(Y - V_2))]]] \\
&= \mathrm{Cl}^\star[\mathrm{Int}^\star[\mathrm{Cl}^\star[F^-(Y - s\mathrm{Cl}_{\mathscr{I}^\star}(V_1)) \cup F^+(Y - s\mathrm{Cl}_{\mathscr{I}^\star}(V_2))]]] \\
&= \mathrm{Cl}^\star[\mathrm{Int}^\star[\mathrm{Cl}^\star[(X - F^+(s\mathrm{Cl}_{\mathscr{I}^\star}(V_1))) \cup (X - F^-(s\mathrm{Cl}_{\mathscr{I}^\star}(V_2)))]]] \\
&= \mathrm{Cl}^\star[\mathrm{Int}^\star[\mathrm{Cl}^\star[X - (F^+(s\mathrm{Cl}_{\mathscr{I}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{I}^\star}(V_2)))]]] \\
&= X - \mathrm{Int}^\star[\mathrm{Cl}^\star[\mathrm{Int}^\star[F^+(s\mathrm{Cl}_{\mathscr{I}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{I}^\star}(V_2))]]].
\end{aligned}$$

Thus, $F^+(V_1) \cap F^-(V_2) \subseteq \mathrm{Int}^\star[\mathrm{Cl}^\star[\mathrm{Int}^\star[F^+(s\mathrm{Cl}_{\mathscr{I}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{I}^\star}(V_2))]]]$.

(12) $\Rightarrow$ (1) Let $x \in X$ and V_1, V_2 be any $\star$-open sets of Y such that $F(x) \in V_1^+ \cap V_2^-$. By (12), we have

$$x \in F^+(V_1) \cap F^-(V_2) \subseteq \mathrm{Int}^\star[\mathrm{Cl}^\star[\mathrm{Int}^\star[F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))]]]$$

and hence F is almost $\star(\alpha)$-continuous at x by Theorem 3.9. Consequently, we obtain F is almost $\star(\alpha)$-continuous. □

Definition 3.13. A function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be *almost $\star(\alpha)$-continuous* if $f^{-1}(V)$ is α-$\mathscr{I}^\star$-open in X for every R-$\mathscr{J}^\star$-open set V of Y.

Corollary 3.14. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is almost $\star(\alpha)$-continuous;*
(2) *for each $x \in X$ and every $\star$-open set V of Y containing $f(x)$, there exists an α-$\mathscr{I}^\star$-open set U of X containing x such that $f(U) \subseteq s\mathrm{Cl}_{\mathscr{J}^\star}(V)$;*
(3) *for each $x \in X$ and every R-$\mathscr{J}^\star$-open sets V of Y containing $f(x)$, there exists an α-$\mathscr{I}^\star$-open set U of X containing x such that $f(U) \subseteq V$;*
(4) *$f^{-1}(V)$ is α-$\mathscr{I}^\star$-open in X for every R-$\mathscr{J}^\star$-open set V of Y;*
(5) *$f^{-1}(K)$ is α-$\mathscr{I}^\star$-closed in X for every R-$\mathscr{J}^\star$-closed set K of Y;*
(6) *$f^{-1}(V) \subseteq \alpha\mathrm{Int}_{\mathscr{I}^\star}(f^{-1}(s\mathrm{Cl}_{\mathscr{J}^\star}(V))$ for every $\star$-open set V of Y;*
(7) *$\alpha\mathrm{Cl}_{\mathscr{I}^\star}[f^{-1}(s\mathrm{Int}_{\mathscr{J}^\star}(K))] \subseteq f^{-1}(K)$ for every $\star$-closed set K of Y;*
(8) *$\alpha\mathrm{Cl}_{\mathscr{I}^\star}[f^{-1}(\mathrm{Cl}^\star(\mathrm{Int}^\star(K)))] \subseteq f^{-1}(K)$ for every every $\star$-closed sets K of Y;*
(9) *$\alpha\mathrm{Cl}_{\mathscr{I}^\star}[f^{-1}(\mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(B))))] \subseteq f^{-1}(\mathrm{Cl}^\star(B))$ for every subset B of Y;*
(10) *$\mathrm{Cl}^\star[\mathrm{Int}^\star[\mathrm{Cl}^\star[f^{-1}(\mathrm{Cl}^\star(\mathrm{Int}^\star(K_1)))]]] \subseteq f^{-1}(K)$ for every $\star$-closed set K of Y;*
(11) *$\mathrm{Cl}^\star[\mathrm{Int}^\star[\mathrm{Cl}^\star[f^{-1}(s\mathrm{Int}_{\mathscr{J}^\star}(K_1))]]] \subseteq f^{-1}(K)$ for every $\star$-closed set K of Y;*
(12) *$f^{-1}(V) \subseteq \mathrm{Int}^\star[\mathrm{Cl}^\star[\mathrm{Int}^\star[f^{-1}(s\mathrm{Cl}_{\mathscr{J}^\star}(V))]]]$ for every $\star$-open set V of Y.*

Theorem 3.15. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *F is almost $\star(\alpha)$- continuous;*
(2) *$\alpha\mathrm{Cl}_{\mathscr{I}^\star}(F^-(V_1) \cup F^+(V_2)) \subseteq F^-(\mathrm{Cl}^\star(V_1)) \cup F^+(\mathrm{Cl}^\star(V_2))$ for every semi-$\mathscr{J}^\star$-preopen sets V_1, V_2 of Y;*
(3) *$\alpha\mathrm{Cl}_{\mathscr{I}^\star}(F^-(V_1) \cup F^+(V_2)) \subseteq F^-(\mathrm{Cl}^\star(V_1)) \cup F^+(\mathrm{Cl}^\star(V_2))$ for every semi-$\mathscr{J}^\star$-open sets V_1, V_2 of Y;*
(4) *$F^+(V_1) \cap F^-(V_2) \subseteq \alpha\mathrm{Int}_{\mathscr{I}^\star}(F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2)))$ for every $\mathscr{J}^\star$-preopen sets V_1, V_2 of Y.*

Proof. (1) $\Rightarrow$ (2) Let V_1, V_2 be any semi-$\mathscr{J}^\star$-preopen sets of Y. Since $\mathrm{Cl}^\star(V_1)$ and $\mathrm{Cl}^\star(V_1)$ are R-$\mathscr{J}^\star$-closed, by Theorem 3.12,

$$F^-(\mathrm{Cl}^\star(V_1)) \cup F^+(\mathrm{Cl}^\star(V_2))$$

is α-$\mathscr{I}^\star$-closed in X. Since $F^-(V_1) \cup F^+(V_2) \subseteq F^-(\mathrm{Cl}^\star(V_1)) \cup F^+(\mathrm{Cl}^\star(V_2))$, $\alpha\mathrm{Cl}_{\mathscr{I}^\star}(F^-(V_1) \cup F^+(V_2)) \subseteq F^-(\mathrm{Cl}^\star(V_1)) \cup F^+(\mathrm{Cl}^\star(V_2))$.

(2) $\Rightarrow$ (3) This is obvious since every semi-$\mathscr{J}^\star$-open set is semi-$\mathscr{J}^\star$-preopen.

(3) $\Rightarrow$ (1) Let K_1, K_2 be any R-$\mathscr{J}^\star$-closed sets of Y. Then K_1, K_2 are semi-$\mathscr{J}^\star$-open in Y. By (3), we have

$$\begin{aligned}\alpha\mathrm{Cl}_{\mathscr{I}^\star}(F^-(K_1) \cup F^+(K_2)) &\subseteq F^-(\mathrm{Cl}^\star(K_1)) \cup F^+(\mathrm{Cl}^\star(K_2))\\ &\subseteq F^-(K_1) \cup F^+(K_2)\end{aligned}$$

and hence $F^-(K_1) \cup F^+(K_2)$ is α-$\mathscr{I}^\star$-closed in X. Thus, F is almost $\star(\alpha)$-continuous by Theorem 3.12.

(1) $\Rightarrow$ (4) Let V_1, V_2 be any $\mathscr{J}^\star$-preopen sets of Y. Since $\mathrm{Int}^\star(\mathrm{Cl}^\star(V_1))$ and $\mathrm{Int}^\star(\mathrm{Cl}^\star(V_2))$ are R-$\mathscr{J}^\star$-open in Y, $\mathrm{Int}^\star(\mathrm{Cl}^\star(V_1)) = s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)$ and

$$\mathrm{Int}^\star(\mathrm{Cl}^\star(V_2)) = s\mathrm{Cl}_{\mathscr{J}^\star}(V_2),$$

by Theorem 3.12, we have $F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))$ is α-$\mathscr{I}^\star$-open in X and hence

$$\begin{aligned}F^+(V_1) \cap F^-(V_2) &\subseteq F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))\\ &= \alpha\mathrm{Int}_{\mathscr{I}^\star}[F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))].\end{aligned}$$

(4) $\Rightarrow$ (1) Let V_1, V_2 be any R-$\mathscr{J}^\star$-open sets of Y. Since V_1, V_2 are $\mathscr{J}^\star$-preopen in Y, $F^+(V_1) \cap F^-(V_2) \subseteq \alpha\mathrm{Int}_{\mathscr{I}^\star}(F^+(s\mathrm{Cl}_{\mathscr{J}^\star}(V_1)) \cap F^-(s\mathrm{Cl}_{\mathscr{J}^\star}(V_2))) = \alpha\mathrm{Int}_{\mathscr{I}^\star}(F^+(V_1) \cap F^-(V_2))$. Thus, $F^+(V_1) \cap F^-(V_2)$ is α-$\mathscr{I}^\star$-open in X. It follows from Theorem 3.12 that F is almost $\star(\alpha)$-continuous. $\square$

COROLLARY 3.16. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) *f is almost $\star(\alpha)$-continuous;*
(2) *$\alpha\mathrm{Cl}_{\mathscr{I}^\star}(f^{-1}(V)) \subseteq f^{-1}(\mathrm{Cl}^\star(V))$ for every semi-$\mathscr{J}^\star$-preopen set V of Y;*
(3) *$\alpha\mathrm{Cl}_{\mathscr{I}^\star}(f^{-1}(V_1)) \subseteq f^{-1}(\mathrm{Cl}^\star(V))$ for every semi-$\mathscr{J}^\star$-open set V of Y;*
(4) *$f^{-1}(V) \subseteq \alpha\mathrm{Int}_{\mathscr{I}^\star}(f^{-1}(s\mathrm{Cl}_{\mathscr{J}^\star}(V)))$ for every $\mathscr{J}^\star$-preopen set V of Y.*

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