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A GENERALIZATION OF WEIGHTED BILINEAR HARDY INEQUALITY

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Abstract. In this paper, we give some new generalizations of the weighted bilinear Hardy inequality by using weighted mean operators $S := (Sf)_g^w$, where f nonnegative integrable function with two variables on $\triangle = (0,+\infty)\times(0,+\infty)$, defined by

$$S(x,y) = \int_a^x \int_c^y \frac{w(t)w(s)}{W(t)W(s)} g(f(t,s))\mathrm{d}s\mathrm{d}t,$$

with

$$W(z) = \int_0^z w(r)\mathrm{d}r, \quad for\ z \in (0,+\infty),$$

where w is a weight function and g is a nonnegative continuous function on $(0,+\infty)$.

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Key words. Hölder's inequality, Hardy-Type Integral Inequality, weight function.

1. INTRODUCTION

The inequality

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p \mathrm{d}x \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\mathrm{d}x, \tag{1}$$

where $F(x) = \int_0^x f(t)\mathrm{d}t$, known as Hardy's inequality, is satisfied for all functions f non-negative and measurable on $(0,\infty)$ with $p > 1$. The constant $\left(\frac{p}{p-1}\right)^p$ is the best possible.

In 1928, Hardy proved the following inequality [5]. Let f non-negative measurable function on $(0,\infty)$,

$$F(x) = \begin{cases} \int_0^x f(t)\mathrm{d}t, & \text{for } \alpha < p-1, \\ \int_x^\infty f(t)\mathrm{d}t, & \text{for } \alpha > p-1. \end{cases}$$

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Then

$$(2)\qquad \int_0^\infty x^{\alpha-p}F^p(x)\mathrm{d}x \le \left(\frac{p}{|p-1-\alpha|}\right)^p \int_0^\infty x^\alpha f^p(x)\mathrm{d}x, \quad \text{for } p>1.$$

Some works are devoted to this inequality in dimension two. Hardy-type inequalities for various integral operators in dimension two have been studied in [1-4, 7-14] and the references therein. The objective of this paper is to give some new generalizations of the weighted bilinear Hardy inequality by using some elementary methods of analysis and Sarikaya operator $S := Sf_g^w$.

2. PRELIMINARIES

In this section we give some lemmas which will be used in the proof of main theorems. Let $W(z) = \int_0^z w(r)\mathrm{d}r$, for $z \in (0,+\infty)$ and $\triangle = (0,+\infty)\times(0,+\infty)$.

LEMMA 2.1. *Suppose f nonnegative integrable on $\triangle$, g nonnegative continuous on $(0,+\infty)$ and $p>1$, $\alpha<p-1$. Let*

$$S(x,y) = \int_a^x \int_c^y \frac{w(t)w(s)}{W(t)W(s)} g(f(t,s))\mathrm{d}s\mathrm{d}t,$$

$$G(x,y) = \int_a^x \frac{w(t)}{W(t)} g(f(t,y))\mathrm{d}t.$$

Fix $x>a$. Then

$$(3)\qquad \int_c^d \frac{w(y)}{W^{p-\alpha}(y)} S^p(x,y)\mathrm{d}y \le \left(\frac{p}{p-\alpha-1}\right)^p \int_c^d \frac{w(y)}{W^{p-\alpha}(y)} G^p(x,y)\mathrm{d}y.$$

Proof. Let be x fixed in (3) and using Fubini's theorem, we have

$$S(x,y) = \int_c^y \frac{w(s)}{W(s)} \left(\int_a^x \frac{w(t)}{W(t)} g(f(t,s))\mathrm{d}t\right)\mathrm{d}s,$$

then $S'(x,y) = \frac{\partial S(x,y)}{\partial y} = \frac{w(y)}{W(y)}G(x,y)$. Let $I(x) = \int_c^d \frac{w(y)}{W^{p-\alpha}(y)} S^p(x,y)\mathrm{d}y$. Integrating by parts $I(x)$, we get

$$\begin{aligned} I(x) &= \left[-\frac{S^p(x,y)}{(p-\alpha-1)W^{p-\alpha-1}(y)}\right]_c^d + \frac{p}{p-\alpha-1} \\ &\times \int_c^d \frac{w(y)}{W^{p-\alpha}(y)} G(x,y)S^{p-1}(x,y)\mathrm{d}y, \\ &= \left[-\frac{S^p(x,d)}{(p-\alpha-1)W^{p-\alpha-1}(d)}\right] + \frac{p}{p-\alpha-1} \\ &\times \int_c^d \frac{w(y)}{W^{p-\alpha}(y)} G(x,y)S^{p-1}(x,y)\mathrm{d}y. \end{aligned}$$

Since $p-\alpha-1>0$ and $S(x,d)\geq 0$, we have

$$I(x)\leq \frac{p}{p-\alpha-1}\int_c^d \frac{w(y)}{W^{p-\alpha}(y)}G(x,y)S^{p-1}(x,y)\mathrm{d}y.$$

From Hölder integral inequality for $\frac{1}{p}+\frac{1}{q}=1$, we obtain

$$I(x)\leq \frac{p}{p-\alpha-1}\left(\int_c^d \frac{w(y)}{W^{p-\alpha}(y)}G^p(x,y)\mathrm{d}y\right)^{\frac{1}{p}}\left(\int_c^d \frac{w(y)}{W^{p-\alpha}(y)}S^p(x,y)\mathrm{d}y\right)^{\frac{1}{q}},$$

and thus on simplification, we get

$$\int_c^d \frac{w(y)}{W^{p-\alpha}(y)}S^p(x,y)\mathrm{d}y\leq \left(\frac{p}{p-\alpha-1}\right)^p\int_c^d \frac{w(y)}{W^{p-\alpha}(y)}G^p(x,y)\mathrm{d}y,$$

which gives the required inequality. □

Lemma 2.2. *Suppose f nonnegative integrable on $\triangle$, g nonnegative continuous on $(0,+\infty)$ and $p>1$, $\alpha<p-1$. Let $G(x,y)=\int_a^x \frac{w(t)}{W(t)}g(f(t,y))\mathrm{d}t$. Fix $y>c$. Then*

$$\int_a^b \frac{w(x)}{W^{p-\alpha}(x)}G^p(x,y)\mathrm{d}x\leq \left(\frac{p}{p-\alpha-1}\right)^p\int_a^b \frac{w(x)}{W^{p-\alpha}(x)}g^p(f(x,y))\mathrm{d}x. \tag{4}$$

Proof. Let be y fixed in (4). Then

$$G'(x,y)=\frac{\partial G(x,y)}{\partial x}=\frac{w(x)}{W(x)}g(f(x,y)).$$

Integrating by parts $I(y)=\int_a^b \frac{w(x)}{W^{p-\alpha}(x)}G^p(x,y)\mathrm{d}x$, it follows that

$$\begin{aligned}I(y)&=\left[-\frac{G^p(x,y)}{(p-\alpha-1)W^{p-\alpha-1}(x)}\right]_a^b+\frac{p}{p-\alpha-1}\\&\quad\times\int_a^b \frac{w(x)}{W^{p-\alpha}(x)}g(f(x,y))G^{p-1}(x,y)\mathrm{d}x\\&=\left[-\frac{G^p(b,y)}{(p-\alpha-1)W^{p-\alpha-1}(b)}\right]+\frac{p}{p-\alpha-1}\\&\quad\times\int_a^b \frac{w(x)}{W^{p-\alpha}(x)}g(f(x,y))G^{p-1}(x,y)\mathrm{d}x.\end{aligned}$$

Since $p-\alpha>1$ and $G(b,y)\geq 0$, we have

$$I(y)\leq \frac{p}{p-\alpha-1}\int_a^b \frac{w(x)}{W^{p-\alpha}(x)}g(f(x,y))G^{p-1}(x,y)\mathrm{d}x.$$

By Hölder integral inequality for $\frac{1}{p}+\frac{1}{q}=1$, we obtain

$$I(y)\frac{p}{p-\alpha-1}\left(\int_a^b \frac{w(x)}{W^{p-\alpha}(x)}g^p(f(x,y))\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_a^b \frac{w(x)}{W^{p-\alpha}(x)}G^p(x,y)\mathrm{d}x\right)^{\frac{1}{q}},$$

and thus on simplification, we get

$$\int_a^b \frac{w(x)}{W^{p-\alpha}(x)} G^p(x,y)\mathrm{d}x \le \left(\frac{p}{p-\alpha-1}\right)^p \int_a^b \frac{w(x)}{W^{p-\alpha}(x)} g^p(f(x,y))\mathrm{d}x.$$

□

Lemma 2.3. *Suppose f nonnegative integrable on $\triangle$, g nonnegative continuous on $(0,+\infty)$ and $p>1$, $\alpha>p-1$. Let*

$$S(x,y) = \int_x^b \int_y^d \frac{w(t)w(s)}{W(t)W(s)} g(f(t,s))\mathrm{d}s\mathrm{d}t,$$

$$H(x,y) = \int_x^b \frac{w(t)}{W(t)} g(f(t,y))\mathrm{d}t.$$

Let $x>a$ fixed. We get

$$\int_c^d \frac{w(y)}{W^{p-\alpha}(y)} S^p(x,y)\mathrm{d}y \le \left(\frac{p}{1-p+\alpha}\right)^p \int_c^d \frac{w(y)}{W^{p-\alpha}(y)} H^p(x,y)\mathrm{d}y. \tag{5}$$

Proof. The proof is similar to the proof of Lemma 2.1. □

Lemma 2.4. *Suppose f nonnegative integrable on $\triangle$, g nonnegative continuous on $(0,+\infty)$ and $p>1$, $\alpha>p-1$. Let $H(x,y) = \int_x^b \frac{w(t)}{W(t)} g(f(t,y))\,\mathrm{d}t$. Let $y>c$ be fixed. Then*

$$\int_a^b \frac{w(x)}{W^{p-\alpha}(x)} H^p(x,y)\mathrm{d}x \le \left(\frac{p}{1-p+\alpha}\right)^p \int_a^b \frac{w(x)}{W^{p-\alpha}(x)} g^p(f(x,y))\mathrm{d}x. \tag{6}$$

Proof. The proof is similar to the proof of Lemma 2.2. □

3. MAIN RESULTS

Let $0<a<b<+\infty$ and $0<c<d<+\infty$. Throughout the paper, we will assume that the functions f and g are nonnegative integrable on $\triangle = (0,+\infty)\times(0,+\infty)$ and $(0,+\infty)$, the integrals throughout are assumed to exist and are finite (i.e., convergent), $w \in L_p(0,\infty)$ and $W(z) = \int_0^z w(r)\mathrm{d}r$.

Theorem 3.1. *Suppose f nonnegative integrable on $\triangle$, g nonnegative continuous on $(0,+\infty)$ and $p>1$, $\alpha<p-1$. Let*

$$S(x,y) = \int_a^x \int_c^y \frac{w(t)w(s)}{W(t)W(s)} g(f(t,s))\,\mathrm{d}s\mathrm{d}t.$$

Then

$$\begin{aligned}&\int_a^b \int_c^d \frac{w(x)w(y)}{W^{p-\alpha}(x)W^{p-\alpha}(y)} S^p(x,y)\mathrm{d}y\mathrm{d}x \\ &\quad\le \left(\frac{p}{p-\alpha-1}\right)^{2p} \int_a^b \int_c^d \frac{w(x)w(y)}{W^{p-\alpha}(x)W^{p-\alpha}(y)} g^p(f(x,y))\mathrm{d}y\mathrm{d}x.\end{aligned} \tag{7}$$

Proof. We denote by "Lhs" the left hand side of inequality (7). By using Fubini's theorem, Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned}
Lhs &= \int_a^b \frac{w(x)}{W^{p-\alpha}(x)} \left(\int_c^d \frac{w(y)}{W^{p-\alpha}(y)} S^p(x,y)\mathrm{d}y \right) \mathrm{d}x \\
&\le \left(\frac{p}{p-\alpha-1} \right)^p \int_a^b \frac{w(x)}{W^{p-\alpha}(x)} \left(\int_c^d \frac{w(y)}{W^{p-\alpha}(y)} G^p(x,y)\mathrm{d}y \right) \mathrm{d}x \\
&= \left(\frac{p}{p-\alpha-1} \right)^p \int_c^d \frac{w(y)}{W^{p-\alpha}(y)} \left(\int_a^b \frac{w(x)}{W^{p-\alpha}(x)} G^p(x,y)\mathrm{d}x \right) \mathrm{d}y \\
&\le \left(\frac{p}{p-\alpha-1} \right)^{2p} \int_c^d \frac{w(y)}{W^{p-\alpha}(y)} \left(\int_a^b \frac{w(x)}{W^{p-\alpha}(x)} g^p(f(x,y))\mathrm{d}x \right) \mathrm{d}y \\
&= \left(\frac{p}{p-\alpha-1} \right)^{2p} \int_a^b \int_c^d \frac{w(x)w(y)}{W^{p-\alpha}(x)W^{p-\alpha}(y)} g^p(f(x,y))\mathrm{d}y\mathrm{d}x
\end{aligned}$$

which completes the proof. □

Theorem 3.2. *Suppose f nonnegative integrable on $\triangle$, g nonnegative continuous on $(0,+\infty)$ and $p > 1$, $\alpha > p-1$. Let*

$$S(x,y) = \int_x^b \int_y^d \frac{w(t)w(s)}{W(t)W(s)} g(f(t,s))\, \mathrm{d}s\mathrm{d}t.$$

Then

$$\begin{aligned}
&\int_a^b \int_c^d \frac{w(x)w(y)}{W^{p-\alpha}(x)W^{p-\alpha}(y)} S^p(x,y)\mathrm{d}y\mathrm{d}x \\
&\quad \le \left(\frac{p}{1-p+\alpha} \right)^{2p} \int_a^b \int_c^d \frac{w(x)w(y)}{W^{p-\alpha}(x)W^{p-\alpha}(y)} g^p(f(x,y))\mathrm{d}y\mathrm{d}x.
\end{aligned} \tag{8}$$

Proof. By using Lemmas 2.3 and Lemma 2.4, the proof is similar to Theorem 3.1. □

4. APPLICATIONS

If we put $W(x) = x$ and $g(f(x,y)) = xyf(x,y)$ in Theorem 3.1 and Theorem 3.2, we have the following corollary, the **weighted bilinear Hardy inequality:**

Corollary 4.1. *Suppose $p > 1$, $\alpha < p-1$ and f be nonnegative integrable function on $\triangle$. Let $F(x,y) = \int_a^x \int_c^y f(t,s)\, \mathrm{d}s\mathrm{d}t$. Then*

$$\begin{aligned}
&\int_a^b \int_c^d (xy)^{\alpha-p} F^p(x,y)\mathrm{d}y\mathrm{d}x \\
&\quad \le \left(\frac{p}{p-\alpha-1} \right)^{2p} \int_a^b \int_c^d (xy)^{\alpha} f^p(x,y)\mathrm{d}y\mathrm{d}x.
\end{aligned} \tag{9}$$

Corollary 4.2. *Suppose $p > 1$, $\alpha > p-1$ and f be nonnegative integrable function on $\triangle$. Let $F(x,y) = \int_x^b \int_y^d f(t,s)\,\mathrm{d}s\mathrm{d}t$. Then*

$$\begin{aligned} &\int_a^b \int_c^d (xy)^{\alpha-p} F^p(x,y)\mathrm{d}y\mathrm{d}x \\ &\leq \left(\frac{p}{1-p+\alpha}\right)^{2p} \int_a^b \int_c^d (xy)^{\alpha} f^p(x,y)\mathrm{d}y\mathrm{d}x. \end{aligned} \tag{10}$$

- Function with two independent variables:

Suppose $f(x,y) = f_1(x).f_2(y)$ where $f_1,\ f_2$ are nonnegative integrable functions on $(0,\infty)$. From Corollary 4.1 and Corollary 4.2, we obtain:

Corollary 4.3. *Let $p > 1$, $\alpha < p-1$ and*

$$F(x,y) = \left(\int_a^x f_1(t)\,\mathrm{d}t\right)\left(\int_c^y f_2(s)\,\mathrm{d}s\right).$$

Then

$$\begin{aligned} &\int_a^b \int_c^d (xy)^{\alpha-p} F^p(x,y)\mathrm{d}y\mathrm{d}x \\ &\leq \left(\frac{p}{p-\alpha-1}\right)^{2p} \left(\int_a^b x^{\alpha} f_1^p(x)\mathrm{d}x\right)\left(\int_c^d y^{\alpha} f_2^p(y)\mathrm{d}y\right). \end{aligned} \tag{11}$$

Corollary 4.4. *Let $p > 1$, $\alpha > p-1$ and*

$$F(x,y) = \left(\int_x^b f_1(t)\,\mathrm{d}t\right)\left(\int_y^d f_2(s)\,\mathrm{d}s\right).$$

Then

$$\begin{aligned} &\int_a^b \int_c^d (xy)^{\alpha-p} F^p(x,y)\mathrm{d}y\mathrm{d}x \\ &\leq \left(\frac{p}{1-p+\alpha}\right)^{2p} \left(\int_a^b x^{\alpha} f_1^p(x)\mathrm{d}x\right)\left(\int_c^d y^{\alpha} f_2^p(y)\mathrm{d}y\right). \end{aligned} \tag{12}$$

If we choose $f_1 = f_2,\ x = y,\ a = c,\ b = d$, we deduce the **weighted Hardy integral inequality:**

Corollary 4.5. *Suppose $p > 1$, $\alpha < p-1$ and f nonnegative integrable on $(0,\infty)$. Let $F(x) = \int_a^x f(t)\,\mathrm{d}t$. Then*

$$\int_a^b x^{\alpha-p} F^p(x)\mathrm{d}x \leq \left(\frac{p}{p-\alpha-1}\right)^{p} \int_a^b x^{\alpha} f^p(x)\mathrm{d}x. \tag{13}$$

Corollary 4.6. *Suppose $p > 1$, $\alpha > p-1$ and f nonnegative integrable on $(0,\infty)$. Let $F(x) = \int_x^b f(t)\,\mathrm{d}t$. Then*

$$\int_a^b x^{\alpha-p} F^p(x)\mathrm{d}x \leq \left(\frac{p}{1-p+\alpha}\right)^{p} \int_a^b x^{\alpha} f^p(x)\mathrm{d}x. \tag{14}$$

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