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A NOTE ON THE DIOPHANTINE EQUATION $x^2 - kxy + ky^2 + ly = 0$

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Abstract. We investigate the Diophantine equation $x^2 - kxy + ky^2 + ly = 0$ for integers k and l with k even. We give a characterization of the positive solutions of this equation in terms of k and l. We also consider the same equation when $l = p^n$ and $k \equiv 2 \pmod p$ for $p \equiv 3 \pmod 4$; $l = 2^r 3^s$ and $k = 2k' + 1$ with $k' \equiv 2 \pmod 3$ where n, s, t are non-negative integers.

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1. INTRODUCTION

There have been many works on the Diophantine equation from the title which is related to the theory of Pell's equation. In [2], Hu and Le studied the equation

$$x^2 - kxy + y^2 + lx = 0, \quad k, l \in \mathbb{Z}. \tag{1}$$

For $l = 1$, Marlewski and Zarzycki showed in [8] that this equation has infinitely many positive integer solutions if and only if $k = 3$. Further, they asked whether there are other values of k for which the equation (1) has infinitely many integer solutions. In [4], Keskin dealt with equation (1) for $l = \pm 1$. He showed that for $k > 3$ and $l = 1$, the above equation has no positive integer solutions, but for $k > 3$, $l = -1$, it has infinitely many positive integer solutions. Subsequently, in [11], Yuan and Hu answered the question posed in [8] by proving that when $l = 1$, equation (1) has infinitely many integer solutions if and only if $k \notin \{-1, 0, 1\}$. They also considered equation (1) for $l = 2$ and $l = 4$ and determined which values of the positive integer k yield an equation with infinitely many positive integer solutions. In [5] and [6], Keskin dealt with the equation $x^2 - kxy + y^2 \pm 2^n = 0$ and determined the circumstances

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under which it has infinitely many positive integer solutions for $0 \leq n \leq 10$. In [3], Karaatli and Şiar considered the Diophantine equation

$$x^2 - kxy + ky^2 + ly = 0 \tag{2}$$

for $l = 2^\kappa, \kappa \in \{0, 1, 2, 3\}$, and determined positive integer values of k for which these equations have infinitely many positive integer solutions.

Recently, Mavecha [9] considered the equation (2) for $l = 2^n$, where n is a non-negative integer and k is odd, and proved that this equation has infinitely many positive integer solutions x and y if and only if $k = 5$.

In this paper, we deal with equation (2) in integers k and l. We determine conditions under which this equation has infinitely many positive integer solutions. We do the same thing for this equation when $l = 3^n$ and $k \equiv 2 \pmod 3$; $l = 2^r 3^s$ and $k = 2k' + 1$ with $k' \equiv 2 \pmod 3$ where n, s, t are non-negative integers.

2. PRELIMINARIES

We give, without proof, two theorems on the theory of Pell's equation. These theorems can be found in many books on number theory, in particular that of Nagell [10].

THEOREM 2.1. *Let D be a positive integer which is not a perfect square. The equation*

$$X^2 - DY^2 = 1 \tag{3}$$

has infinitely many integer solutions $x + y\sqrt{D}$. All solutions with positive x and y are obtained from the formula

$$x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n$$

where $x_1 + y_1\sqrt{D}$ is the fundamental solution of equation (3) and n runs through all natural numbers.

For integer N, consider the Diophantine equation below

$$U^2 - DV^2 = N. \tag{4}$$

Let $u + v\sqrt{D}$ and $u' + v'\sqrt{D}$ be solutions of this equation. These solutions are said to be associated if there is solution $x + y\sqrt{D}$ of (3), so that

$$u' + v'\sqrt{D} = (u + v\sqrt{D})(x + y\sqrt{D}).$$

The associated solutions of (4) form a class of solutions. It is well-known that there are finitely many classes of solutions to (4), each of which contains infinitely many solutions.

THEOREM 2.2. *If $u + v\sqrt{D}$ is a fundamental solution for some class of solutions to equation* (4) *and if $x_1 + y_1\sqrt{D}$ is the fundamental solution of equation* (3), *then we have the inequalities*

$$0 \leq v \leq \frac{y_1}{\sqrt{2(x_1+1)}}\sqrt{N} \quad \text{and} \quad 0 < |u| \leq \sqrt{\frac{1}{2}(x_1+1)N}.$$

We will make use of the following theorem.

THEOREM 2.3 ([7, Theorem 4.1]). *Let $k > 1$ be an odd integer. If $0 < t < 2\sqrt{k^2-4}$ is an integer and the equation*

$$x^2 - y^2(k^2-4) = 4t$$

has solutions in coprime positive integers x, y, then $t = 1$ or $t = k + 2$.

3. NEW RESULTS

In this section, we prove the following theorem and some related results.

THEOREM 3.1. *Let k and l be integers with $l^2 < k$. The equation*

$$x^2 - kxy + ky^2 + ly = 0$$

has infinitely many positive integer solutions x and y if and only if

$$(l, k) \in \{(1,5), (2,5), (2,6)\}.$$

Proof. Let us assume that, there exist some integers x and y which satisfy equation (2). We assume first that k is even. By completing the square, we get

$$\left(x - \frac{k}{2}y\right)^2 + \left(k - \frac{k^2}{4}\right)y^2 + ly = 0. \tag{5}$$

Set $a := \left(k - \frac{k^2}{4}\right)$. Multiplying equation (5) by $4a$, we get the following equation

$$a(2x - ky)^2 + (2ay + l)^2 = l^2.$$

Set $u := 2ay + l$ and $v := 2x - ky$. We obtain the equation

$$u^2 - \left(\frac{k^2}{4} - k\right)v^2 = l^2. \tag{6}$$

If $k = 2$ then in equation (6) we get $u^2 + v^2 = l^2$ which has only finitely many solutions in terms of l. If $k = 4$, then $u^2 = l^2$, so in this case the equation also has only finitely many solutions. Therefore, we assume that $k > 4$. In particular, $\left(\frac{k^2}{4} - k\right) > 0$. Set $k =: 2k_0$, $k_1 := k_0 - 1$ and $D := k_1^2 - 1$. Then equation (6) becomes

$$U^2 - DV^2 = l^2.$$

The fundamental solution of the Pell equation

$$u^2 - Dv^2 = 1$$

is $(u_1, v_1) = (k_1, 1)$. Let $u + v\sqrt{D}$ be the fundamental solution of the class of solutions to equation (6). By Theorem 2, we find that $0 \leq v \leq \frac{l}{\sqrt{k}} < 1$, so $v = 0$ and $u = \ell$. Thus, all solutions of equation (6) are given by $(U, V) = (\pm u_n l, \pm v_n l)$ where (u_n, v_n) is the nth solution of $u^2 - Dv^2 = 1$. We have that l and y are positive, that $2ay + l = U = \pm u_n l$ for some n and a is negative. If $U = lu_n$, we then get $l(u_n - 1) = 2k_0(2 - k_0)y$, which is impossible since $k_0 > 2$. If $U = -u_n l$, then we see that $2k_0(k_0 - 2) \mid l(u_n + 1)$. Note that u_n is given by

$$u_n = \frac{(k_1 + \sqrt{D})^n + (k_1 - \sqrt{D})^n}{2} \equiv k_1^n \pmod{D}.$$

If n is even, then $u_n \equiv (k_1^2)^{n/2} \equiv 1 \pmod{k_0}$, hence $k_0 \mid u_n - 1$. Since we know that $2k_0 \mid l(u_n - 1)$, and $\gcd(u_n + 1, u_n - 1) \mid 2$, we see that $k_0 \mid l$, implying $k_0 \leq l < \sqrt{2k_0}$, which is a contradiction.

If n is odd, then $u_n \equiv k_1(k_1^2)^{(n-1)/2} \pmod{D} = k_1 \pmod{k_0(k_0 - 2)}$, therefore we have $u_n + 1 \equiv k_0 \pmod{k_0(k_0 - 2)}$. Hence, $(u_n + 1)/k_0$ is an integer congruent to $1 \pmod{k_0 - 2}$, and so it is coprime to $k_0 - 2$. Since by above we have that $2k_0(k_0 - 2) \mid l(u_n + 1)$, we must have $2(k_0 - 2) \mid l(u_n + 1)/k_0$. If k_0 is even, then so is $k_0 - 2$ and so $2(k_0 - 2)$ divides l, giving $2(k_0 - 2) < \sqrt{2k_0}$, which has no solution for even $k_0 \geq 4$. If k_0 is odd, then since $u_1 = k_1 = k_0 - 1$ is even and n is odd, we have $u_1 \mid u_n$, meaning that u_n is even and $u_n + 1$ is odd and again we get $2(k_0 - 2) \mid l$. This implies that $2(k_0 - 2) \leq \sqrt{2k_0}$, to which the only solution is $k_0 = 3$; i.e., $k = 6$. Noting that $2(k_0 - 2) = 2$ divides l, we see that l is even and $l < \sqrt{k} = \sqrt{6} < 4$, so $l = 2$. Since $u_n \equiv 2 \pmod{3}$ and u_n is even, we see that y_n is a positive integer and $x_n = (6y_n + 2v_n)/2$ is also a positive integer.

Suppose that k is odd. Performing the same algebraic manipulations as above and multiplying by 4, equation (6) becomes

$$U^2 - ((k-2)^2 - 4)V^2 = 4l^2,$$

where $U = k(4 - k)y + 2l$ and $V = 2x - ky$. From this we see $k \geq 5$. Dividing this equation by $l_1 := \gcd(U, V)$, we obtain

$$U_1^2 - ((k-2)^2 - 4)V_1^2 = 4l_2^2, \tag{7}$$

where $U_1 = U/l_1, V_1 = V/l_1$ and $l_2 = l/l_1$. Since for $k \geq 7$ we have

$$l^2 < k < 2\sqrt{(k-2)^2 - 4}$$

and for $k = 5$, we have $l^2 < 5$, giving $l^2 \leq 4 < 2\sqrt{(5-2)^2 - 4}$, Theorem 2.3 tells us that $l_2^2 = 1$ or $l_2^2 = k - 2 + 2 = k$. Since $l_2^2 < k$, we must have $l_2 = 1$. Hence, $(U, V) = l(\pm u_n, v_n)$, where this time (u_n, v_n) is the nth positive integer solution of $u^2 - ((k-2)^2-4)^2v^2 = 4$. If $U = lu_n$, then $l(u_n - 2) = k(4-k)y$, which is impossible for $k \geq 5$ odd. This means we must have $U = -lu_n$ and so $k(k-4) \mid l(u_n+2)$. We now set $k_1 = k-2$ and observe that

$$u_n = \Big(\frac{k_1 + \sqrt{k_1^2-4}}{2}\Big)^n + \Big(\frac{k_1 - \sqrt{k_1^2-4}}{2}\Big)^n.$$

Hence, $u_n \equiv 2(k_1/2)^n \pmod{k_1^2-4}$. Since k is odd, k_1^2-4 is also odd, and so 2 is invertible modulo k_1^2-4. If n is even, we then have

$$u_n \equiv 2(k_1^2\cdot 4^{-1})^{n/2} \equiv 2((k_1^2-4)\cdot 4^{-1}+1)^{n/2} \equiv 2 \pmod{k_1^2-4},$$

which means that $k_1^2 - 4 = k(k-4) \mid u_n - 2$. Since $k(k-4) \mid l(u_n+2)$, $\gcd(u_n-2, u_n+2) \mid 4$ and $k(k-4)$ is odd, we have $k(k-4) \mid l$, so $k(k-4) \leq \sqrt{k}$, which cannot hold for $k \geq 5$. Hence, n must be odd, which means that

$$u_n \equiv k_1(k_1^2/4)^{(n-1)/2} \equiv k-2 \pmod{k_1^2-4},$$

and so $u_n + 2 \equiv k \pmod{k(k-4)}$. From this, we have $k \mid u_n+2$ and $(u_n+2)/k \equiv 1 \pmod{k-4}$, and so u_n+2 is coprime to $k-4$. Since $k-4 \mid l(u_n+2)/k$, we must have $k-4 \mid l$, meaning $k-4 \leq \sqrt{k}$, and so $k=5$. Since $v_1 = 1$ and n is odd, we have that v_n is odd and $y = l(u_n+2)/5$. Since $l < \sqrt{k}$, we get $l = 1, 2$, which are the cases covered in [3] and [9]. Hence, equation (2) has infinitely many solutions if and only if $(l,k) = (1,5), (2,5), (2,6)$. □

We deduce the following corollary.

Corollary 3.2.

(i) *If k is odd and $s \geq 0$ is an integer, the equation $x^2 - kxy + ky^2 + 2^sy = 0$ has infinitely many positive integer solutions x and y if and only if $k = 5$. If k is even and $s \in \{0,1,2\}$, then the equation $x^2 - kxy + ky^2 + 2^sy = 0$ has infinitely many positive integer solutions x and y if and only if $(k,s) \in \{(6,1),(6,2),(8,2)\}$.*

(ii) *The equation $x^2 - kxy + ky^2 + ly = 0$, where $l = p^n$, $p \equiv 3 \pmod 4$ and $k \equiv 2 \pmod p$ are integers and n is a non-negative integer, has infinitely many positive integer solutions x and y if and only if $k = 5$.*

(iii) *The equation $x^2 - kxy + ky^2 + ly = 0$, where $l = 2^a3^b$, $k = 2k'+1$, $k' \equiv 2 \pmod 3$ are integers and a and b are non-negative integers, has infinitely many positive integer solutions x and y if and only if $k = 5$.*

Proof.

(i) Suppose first that k is odd. Then $(k-2)^2 - 4 \equiv 5 \pmod 8$, so reducing equation (7) modulo 8 gives $U_1^2 + 3V_1^2 \equiv 4l_2^2 \pmod 8$. Since $l = 2^s$, the value l_2 must also be a power of 2 and so U_1 and V_1 must be odd and $l_2^2 = 1$. It follows that $U = -lu_n$, and so $k(k-4) \mid l$ if n is even or $(k-4) \mid l$ if n is odd. Since k is odd and l is a power of 2, the only possibility is that n is odd and $k = 5$.

Now suppose that k is even. The cases $s = 0, 1$ follow from Theorem 3.1 so assume that $s = 2$, that is $l = 4$. If $k \equiv 2 \pmod 4$, then $\frac{k(k-4)}{4}$ is a product of two consecutive odd integers so it is congruent to $3 \pmod 4$, and thus equation (6) gives $U^2 + V^2 \equiv l^2 \pmod 4$. From this we see that U and V cannot both be odd and so $l \mid U$. If $k \equiv 0 \pmod 4$, then $\frac{k(k-4)}{4}$ is a multiple of 8, so equation (6) becomes $U^2 - 8bV^2 = 16$ for some integer b which gives $4 \mid U$. Hence, $l = 4 \mid U$ in either case. From the proof of Theorem 3.1 it follows that either n is even, $k = 2k_0$ and k_0 divides l, or n is odd, $k = 2k_0$ and $2(k_0 - 2)$ divides l. The first case gives $k_0 \mid 4$, for which the possibilities are $k_0 = 1, 2, 4$, the first two of which are impossible since $x^2 - kxy + ky^2 + 4y$ is positive for x, y positive and $k = 2, 4$, which leaves $k = 8$ as the only possibility. The second case gives $k_0 - 2 \mid 2$, so $k_0 = 3, 4$, which means that $k = 6, 8$.

(ii) If k is odd, then equation (7) modulo p becomes $U_1^2 + (2V_1)^2 \equiv 4l_2^2 \pmod p$, and since $p \equiv 3 \pmod 4$, we can't have that p divides the sum of two coprime squares, so we must have $l_2^2 = 1$. Hence either n is even and $k(k-4)$ divides l or n is odd and $k - 4$ divides l. In the first instance, since $l = p^n$, we must have that k and $k - 4$ are powers of the same prime. This is only possible for $k = 5$, but then $p = 5$ does not satisfy the congruence $p \equiv 3 \pmod 4$. In the second instance, $k = 5$ satisfies the condition.

If k is even then similarly to before, we get that either n is even and $k_0 \mid l$ or n is odd and $2(k_0 - 2) \mid l$. The second case is impossible since l is a power of an odd prime. In the first case we get that k_0 is a power of p, but since $k \equiv 2 \pmod p$ and $k = 2k_0$, only $k = 2$ works. But in that case the expression $x^2 - 2xy + 2y^2 + ly$ is always positive and so there are no solutions.

(iii) Reducing equation (7) modulo 8 in a similar manner to in part (i) gives that l_2^2 is odd, so since $l = 2^a 3^b$, we must have $2^a \mid U$. Reducing modulo 3, we obtain $l_2^2 \equiv 1 \pmod 3$, and so $3^b \mid U$. Hence, $l \mid U$ and we again obtain that either n is even and $k(k-4) \mid l$ or n is odd and $k - 4 \mid l$. Since $l = 2^a 3^b$ and k is odd, the case when n is even implies that k and $k - 4$ are powers of 3, which is impossible. If n is odd, then we must have $k - 4 = 3^r$ for some r. If $r > 0$, then $k \equiv 1 \pmod 3$, which contradicts $k = 2k' + 1 \equiv 2(2) + 1 \equiv 2 \pmod 3$. Therefore $k = 5$ is the only possibility. □

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