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ON $\mathcal{I}$-STATISTICALLY LIMIT POINTS AND $\mathcal{I}$-STATISTICALLY CLUSTER POINTS OF SEQUENCES OF FUZZY NUMBERS

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Abstract. The main aim of this paper is to introduce $\mathcal{I}$-st limit points and $\mathcal{I}$-st cluster points of a sequence of fuzzy numbers and also study some of its basic properties. Conditions for a $\mathcal{I}$-st limit point of a $\mathcal{I}$-st cluster point are investigated.

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Key words. Limit point, cluster point, ideal, fuzzy number.

1. INTRODUCTION

The idea of convergence of a real sequence had been extended to statistical convergence by Fast [8]. Later on it was further investigated from sequence space point of view and linked with summability theory by Fridy [9] and Salat [23] and many others. Some applications of statistical convergence in number theory and mathematical analysis can be found in [3, 4, 16, 18]. The idea is based on the notion of natural density of subsets of $\mathbb{N}$, the set of all positive integers which is defined as follows: The natural density of a subset A of $\mathbb{N}$ denoted by $\delta(A)$ is defined by $\delta(A) = \lim_{n\to\infty} \frac{1}{n}|\{k < n : k \in A\}|$.

The notion of ideal convergence was introduced by Kostyrko et al. [15] which generalizes and unifies different notion of convergence of sequences including usual convergence and statistical convergence. They used the notion of an ideal $\mathcal{I}$ of subsets of the set $\mathbb{N}$ to define such a concept. For an extensive view of this article we refer to [7, 14, 28].

The idea of $\mathcal{I}$-statistical convergence was introduced by Savas and Das [26] as an extension of ideal convergence. Later on, it was further investigated by Savas and Das [27], Debnath and Debnath [5], Et et al. [11] and many others.

On the other hand the concept of the convergence of the sequence of fuzzy numbers was introduced by Matloka [17] who proved some basic theorems for sequences of fuzzy number. Later on, several mathematician such as Nanda [19], Savas [24] and many others have generalized this concept. Nuray and

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Savas [20] generalized the concept of ordinary convergence and introduced statistical convergence and statistically cauchy sequences of fuzzy number. Later on, it was studied and developed by Aytar and Pehlivan [2] and many others. Aytar [1] extended the concept of statistical limit points and cluster points for sequences of fuzzy number. Kumar et al. [12, 13] introduced $\mathcal{I}$-convergence, $\mathcal{I}$-limit points, $\mathcal{I}$-cluster point for sequence of fuzzy numbers. The concepts of $\mathcal{I}$-statistically convergence for sequences of fuzzy numbers was introduced by Debnath and Debnath [6].

In this paper, we investigate some basic properties of $\mathcal{I}$-statistically convergent sequence of fuzzy numbers and introduce $\mathcal{I}$-statistical limit point, $\mathcal{I}$-statistical cluster point for fuzzy number sequences.

2. DEFINITIONS AND PRELIMINARIES

We first recall some basic notions in the theory of fuzzy numbers. We denote by $\mathcal{D}$ the set of all closed and bounded intervals on real line $\mathbb{R}$, i.e., $\mathcal{D} = \left\{A \subset \mathbb{R} : A = \left[\underline{A}, \overline{A}\right]\right\}$. For $A, B \in \mathcal{D}$ we define $A \leq B$ if and only if $\underline{A} \leq \underline{B}$ and $\overline{A} \leq \overline{B}$ and $d = max\left\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\right\}$. $\left(\mathcal{D}, \overline{d}\right)$ forms a complete metric space.

Definition 2.1. A fuzzy number is a function X from $\mathbb{R}$ to $[0, 1]$, which satisfies the following conditions:
(i) X is normal.
(ii) X is fuzzy convex.
(iii) X is upper semi-continuous.
(iv) The closure of the set $\{x \in \mathbb{R} : X(x) > 0\}$ is compact.

The properties (i)-(iv) imply that for each $\alpha \in (0, 1]$, the α-level set, $X^\alpha = \{x \in \mathbb{R} : X(x) \geq \alpha\} = \left[\underline{X}^\alpha, \overline{X}^\alpha\right]$ is a non-empty compact convex subset of $\mathbb{R}$. The 0-level set is the class of the strong 0-cut, i.e., $cl\,\{x \in R : X(x) > 0\}$. Let $L(\mathbb{R})$ denotes the set of all fuzzy numbers. Define a map $\overline{d}(X, Y) = \sup_{\alpha\in[0,1]} d(X^\alpha, Y^\alpha)$. $\left(L(\mathbb{R}), \overline{d}\right)$ also forms a complete metric space [22].

Definition 2.2 ([17]). A sequence (X_k) of fuzzy numbers is said to be convergent to a fuzzy number X_0 if for each $\varepsilon > 0$ there exist a positive number m such that $\bar{d}(X_n, X_0) < \varepsilon$ for every $n \geq m$. The fuzzy number X_0 is called the ordinary limit of the sequence (X_k) and we write $\lim_{k\to\infty} X_k = X_0$.

Definition 2.3 ([17]). A fuzzy number X_0 is said to be a limit point of a sequence of fuzzy number (X_k) provided that there is a subsequence of (X_k) that converges to X_0. L_X denotes the set of all limit points of the sequence $X = (X_k)$.

Definition 2.4 ([20]). A sequence (X_k) of fuzzy numbers is said to be statistical convergent to a fuzzy number X_0 if for each $\varepsilon > 0$ the set $A\,(\varepsilon) = \left\{k \in \mathbb{N} : \bar{d}(X_k, X_0) \geq \varepsilon\right\}$ has natural density zero. The fuzzy number X_0 is called the statistical limit of the sequence (X_k) and we write St-$\lim_{k\to\infty} X_k = X_0$.

Definition 2.5 ([1]). If $\{X_{k(j)}\}$ be a subsequence of a sequence of fuzzy numbers $X = (X_k)$ and density of $K = \{k(j) : j \in \mathbb{N}\}$ is zero then $\{X_{k(j)}\}$ is called a thin subsequence. Otherwise, $\{X_{k(j)}\}$ is called a non-thin subsequence of X. A fuzzy number X_0 is said to be a statistical limit point of a sequence of fuzzy numbers (X_k), if there exists a non-thin subsequence of (X_k) which conveges to X_0. Let Λ_X denotes the set of all statistical limit points of the sequence (X_k).

Definition 2.6 ([1]). A fuzzy number X_0 is said to be a statistical cluster point of a sequence of fuzzy numbers (X_k) provided that for each $\varepsilon > 0$ the density of the set $\{k \in \mathbb{N} : \bar{d}(X_k, X_0) < \varepsilon\}$ is not equal to 0.

Let Γ_X denotes the set of all statistical cluster points of the sequence (X_k).

Definition 2.7 ([15]). Let X be a non-empty set. A family of subsets $\mathcal{I} \subset P(X)$ is called an ideal on X if and only if

(i) $\emptyset \in \mathcal{I}$;

(ii) for each $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$;

(iii) for each $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$.

Definition 2.8 ([15]). Let X be a non-empty set. A family of subsets $\mathcal{F} \subset P(X)$ is called an filter on X if and only if

(i) $\emptyset \notin \mathcal{F}$;

(ii) for each $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$;

(iii) for each $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. The filter $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X - A : A \in \mathcal{I}\}$ is called the filter associated with the ideal $\mathcal{I}$. A non-trivial ideal $\mathcal{I} \subset P(X)$ is called an admissible ideal in X if and only if $\mathcal{I} \supset \{\{x\} : x \in X\}$.

Definition 2.9 ([12]). Let $\mathcal{I} \subset P(\mathbb{N})$ be a non-trivial ideal on $\mathbb{N}$. A sequence (X_k) of fuzzy numbers is said to be $\mathcal{I}$-convergent to a fuzzy number X_0 if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{k \in \mathbb{N} : \bar{d}(X_k, X_0) \geq \varepsilon\}$ belongs to $\mathcal{I}$. The fuzzy number X_0 is called the $\mathcal{I}$-limit of the sequence (X_k) and we write $\mathcal{I}$-$\lim_{k\to\infty} X_k = X_0$.

Definition 2.10 ([13]). A fuzzy number X_0 is said to be $\mathcal{I}$-limit point of a sequence of fuzzy number (X_k) provided that there is a subset $K = \{k_1 < k_2 < \ldots\} \subset \mathbb{N}$ such that $K \notin \mathcal{I}$ and $\lim X_{k_n} = X_0$.

Let $\mathcal{I}(\Lambda_X)$ denotes the set of all $\mathcal{I}$-limit points of the sequence X.

Definition 2.11 ([13]). A fuzzy number X_0 is said to be $\mathcal{I}$-cluster point of a sequence of fuzzy numbers (X_k) provided that for each $\varepsilon > 0$ the set $\{k \in \mathbb{N} : \bar{d}(X_k, X_0) < \varepsilon\} \notin \mathcal{I}$.

Let $\mathcal{I}(\Gamma_X)$ denotes the set of all $\mathcal{I}$-cluster points of the sequence X.

DEFINITION 2.12 ([6]). A sequence of fuzzy numbers (X_k) is said to be $\mathcal{I}$-statistically convergent to a fuzzy number X_0 if for every $\varepsilon > 0$ and every $\delta > 0$ $\left\{n \in \mathbb{N} : \frac{1}{n}|\left\{k \leq n : \bar{d}(X_k, X_0) \geq \varepsilon\right\}| \geq \delta\right\} \in \mathcal{I}$.

The fuzzy number X_0 is called $\mathcal{I}$-statistical limit of the fuzzy number sequence (X_k) and we write $\mathcal{I}\text{-st}\lim X_k = X_0$.

Throughout the paper we consider $\mathcal{I}$ as an admissible ideal.

3. MAIN RESULT

THEOREM 3.1. *If (X_k) be a sequence of fuzzy numbers such that $\mathcal{I}\text{-st}\lim X_k = X_0$, then X_0 determined uniquely.*

Proof. Let the sequence (X_k) be $\mathcal{I}$-statistically convergent to two different fuzzy numbers X_0 and Y_0, i.e., for any $\varepsilon > 0$, $\delta > 0$, we have

$$A_1 = \left\{n \in \mathbb{N} : \frac{1}{n}|\left\{k \leq n : \bar{d}(X_k, X_0) \geq \varepsilon\right\}| < \delta\right\} \in \mathcal{F}(\mathcal{I}),$$
$$A_2 = \left\{n \in \mathbb{N} : \frac{1}{n}|\left\{k \leq n : \bar{d}(X_k, Y_0) \geq \varepsilon\right\}| < \delta\right\} \in \mathcal{F}(\mathcal{I}).$$

Therefore, $A_1 \cap A_2 \neq \emptyset$, since $A_1 \cap A_2 \in \mathcal{F}(\mathcal{I})$. Let $m \in A_1 \cap A_2$ and take $\varepsilon = \frac{\bar{d}(X_0,Y_0)}{3} > 0$ such that we have $\frac{1}{m}|\left\{k \leq m : \bar{d}(X_k, X_0) \geq \varepsilon\right\}| < \delta$ and it follows that $\frac{1}{m}|\left\{k \leq m : \bar{d}(X_k, Y_0) \geq \varepsilon\right\}| < \delta$, i.e., for maximum $k \leq m$ we have $\bar{d}(X_k, X_0) < \varepsilon$ and $\bar{d}(X_k, Y_0) < \varepsilon$ for a very small $\delta > 0$. Thus we must have

$$\left\{k \leq m : \bar{d}(X_k, X_0) < \varepsilon\right\} \cap \left\{k \leq m : \bar{d}(X_k, Y_0) < \varepsilon\right\} \neq \emptyset,$$

a contradiction, as the nbd of X_0 and Y_0 are disjoint. Hence X_0 is uniquely determined. □

THEOREM 3.2. *Let (X_k) be a fuzzy numbers sequence then st-lim$X_k = X_0$ implies $\mathcal{I}\text{-st}\lim X_k = X_0$.*

Proof. Let st-lim$X_k = X_0$. Then for each $\varepsilon > 0$ the set

$$A(\varepsilon) = \left\{k \leq n : \bar{d}(X_k, X_0) \geq \varepsilon\right\}$$

has natural density zero, i.e., $\lim_{n\to\infty} \frac{1}{n}|\left\{k \leq n : \bar{d}(X_k, X_0) \geq \varepsilon\right\}| = 0$. So, for every $\varepsilon > 0$ and $\delta > 0$, $\left\{n \in \mathbb{N} : \frac{1}{n}|\left\{k \leq n : \bar{d}(X_k, X_0) \geq \varepsilon\right\}| \geq \delta\right\}$ is a finite set and therefore belongs to $\mathcal{I}$, as $\mathcal{I}$ is an admissible ideal. Hence $\mathcal{I}\text{-st}\lim X_k = X_0$. □

THEOREM 3.3. *For any sequence of fuzzy numbers (X_k), $\mathcal{I}$-lim$X_k = X_0$ implies $\mathcal{I}\text{-st}\lim X_k = X_0$.*

Proof. The proof is obvious. □

But the converse is not true. For example, take $\mathcal{I}=\mathcal{I}_f$, the fuzzy number sequence (X_k), where $X_k(t)=\begin{cases}\frac{n+t}{n}, & -n\le t\le 0\\ \frac{n-t}{n} & 0\le t\le n\end{cases}$ for $k=n^2$, $n\in\mathbb{N}$ and $X_k(t)=\begin{cases}1+tn, & -\frac{1}{n}\le t\le 0\\ 1-tn & 0\le t\le \frac{1}{n}\end{cases}$ for $k\neq n^2$, $n\in\mathbb{N}$. Then (X_k) is $\mathcal{I}$-statistically convergent, but not $\mathcal{I}$-convergent.

Theorem 3.4. *Let (X_k) and (Y_k) be two fuzzy numbers sequence. Then*
(i) $\mathcal{I}$-st lim $X_k=X_0$, $c\in R$, *implies* $\mathcal{I}$-st lim $cX_k=cX_0$.
(ii) $\mathcal{I}$-st lim $X_k=X_0$, $\mathcal{I}$-st lim $Y_k=Y_0$ *implies* $\mathcal{I}$-st lim $(X_k+Y_k)=X_0+Y_0$.

Proof. (i) If $c=0$, we have nothing to prove. So, assume that $c\neq 0$. Now

$$\frac{1}{n}\left|\left\{k\le n:\bar{d}\left(cX_k,cX_0\right)\ge\varepsilon\right\}\right|=\frac{1}{n}\left|\left\{k\le n:|c|\bar{d}\left(X_k,X_0\right)\ge\varepsilon\right\}\right|$$
$$\le\frac{1}{n}\left|\left\{k\le n:\bar{d}\left(X_k,X_0\right)\ge\frac{\varepsilon}{|c|}\right\}\right|<\delta.$$

Therefore we have $\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\le n:\bar{d}\left(cX_k,cX_0\right)\ge\varepsilon\right\}\right|<\delta\right\}\in\mathcal{F}\left(\mathcal{I}\right)$, i.e., $\mathcal{I}$-st lim $cX_k=cX_0$.

(ii) We have $A_1=\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\le n:\bar{d}\left(X_k,X_0\right)\ge\frac{\varepsilon}{2}\right\}\right|<\frac{\delta}{2}\right\}\in\mathcal{F}\left(\mathcal{I}\right)$ and $A_2=\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\le n:\bar{d}\left(Y_k,Y_0\right)\ge\frac{\varepsilon}{2}\right\}\right|<\frac{\delta}{2}\right\}\in\mathcal{F}\left(\mathcal{I}\right)$. Since $A_1\cap A_2\neq\emptyset$, therefore for all $n\in A_1\cap A_2$, we have

$$\frac{1}{n}\left|\left\{k\le n:\bar{d}\left(X_k+Y_k,X_0+Y_0\right)\ge\varepsilon\right\}\right|$$
$$\le\frac{1}{n}\left|\left\{k\le n:\bar{d}\left(X_k,X_0\right)\ge\frac{\varepsilon}{2}\right\}\right|+\frac{1}{n}\left|\left\{k\le n:\bar{d}\left(Y_k,Y_0\right)\ge\frac{\varepsilon}{2}\right\}\right|<\delta,$$

i.e., $\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\le n:\bar{d}\left(X_k+Y_k,X_0+Y_0\right)\ge\varepsilon\right\}<\delta\right\}\in\mathcal{F}(\mathcal{I})$. Hence $\mathcal{I}$-st lim $(X_k+Y_k)=(X_0+Y_0)$. □

Definition 3.5. An element $X_0\in L(\mathbb{N})$ is said to be an $\mathcal{I}$-statistical limit point of a fuzzy number sequence $X=(X_k)$ provided that for each $\varepsilon>0$ there is a set $M=\{m_1<m_2<\ldots\}\subset\mathbb{N}$ such that $M\notin\mathcal{I}$ and st-lim $X_{m_k}=X_0$.

$\mathcal{I}$-$S\left(\Lambda_X\right)$ denotes the set of all $\mathcal{I}$-statistical limit points of fuzzy number sequence (X_k).

Theorem 3.6. *If (X_k) is a sequence of fuzzy numbers such that $\mathcal{I}$-st lim $X_k=X_0$, then $\mathcal{I}$-$S\left(\Lambda_X\right)=\{X_0\}$.*

Proof. Since (X_k) is $\mathcal{I}$-statistically convergent to a fuzzy number X_0, for each $\varepsilon>0$ and $\delta>0$, the set $A=\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\le n:\bar{d}\left(X_k,X_0\right)\ge\varepsilon\right\}\right|\ge\delta\right\}\in\mathcal{I}$, where $\mathcal{I}$ is an admissible ideal.

Suppose $\mathcal{I}$-$S\left(\Lambda_X\right)$ contains Y_0 different from X_0, i.e., $Y_0\in\mathcal{I}$-$S\left(\Lambda_X\right)$. So there exists a $M\subset\mathbb{N}$ such that $M\notin\mathcal{I}$ and st-lim $X_{m_k}=Y_0$.

Let $B=\left\{n\in M:\frac{1}{n}\left|\left\{k\le n:\bar{d}\left(X_k,Y_0\right)\ge\varepsilon\right\}\right|\ge\delta\right\}$. So B is a finite set and therefore $B\in\mathcal{I}$. So $B^c=\left\{n\in M:\frac{1}{n}\left|\left\{k\le n:\bar{d}\left(X_k,Y_0\right)\ge\varepsilon\right\}\right|<\delta\right\}\in$

$\mathcal{F}(\mathcal{I})$. Again let $A_1 = \{n \in M : \frac{1}{n}|\{k \leq n : \bar{d}(X_k, X_0) \geq \varepsilon\}| \geq \delta\}$. So $A_1 \subset A \in \mathcal{I}$, i.e., $A_1^c \in \mathcal{F}(\mathcal{I})$. Therefore, $A_1^c \cap B^c \neq \emptyset$, since $A_1^c \cap B^c \in \mathcal{F}(\mathcal{I})$.

Let $p \in A_1^c \cap B^c$ and take $\varepsilon = \frac{\bar{d}(X_0,Y_0)}{3} > 0$, so $\frac{1}{p}|\{k \leq p : \bar{d}(X_k, X_0) \geq \varepsilon\}| < \delta$ and $\frac{1}{p}|\{k \leq p : \bar{d}(X_k, Y_0) \geq \varepsilon\}| < \delta$, i.e., for maximum $k \leq p$ will satisfy $\bar{d}(X_k, X_0) < \varepsilon$ and $\bar{d}(X_k, Y_0) < \varepsilon$ for a very small $\delta > 0$. Thus we must have $\{k \leq p : \bar{d}(X_k, X_0) < \varepsilon\} \cap \{k \leq p : \bar{d}(X_k, Y_0) < \varepsilon\} \neq \emptyset$, a contradiction, as the nbd of X_0 and Y_0 are disjoint. Hence $\mathcal{I}\text{-}S(\Lambda_X) = \{X_0\}$. □

Definition 3.7. An element $X_0 \in L(R)$ is said to be an $\mathcal{I}$-statistical cluster point of a fuzzy number sequence $X = (X_k)$ if for each $\varepsilon > 0$ and $\delta > 0$ the set $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \bar{d}(X_k, X_0) \geq \varepsilon\}| < \delta\} \notin \mathcal{I}$.

$\mathcal{I}\text{-}S(\Gamma_X)$ denotes the set of all $\mathcal{I}$-statistical cluster points of fuzzy number sequence (X_k).

Theorem 3.8. *For any sequence $X = (X_k)$ of fuzzy numbers $\mathcal{I}\text{-}S(\Gamma_X)$ is closed.*

Proof. Let the fuzzy number Y_0 be a limit point of the set $\mathcal{I}\text{-}S(\Gamma_X)$. Then for any $\varepsilon > 0$, $\mathcal{I}\text{-}S(\Gamma_X) \cap B(Y_0, \varepsilon) \neq 0$, where $B(Y_0, \varepsilon) = \{U \in L(\mathbb{R}) : \bar{d}(U, Y_0) < \varepsilon\}$. Let $Z_0 \in \mathcal{I}\text{-}S(\Gamma_X) \cap B(Y_0, \varepsilon)$ and choose $\varepsilon_1 > 0$ such that $B(Z_0, \varepsilon_1) \subseteq B(Y_0, \varepsilon)$. Then we have $\{k \leq n : \bar{d}(X_k, Z_0) \geq \varepsilon_1\} \supseteq \{k \leq n : \bar{d}(X_k, Y_0) \geq \varepsilon\}$, which implies $\frac{1}{n}|\{k \leq n : \bar{d}(X_k, Z_0) \geq \varepsilon_1\}| \geq \frac{1}{n}|\{k \leq n : \bar{d}(X_k, Y_0) \geq \varepsilon\}|$.

Now, for any $\delta > 0$, we have $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \bar{d}(X_k, Z_0) \geq \varepsilon_1\}| < \delta\} \subseteq \{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \bar{d}(X_k, Y_0) \geq \varepsilon\}| < \delta\}$. Since $Z_0 \in \mathcal{I}\text{-}S(\Gamma_X)$, we have $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \bar{d}(X_k, Y_0) \geq \varepsilon\}| < \delta\} \notin \mathcal{I}$, i.e., $Y_0 \in \mathcal{I}\text{-}S(\Gamma_X)$. This ends the proof. □

Theorem 3.9. *For any fuzzy number sequence (X_k), $\mathcal{I}\text{-}S(\Lambda_X) \subseteq \mathcal{I}\text{-}S(\Gamma_X)$.*

Proof. Let $X_0 \in \mathcal{I}\text{-}S(\Lambda_X)$. Then there exists a set $M = \{m_1 < m_2 < \ldots\} \notin \mathcal{I}$ such that st-lim $X_{m_k} = X_0 \Rightarrow \lim_{k\to\infty} \frac{1}{k}|\{m_i \leq k : \bar{d}(X_{m_i}, X_0) \geq \varepsilon\}| = 0$. Take $\delta > 0$, so there exists $k_0 \in \mathbb{N}$ such that for $n > k_0$ we have $\frac{1}{n}|\{m_i \leq n : \bar{d}(X_{m_i}, X_0) \geq \varepsilon\}| < \delta$.

Let $A = \{n \in \mathbb{N} : \frac{1}{n}|\{m_i \leq n : \bar{d}(X_{m_i}, X_0) \geq \varepsilon\}| < \delta\}$. Also, we have $A \supset M/\{m_1 < m_2 < \ldots < m_{k_0}\}$. Since $\mathcal{I}$ is an admissible ideal and $M \notin \mathcal{I}$, therefore $A \notin \mathcal{I}$. So by definition of $\mathcal{I}$-statistical cluster point $X_0 \in \mathcal{I}\text{-}S(\Gamma_X)$. This ends the proof. □

Theorem 3.10. *If (X_k) and (Y_k) are two sequences of fuzzy numbers such that $\{n \in N : X_n \neq Y_n\} \in \mathcal{I}$, then*

(i) $\mathcal{I}\text{-}S(\Lambda_X) = \mathcal{I}\text{-}S(\Lambda_Y)$.

(ii) $\mathcal{I}\text{-}S(\Gamma_X) = \mathcal{I}\text{-}S(\Gamma_Y)$.

Proof. (i) Let $X_0 \in \mathcal{I}\text{-}S(\Lambda_X)$. So by definition there exist a set $K = \{k_1 < k_2 < k_3 < \cdots\}$ of N such that $K \notin \mathcal{I}$ and *st-lim*$X_{k_n} = X_0$. Since $\{n \in K : X_n \neq Y_n\} \subset \{n \in N : X_n \neq Y_n\} \in \mathcal{I}$, $K' = \{n \in K : X_n = Y_n\} \notin \mathcal{I}$

and $K' \subseteq K$. So, we have st-lim $Y_{k'_n} = X_0$. This shows that $X_0 \in \mathcal{I}\text{-}S\,(\Lambda_Y)$ and therefore $\mathcal{I}\text{-}S\,(\Lambda_X) \subseteq \mathcal{I}\text{-}S\,(\Lambda_Y)$. By symmetry, $\mathcal{I}\text{-}S\,(\Lambda_Y) \subseteq \mathcal{I}\text{-}S\,(\Lambda_X)$. Hence $\mathcal{I}\text{-}S\,(\Lambda_X) = \mathcal{I}\text{-}S\,(\Lambda_Y)$.

(ii) Let $X_0 \in \mathcal{I}\text{-}S\,(\Gamma_X)$. So, by definition for each $\varepsilon > 0$ we have

$$A = \left\{ n \in \mathbb{N} \,:\, \frac{1}{n} \big| \left\{ k \leq n \,:\, \bar{d}\,(X_k, X_0) \geq \varepsilon \right\} \big| < \delta \right\} \notin \mathcal{I}.$$

Let $B = \left\{ n \in \mathbb{N} \,:\, \frac{1}{n} \big| \left\{ k \leq n \,:\, \bar{d}\,(Y_k, X_0) \geq \varepsilon \right\} < \delta \right\}$. We have to prove that $B \notin \mathcal{I}$. Suppose $B \in \mathcal{I}$, So $B^c = \left\{ n \in \mathbb{N} \,:\, \frac{1}{n} \big| \left\{ k \leq n \,:\, \bar{d}\,(Y_k, X_0) \geq \varepsilon \right\} \geq \delta \right\} \in \mathcal{F}(\mathcal{I})$. By hypothesis, $C = \{ n \in \mathbb{N} : X_n = Y_n \} \in \mathcal{F}(\mathcal{I})$. Therefore $B^c \cap C \in \mathcal{F}(\mathcal{I})$. Also, it is clear that $B^c \cap C \subset A^c \in \mathcal{F}(\mathcal{I})$, i.e., $A \in \mathcal{I}$, which is a contradiction. Hence $B \notin \mathcal{I}$ and thus the result proved. □

4. CONCLUSION

$\mathcal{I}$-statistical convergence is a generalization of many other convergences such as $\mathcal{I}$-convergence, statistical convergence etc. In this paper, some existing theories on convergence of fuzzy number sequences are extended to $\mathcal{I}$-statistical convergence of fuzzy number sequence. We also introduce the notion of $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points for a sequence of fuzzy numbers and investigate the relation between them.

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