MATHEMATICA, 63 (86), N° 1, 2021, pp. 98–110

ON TRIPLE SEQUENCE SPACE OF BERNSTEIN STANCU CHENEY AND SHARMA OPERATOR OF ROUGH I_λ-CONVERGENCE OF WEIGHT g

ARULMANI INDUMATHI, AYHAN ESI, and NAGARAJAN SUBRAMANIAN

Abstract. We introduce and study some basic properties of rough I_λ-convergence of weight g, where $g : \mathbb{N}^3 \to [0, \infty)$ is a function statisying $g(n) \to \infty$ and $g(n) \not\to 0$ as $n \to \infty$ of a triple sequence of Bernstein Stancu Cheney and Sharma operators and also investigate certain properties of rough I_λ-convergence of weight g.

MSC 2010. 40F05, 40J05, 40G05.

Key words. Triple sequences, rough convergence, closed and convex, cluster points and rough limit points, Bernstein Cheney and Sharma operators, I-statistical convergence of order g, I_λ-statistical convergence of weight g.

1. INTRODUCTION

In this study, Bernstein Stancu Chenney and Sharma operator of rough I_λ-convergence of weight g is applied for triple sequence spaces. The paper [18] discussed the Stancu type extension of the well known Cheney and Sharma operators, the definition of new rough statistical convergence with the Pascal Fibonacci binomial matrix of rough statistical convergence and the approximation theory as a rate of the rough statistical convergence of Stancu type of Cheney and Sharma operators. But this paper only studied some general properties of Cheney and Sharma operators of weight.

The idea of rough convergence was first introduced by [14] in finite dimensional normed spaces, where it is showed that the set LIM_x^r is bounded, closed and convex, the notion of rough Cauchy sequence is introduced and the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree r is investigated.

Aytar [2] studied the rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained statistical convergence criteria associated with this set and proved that this set is closed and convex. Also, [2] studied that the r-limit set of the sequence is equal to intersection of

The authors are extremely grateful to the anonymous learned referee(s) for their keen reading, valuable suggestion and constructive comments for the improvement of the manuscript.

DOI: 10.24193/mathcluj.2021.1.09

these sets and that r-core of the sequence is equal to the union of these sets. Dündar and [10] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence. The notion of I-convergence of a triple sequence which is based on the structure of the ideal I of subsets of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

In this paper we investigate some basic properties of rough I-convergence of a triple sequence of Bernstein polynomials in three dimensional cases. We study the set of all rough I-limits of a triple sequence of Bernstein polynomials and also the relation between analytic and rough I-convergence of a triple sequence of Bernstein polynomials.

Let K be a subset of the set of positive integers $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and let us denote the set $K_{ij\ell}=\{(m,n,k)\in K : m\leq i, n\leq j, k\leq \ell\}$. Then the natural density of K is given by

$$\delta(K)=\lim_{i,j,\ell\to\infty}\frac{|K_{ij\ell}|}{ij\ell},$$

where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$.

Let β be a nonnegative real number and consider the following formulae.

$$\begin{aligned}&(x+y+u\beta)^u(x+y+v\beta)^v(x+y+w\beta)^w\\&=\sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w}\binom{u}{m}\binom{v}{n}\binom{w}{k}u^3(u+m\beta)^{m-1}(u+n\beta)^{n-1}(u+k\beta)^{k-1}\\&\qquad[v+(u-m)(v-n)(w-k)]^{(u-m)+(v-n)+(w-k)}\end{aligned}$$

$$\begin{aligned}&(x+y+u\beta)^u(x+y+v\beta)^v(x+y+w\beta)^w\\&=\sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w}\binom{u}{m}\binom{v}{n}\binom{w}{k}(u+m\beta)^{m}(u+n\beta)^{n}(u+k\beta)^{k}\\&\qquad v[v+(u-m)(v-n)(w-k)]^{(u-m-1)+(v-n-1)+(w-k-1)}\end{aligned}$$

$$\begin{aligned}&(x+y)(x+y+u\beta)^{u-1}(x+y+v\beta)^{v-1}(x+y+w\beta)^{w-1}\\&=\sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w}\binom{u}{m}\binom{v}{n}\binom{w}{k}u^3(u+m\beta)^{m-1}(u+n\beta)^{n-1}(u+k\beta)^{k-1}\\&\qquad v[v+(u-m)(v-n)(w-k)]^{(u-m-1)+(v-n-1)+(w-k-1)},\end{aligned}$$

where $u,v\in\mathbb{R}$ and $u,v,w\geq 1$.

Cheney and Sharma generalized Bernstein polynomials by taking $\beta\geq 0$, $u=x$ and $v=1-x$, $x\in[0,1]$, and $u=r;v=s;w=t\in\mathbb{N}$ as in the following forms:

$$P_{rst}^{\beta}(f,x) = (1+r\beta)^{-r}(1+s\beta)^{-s}(1+t\beta)^{-t}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k}$$
$$x^3(x+m\beta)^{m-1}(x+n\beta)^{n-1}(x+k\beta)^{k-1}$$
$$[1-x+(r-m)(s-n)(t-k)\beta]^{(r-m)+(s-n)+(t-k)} f\left(\frac{mnk}{rst}\right)$$

and

$$G_{rst}^{\beta}(f,x) = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} P_{rst,mnk}^{\beta}(x) f\left(\frac{mnk}{rst}\right)$$

where

$$P_{rst,mnk}^{\beta}(x) = (1+m\beta)^{1-m}(1+n\beta)^{1-n}(1+k\beta)^{1-k}$$
$$\binom{r}{m}\binom{s}{n}\binom{t}{k}x^3(x+m\beta)^{m-1}(x+n\beta)^{n-1}(x+k\beta)^{k-1}$$
$$(1-x)[1-x+(r-m)(s-n)(t-k)\beta]^{(r-m-1)+(s-n-1)+(t-k-1)}$$

for $f \in C[0,1]$, the space of real valued continuous functions on $[0,1]$. Denoting $e_\gamma(t) := t^\gamma, t \in [0,1], \gamma = 0,1,2,\ldots$, it is obvious that

$$G_{rst}^{\beta}(e_0;x) = 1$$

we have

$$G_{rst}^{\beta}(e_1;x) = x$$

since $\beta \geq 0$, these operators are linear and positive and called as Bernstein type Cheney and Sharma operators. The reduction formula

$$S(mnk,rst,x,y) = x\,S((m-1)(n-1)(k-1),rst,x,y) + (rst)\beta$$
$$S(mnk,(r-1)(s-1)(t-1),x+\beta y),$$

where

$$S(mnk,rst,x,y) :=$$
$$\sum_{\alpha=0}^{r}\sum_{\beta=0}^{s}\sum_{\gamma=0}^{t}\binom{r}{\alpha}\binom{s}{\beta}\binom{t}{\gamma}(x+\alpha)^{\alpha+m-1}(x+\beta)^{\beta+n-1}(x+\gamma)^{\gamma+k-1}$$
$$(y+(r-\alpha)(s-\beta)(t-\gamma)\beta)^{(r-\alpha)+(s-\beta)+(t-\gamma)},$$

the authors proved uniform convergence of each sequence operators $P_{rst}^{\beta}(f)$ and $G_{rst}^{\beta}(f)$ to f on $[0,1]$ by taking β as a sequence of nonnegative real numbers satisfying $\beta = O\left(\frac{1}{rst}\right)$, $r,s,t \to \infty$. It is obvious that $P_{rst}^{0} = G_{rst}^{0} = B_{rst}$, where B_{rst} is the $(r,s,t)^{th}$ Bernstein operator.

In the present paper, we consider the Stancu operators $L_{rst,uvw}$ in the basis of the Bernstein type Cheney and Sharma operators G_{rst}^{β} given by

$$L^{\beta}_{rst,uvw}(f;x) = \sum_{m=0}^{r-u}\sum_{n=0}^{s-v}\sum_{k=0}^{t-w} P^{\beta}_{(r-u,m)+(s-v,n)+(t-w,k)}(x)$$
$$\left[(1-x)f\left(\frac{mnk}{rst}\right) + xf\left(\frac{(m+u)(n+v)(k+w)}{rst}\right)\right],$$

for $f \in C[0,1]$ and (u,v,w) is a nonnegative integer parameter with $r \geq 2u$, $s \geq 2v$, $t \geq 2w$, $r,s,t \in \mathbb{N}$, where $P^{\beta}_{(r-u,m)+(s-v,n)+(t-w,k)}$ with $(r-u)(s-v)(t-w)$ in places of (r,s,t). We shall call these operators as Stancu type extensions of the Cheney and Sharma operators.

Throughout the paper, we consider the real three dimensional space with the metric (X,d). Consider a triple sequence of the Bernstein Stancu Cheney and Sharma polynomials $\left(L^{\beta}_{rst,uvw}(f,x)\right)$ such that $\left(L^{\beta}_{rst,uvw}(f,x)\right) \in \mathbb{R}$, $m,n,k \in \mathbb{N}$. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein Stancu Cheney and Sharma polynomials $\left(L^{\beta}_{rst,uvw}(f,x)\right)$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st_3 - \lim L^{\beta}_{rst,uvw}(f,x) = f(x)$, provided that the set

$$K_{\epsilon} := \left\{(m,n,k) \in \mathbb{N}^3 : \left|L^{\beta}_{rst,uvw}(f,x) - (f,x)\right| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein Stancu Cheney and Sharma polynomials, i.e., $\delta_3(K_{\epsilon}) = 0$. That is,

$$\lim_{rst\to\infty} \frac{1}{rst}\left|\left\{(m,n,k) \leq (r,s,t) : \left|L^{\beta}_{rst,uvw}(f,x) - (f,x)\right| \geq \epsilon\right\}\right| = 0.$$

In this case, we write $\delta_3 - \lim L^{\beta}_{rst,uvw}(f,x) = (f,x)$ or $L^{\beta}_{rst,uvw}(f,x) \xrightarrow{st_3} (f,x)$.

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers, χ_A-the characteristic function of $A \subset \mathbb{N}$. A subset A of $\mathbb{N}$ is said to have asymptotic density $d(A)$ if

$$d_3(A) = \lim_{ij\ell\to\infty} \frac{1}{ij\ell}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell} \chi_A(K).$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N}^3 \to \mathbb{R}(\mathbb{C})$. The different types of notions of triple sequence was introduced and investigated at the beginning by [7, 9, 8, 11, 16, 18] and many others.

2. DEFINITIONS AND PRELIMINARIES

Throughout the paper $\mathbb{R}^3$ denotes the real three dimensional space with the Euclid metric. Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}$; $m, n, k \in \mathbb{N}$.

Definition 2.1 ([7]). Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein Stancu Cheney and Sharma operators $\left(L^{\beta}_{rst,uvw}(f, x)\right)$ is said to be statistically convergent to $f(x)$, denoted by $L^{\beta}_{rst,uvw}(f, x) \rightarrow^{st-\lim x} (f, x)$, if for any $\epsilon > 0$ we have $d(A(\epsilon)) = 0$, where

$$A(\epsilon) = \left\{(m, n, k) \in \mathbb{N}^3 : \left|L^{\beta}_{rst,uvw}(f, x) - (f, x)\right| \geq \epsilon\right\}.$$

Definition 2.2 ([7]). Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein Stancu Cheney and Sharma operators $\left(L^{\beta}_{rst,uvw}(f, x)\right)$ is said to be statistically convergent to (f, x), denoted by $L^{\beta}_{rst,uvw}(f, x) \rightarrow^{st-\lim x} (f, x)$, provided that the set

$$\left\{(m, n, k) \in \mathbb{N}^3 : \left|L^{\beta}_{rst,uvw}(f, x) - (f, x)\right| \geq \epsilon\right\},$$

has natural density zero for every $\epsilon > 0$.

In this case, (f, x) is called the statistical limit of the sequence of Bernstein Stancu Cheney and Sharma operators.

Definition 2.3 ([7]). Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein Stancu Cheney and Sharma operators $\left(L^{\beta}_{rst,uvw}(f, x)\right)$ in a metric space $(X, |.,.|)$ and r be a non-negative real number is said to be r-convergent to (f, x), denoted by $L^{\beta}_{rst,uvw}(f, x) \rightarrow^{r} (f, x)$, if for any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}^3$ such that for all $m, n, k \geq N_\epsilon$ we have

$$\left|L^{\beta}_{rst,uvw}(f, x) - (f, x)\right| < r + \epsilon$$

In this case $L^{\beta}_{rst,uvw}(f, x)$ is called an r-limit of (f, x).

Remark 2.4. We consider r-limit set $L^{\beta}_{rst,uvw}(f, x)$ which is denoted by $\mathrm{LIM}^{r}_{L^{\beta}_{rst,uvw}(f,x)}$ and is defined by

$$\mathrm{LIM}^{r}_{L^{\beta}_{rst,uvw}(f,x)} = \left\{L^{\beta}_{rst,uvw}(f, x) \in X : L^{\beta}_{rst,uvw}(f, x) \rightarrow^{r} (f, x)\right\}.$$

Definition 2.5 ([11]). Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein Stancu Cheney and Sharma operators $\left(L^{\beta}_{rst,uvw}(f, x)\right)$ is said to be $r-$ convergent if $\mathrm{LIM}^{r}_{L^{\beta}_{rst,uvw}(f,x)} \neq \phi$ and r is called a rough convergence degree of $L^{\beta}_{rst,uvw}(f, x)$. If $r = 0$ then it is ordinary convergence of triple sequence of Bernstein Stancu Cheney and Sharma operators.

Definition 2.6 ([7]). Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein Stancu Cheney and Sharma operators $\left(L^{\beta}_{rst,uvw}(f,x)\right)$ in a metric space $(X,|.,.|)$ and r be a non-negative real number is said to be r-statistically convergent to (f,x), denoted by $L^{\beta}_{rst,uvw}(f,x) \rightarrow^{r-st_3} (f,x)$, if for any $\epsilon > 0$ we have $d(A(\epsilon)) = 0$, where

$$A(\epsilon) = \left\{(m,n,k) \in \mathbb{N}^3 : \left|L^{\beta}_{rst,uvw}(f,x) - (f,x)\right| \geq r + \epsilon\right\}.$$

In this case (f,x) is called r-statistical limit of $L^{\beta}_{rst,uvw}(f,x)$. If $r = 0$ then it is ordinary statistical convergent of triple sequence of Bernstein Stancu Cheney and Sharma operators.

Definition 2.7 ([11]). A class I of subsets of a nonempty set X is said to be an ideal in X provided that
(i) $\phi \in I$.
(ii) $A, B \in I$ implies $A \bigcup B \in I$.
(iii) $A \in I$, $B \subset A$ implies $B \in I$.
I is called a nontrivial ideal if $X \notin I$.

Definition 2.8 ([11]). A nonempty class F of subsets of a nonempty set X is said to be a filter in X provided that
(i) $\phi \in F$.
(ii) $A, B \in F$ implies $A \bigcap B \in F$.
(iii) $A \in F$, $A \subset B$ implies $B \in F$.

Definition 2.9 ([11]). I is a non trivial ideal in X, $X \neq \phi$, then the class

$$F(I) = \{M \subset X : M = X \backslash A \text{ for some } A \in I\}$$

is a filter on X, called the filter associated with I.

Definition 2.10 ([11]). A non trivial ideal I in X is called admissible if $\{x\} \in I$ for each $x \in X$.

Remark 2.11 ([11]). If I is an admissible ideal, then usual convergence in X implies I convergence in X.

Remark 2.12 ([11]). If I is an admissible ideal, then usual rough convergence implies rough I-convergence.

Definition 2.13 ([11]). Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein Stancu Cheney and Sharma operators $\left(L^{\beta}_{rst,uvw}(f,x)\right)$ in a metric space $(X,|.,.|)$ and r be a non-negative real number is said to be rough ideal convergent of weight g or rI_{λ}-convergent to (f,x), denoted by $L^{\beta}_{rst,uvw} \rightarrow^{rI^{g}_{\lambda}} (f,x)$, if for any $\epsilon > 0$ we have

$$\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})}\left|L^{\beta}_{rst,uvw}(f,x) - (f,x)\right| \geq r + \epsilon\right\} \in I.$$

In this case $\left(L^{\beta}_{rst,uvw}(f,x)\right)$ is called rI_{λ}-limit of (f,x) and a triple sequence of Bernstein Stancu Cheney and Sharma operators $\left(L^{\beta}_{rst,uvw}(f,x)\right)$ is called rough I_{λ}-convergent weight g to (f,x) with r as roughness of degree. If $r=0$, then it is ordinary I_{λ}-convergent of weight g.

Remark 2.14 ([18]). Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein Stancu Cheney and Sharma operators $\left(L^{\beta}_{rst,uvw}(f,y)\right)$ is not I_{λ}-convergent of weight g in usual sense and

$$\left|L^{\beta}_{rst,uvw}(f,x)-L^{\beta}_{rst,uvw}(f,y)\right|\leq r \text{ for all } (m,n,k)\in\mathbb{N}$$

or

$$\left\{(p,q,j)\in\mathbb{N}^3:\frac{1}{g(\lambda_{pqj})}\left|L^{\beta}_{rst,uvw}(f,x)-L^{\beta}_{rst,uvw}(f,y)\right|\geq r\right\}\in I \text{ for some } r>0.$$

Then the triple sequence of Bernstein Stancu Cheney and Sharma operators $\left(L^{\beta}_{rst,uvw}(f,x)\right)$ is rI_{λ} -convergent of weight g.

Remark 2.15 ([18]). It is clear that rI^{g}_{λ}-limit of (f,x) is not necessarily unique.

Definition 2.16 ([7]). Consider rI^{g}_{λ}-limit set of (f,x), which is denoted by

$$I^{g}_{\lambda}-\mathrm{LIM}^{r}_{L^{\beta}_{rst,uvw}(f,x)}=\left\{(f,x)\in X: L^{\beta}_{rst,uvw}(f,x)\to^{rI^{g}_{\lambda}}(f,x)\right\},$$

then the triple sequence of Bernstein Stancu Cheney and Sharma operators $\left(L^{\beta}_{rst,uvw}(f,x)\right)$ is said to be rI_{λ}-convergent of weight g, if $I^{g}_{\lambda}-\mathrm{LIM}^{r}_{L^{\beta}_{rst,uvw}(f,x)}\neq\phi$ and r is called a rough I_{λ}-convergence of weight g degree of $L^{\beta}_{rst,uvw}(f,x)$.

Remark 2.17. Let $\lambda=(\lambda_{pqj})_{(p,q,j)\in\mathbb{N}^3}$ be a non-decreasing sequence of positive numbers tending to ∞ such that$\lambda_{(pqj)+1}\leq\lambda_{pqj}+1,\lambda_{111}=1$. The collection of such sequences λ will be denoted by η.

We define the generalized de la Valée-Poussin mean of weight g by

$$t_{pqj}(x)=\frac{1}{g(\lambda_{pqj})}\sum_{(m,n,k)\in I_{pqj}}x_{mnk},$$

where $I_{rst}=[p-\lambda_{p+1,q,j},\ p]\times[q-\lambda_{p,q+1,j},\ q]\times[j-\lambda_{p,q,j+1},\ j]$.

Definition 2.18 ([11]). Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein Stancu Cheney and Sharma operators $\left(L^{\beta}_{rst,uvw}(f,x)\right)$ is said to be $[V,\lambda](I)^{g}$-summable to (f,x), if

$$I-\lim_{p,q,j}t_{pqj}\left(L^{\beta}_{rst,uvw}(f,x)\right)\to(f,x).$$

i.e., for any $\delta > 0, \left\{ (p,q,j) \in \mathbb{N}^3 : \left| t_{pqj}\left(L^{\beta}_{rst,uvw}(f,x) \right) - (f,x) \right| \geq \delta \right\} \in I$ and it is denoted by $[V,\lambda](I)^g$.

Definition 2.19 ([7]). Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein Stancu Cheney and Sharma operators $\left(L^{\beta}_{rst,uvw}(f,x) \right)$ is said to be I_λ-statistically convergent of weight g, if for every $\epsilon > 0$ and $\delta > 0$

$$\left\{ (p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} \left| \left\{ (m,n,k) \in I_{pqj} : \left| L^{\beta}_{rst,uvw}(f,x) - (f,x) \right| \geq r + \epsilon \right\} \right| \geq \delta \right\} \in I.$$

In this case we write $(I_\lambda)^g - \lim L^{\beta}_{rst,uvw}(f,x) = (f,x)$, or $L^{\beta}_{rst,uvw}(f,x) \to (f,x)\,(I_\lambda)^g$.

3. MAIN RESULTS

Theorem 3.1. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein Stancu Cheney and Sharma operators of $\left(L^{\beta}_{rst,uvw}(f,x) \right)$ of real numbers and $g_1, g_2 \in G$ be such that there exist $M > 0$ and $(u_0, v_0, w_0) \in \mathbb{N}^3$ such that $\frac{g_1(\lambda_{pqj})}{g_2(\lambda_{pqj})} \leq M$ for all $p \geq u_0$, $q \geq v_0$, $j \geq w_0$. Then $(I_\lambda)^{g_1} \subset (I_\lambda)^{g_2}$.*

Proof. For any $\epsilon > 0$,

$$\begin{aligned} &\frac{\left| \left\{ (m,n,k) \in I_{pqj} : \left| L^{\beta}_{rst,uvw}(f,x) - (f,x) \right| \geq r + \epsilon \right\} \right|}{g_2(\lambda_{pqj})} \\ &\quad = \frac{g_1(\lambda_{pqj}) \left| \left\{ (m,n,k) \in I_{pqj} : \left| L^{\beta}_{rst,uvw}(f,x) - (f,x) \right| \geq r + \epsilon \right\} \right|}{g_2(\lambda_{pqj})\, g_1(\lambda_{pqj})} \\ &\quad \leq M \frac{\left| \left\{ (m,n,k) \in I_{pqj} : \left| L^{\beta}_{rst,uvw}(f,x) - (f,x) \right| \geq r + \epsilon \right\} \right|}{g_1(\lambda_{pqj})} \end{aligned}$$

for $p \geq u_0$, $q \geq v_0$, $j \geq w_0$. Hence for any $\delta > 0$,

$$\begin{aligned} &\left\{ (p,q,j) \in \mathbb{N}^3 : \frac{1}{g_2(\lambda_{pqj})} \left| \left\{ (m,n,k) \in I_{pqj} : \right.\right.\right. \\ &\qquad \left.\left.\left. \left| L^{\beta}_{rst,uvw}(f,x) - (f,x) \right| \geq r + \epsilon \right\} \right| \geq \delta \right\} \\ &\quad \subset \left\{ (p,q,j) \in \mathbb{N}^3 : \frac{1}{g_1(\lambda_{pqj})} \left| \left\{ (m,n,k) \in I_{pqj} : \right.\right.\right. \\ &\qquad \left.\left.\left. \left| L^{\beta}_{rst,uvw}(f,x) - (f,x) \right| \geq r + \epsilon \right\} \right| \geq \frac{\delta}{M} \right\} \\ &\qquad\qquad \bigcup \left\{ 1, 2, \ldots, (u_0, v_0, w_0) \right\}. \end{aligned}$$

If $x = (x_{mnk}) \in (I_\lambda)^{g_1}$. Hence $(I_\lambda)^{g_1} \subset (I_\lambda)^{g_2}$. □

THEOREM 3.2. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein Stancu Cheney and Sharma operators of $\left(L^{\beta}_{rst,uvw}(f,x)\right)$ of real numbers, $(I)^g \subset (I_\lambda)^g$ if $\lim_{pqj} \inf \frac{g(\lambda_{pqj})}{g(pqj)} > 0$.*

Proof. Since $\lim_{p,q,j} \inf \frac{g(\lambda_{pqj})}{g(pqj)} > 0$, so we can find a $M > 0$ such that for sufficiently large p, q and j, we have $\lim_{p,q,j} \inf \frac{g(\lambda_{pqj})}{g(pqj)} > 0$.

Since $L^{\beta}_{rst,uvw}(f,x) \to (f,x)\,(I_\lambda)^g$, hence for every $\epsilon > 0$ and sufficiently large (r,s,t),

$$\begin{aligned}
&\frac{1}{g(pqj)}\left|\left\{(m,n,k) \le (p,q,j) : \left|L^{\beta}_{rst,uvw}(f,x) - (f,x)\right| \ge r + \epsilon\right\}\right| \\
&\quad \ge \frac{1}{g(\lambda_{pqj})}\left|\left\{(m,n,k) \in I_{pqj} : \left|L^{\beta}_{rst,uvw}(f,x) - (f,x)\right| \ge r + \epsilon\right\}\right| \\
&\quad \ge M\frac{1}{g(\lambda_{pqj})}\left|\left\{(m,n,k) \in I_{pqj} : \left|L^{\beta}_{rst,uvw}(f,x) - (f,x)\right| \ge r + \epsilon\right\}\right|.
\end{aligned}$$

For $\delta > 0$,

$$\begin{aligned}
&\left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})}\left|\left\{(m,n,k) \in I_{pqj} :\right.\right.\right. \\
&\qquad\qquad \left.\left.\left.\left|L^{\beta}_{rst,uvw}(f,x) - (f,x)\right| \ge r + \epsilon\right\}\right| \ge \delta\right\} \\
&\quad \subset \left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(pqj)}\left|\left\{(m,n,k) \in I_{pqj} :\right.\right.\right. \\
&\qquad\qquad \left.\left.\left.\left|L^{\beta}_{rst,uvw}(f,x) - (f,x)\right| \ge r + \epsilon\right\}\right| \ge M\delta\right\} \in I,
\end{aligned}$$

since I is admissible ideal of weight g. □

THEOREM 3.3. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein Stancu Cheney and Sharma operators of $\left(L^{\beta}_{rst,uvw}(f,x)\right)$ of real numbers, if $\lambda \in \eta$ be such that $\lim_{pqj} \frac{(pqj)-\lambda_{pqj}}{g(pqj)} = 0$ then $(I_\lambda)^g \subset (I)^g$.*

Proof. Let $\delta > 0$ be given. Since $\lim_{p,q,j} \frac{(pqj)-\lambda_{pqj}}{g(pqj)} = 0$, we can choose $(u,v,w) \in \mathbb{N}^3$ such that $\lim_{p,q,j} \frac{(pqj)-\lambda_{pqj}}{g(pqj)} < \frac{\delta}{2}$, for all $p \ge u$, $q \ge v$, $j \ge w$. Now observe that, for $\epsilon > 0$

$$\begin{aligned}
&\frac{1}{g(\lambda_{pqj})}\left|\left\{(m,n,k) \le (p,q,j) : \left|L^{\beta}_{rst,uvw}(f,x) - (f,x)\right| \ge r + \epsilon\right\}\right| \\
&= \frac{1}{g(\lambda_{pqj})}\left|\left\{(m,n,k) \le (p,q,j) - \lambda_{pqj} : \left|L^{\beta}_{rst,uvw}(f,x) - (f,x)\right| \ge r + \epsilon\right\}\right| +
\end{aligned}$$

$$\frac{1}{g\left(\lambda_{pqj}\right)}\left|\left\{(m,n,k)\in I_{pqj}:\left|L^{\beta}_{rst,uvw}\left(f,x\right)-\left(f,x\right)\right|\geq r+\epsilon\right\}\right|$$
$$\leq\frac{(p,q,j)-\lambda_{pqj}}{g(pqj)}+\frac{1}{g(\lambda_{pqj})}\left|\left\{(m,n,k)\in I_{pqj}:\left|L^{\beta}_{rst,uvw}(f,x)-(f,x)\right|\geq r+\epsilon\right\}\right|$$
$$\leq\frac{\delta}{2}+\frac{1}{g\left(\lambda_{pqj}\right)}\left|\left\{(m,n,k)\in I_{pqj}:\left|L^{\beta}_{rst,uvw}\left(f,x\right)-\left(f,x\right)\right|\geq r+\epsilon\right\}\right|,$$

for all $(p,q,j)\geq(u,v,w)$. Hence

$$\left\{(p,q,j)\in\mathbb{N}:\frac{1}{g\left(\lambda_{pqj}\right)}\left|\left\{(m,n,k)\leq(p,q,j)\in I_{pqj}:\right.\right.\right.$$
$$\left.\left.\left.\left|L^{\beta}_{rst,uvw}\left(f,x\right)-\left(f,x\right)\right|\geq r+\epsilon\right\}\right|\geq\delta\right\}$$

$$\subset\left\{(p,q,j)\in\mathbb{N}:\frac{1}{g\left(\lambda_{pqj}\right)}\left|\left\{(m,n,k)\in I_{pqj}:\right.\right.\right.$$
$$\left.\left.\left.\left|L^{\beta}_{rst,uvw}\left(f,x\right)-\left(f,x\right)\right|\geq r+\epsilon\right\}\right|\geq\frac{\delta}{2}\right\}\bigcup\left\{1,2,3,\ldots,(u,v,w)\right\}.$$

If $(I_\lambda)^g-\lim L^{\beta}_{rst,uvw}\left(f,x\right)=(f,x)\in I$. Hence a triple sequence of Bernstein Stancu Cheney and Sharma operators of $L^{\beta}_{rst,uvw}\left(f,x\right)$ is I statistically convergent of weight g to $f\left(x\right)$. □

Theorem 3.4. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein Stancu Cheney and Sharma operators of $\left(L^{\beta}_{rst,uvw}\left(f,x\right)\right)$ of real numbers, $g_1,g_2\in G$ and let $\lambda=(\lambda_{pqj})$, $\mu=(\mu_{rst})$ be two sequences in η such that $\lambda_{pqj}\leq\mu_{pqj}$ for all $p,q,j\in\mathbb{N}^3$. If*

$$\liminf_{p,q,j}\frac{g_2\left(\lambda_{pqj}\right)}{g_1\left(\mu_{pqj}\right)}>0 \tag{1}$$

then $(I_\mu)^{g_2}\subset(I_\lambda)^{g_1}$.

Proof. Suppose that $\lambda_{pqj}\leq\mu_{pqj}$ for all $(p,q,j)\in\mathbb{N}^3$ and let (1) be satisfied. For given $\epsilon>0$ we have

$$\left\{(m,n,k)\in J_{pqj}:\left|L^{\beta}_{rst,uvw}\left(f,x\right)-\left(f,x\right)\right|\geq r+\epsilon\right\}$$
$$\supseteq\left\{(m,n,k)\in I_{pqj}:\left|L^{\beta}_{rst,uvw}\left(f,x\right)-\left(f,x\right)\right|\geq r+\epsilon\right\}$$

where $I_{pqj}=[(pqj)-\lambda_{pqj}+1,(pqj)]$ and $J_{pqj}=[(pqj)-\mu_{pqj}+1,(pqj)]$. Therefore we can write

$$\frac{1}{g_2\left(\mu_{pqj}\right)}\left|\left\{(m,n,k)\in J_{pqj}:\left|L^{\beta}_{rst,uvw}\left(f,x\right)-\left(f,x\right)\right|\geq r+\epsilon\right\}\right|$$
$$\geq\frac{g_1\left(\lambda_{pqj}\right)}{g_2\left(\mu_{pqj}\right)}\frac{1}{g_2\left(\lambda_{pqj}\right)}\left|\left\{(m,n,k)\in I_{pqj}:\left|L^{\beta}_{rst,uvw}\left(f,x\right)-\left(f,x\right)\right|\geq r+\epsilon\right\}\right|,$$

for all $p, q, j \in \mathbb{N}$ we have

$$\Big\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g_1(\lambda_{pqj})} \Big|\Big\{(m,n,k) \in I_{pqj} : \Big|L^{\beta}_{rst,uvw}(f,x) - (f,x)\Big| \geq r+\epsilon\Big\}\Big| \geq \delta\Big\}$$

$$\subseteq \Big\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g_2(\lambda_{pqj})} \Big|\Big\{(m,n,k) \leq (p,q,j) \in J_{pqj} : \Big|L^{\beta}_{rst,uvw}(f,x) - (f,x)\Big| \geq r+\epsilon\Big\}\Big| \geq \delta \frac{g_2(\lambda_{pqj})}{g_1(\mu_{pqj})}\Big\} \in I.$$

Hence $(I_\mu)^{g_2} \subset (I_\lambda)^{g_1}$. ☐

THEOREM 3.5. *Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein Stancu Cheney and Sharma operators of $\left(L^{\beta}_{rst,uvw}(f,x)\right)$ of real numbers, if $\{\lambda_{pqj}\} \in \eta$ then $L^{\beta}_{rst,uvw}(f,x) \to (f,x)[V,\lambda](I)^g \Longrightarrow L^{\beta}_{rst,uvw}(f,x) \to (f,x)(I_\lambda)^g$ is a proper ideal of I.*

Proof. Let $\epsilon > 0$ and $L^{\beta}_{rst,uvw}(f,x) \to (f,x)[V,\lambda](I)^g$, we have

$$\sum_{(m,n,k)\in I_{pqj}} \Big|L^{\beta}_{rst,uvw}(f,x) - (f,x)\Big| \geq \sum_{(m,n,k)\in I_{pqj} \text{ and } \left|L^{\beta}_{rst,uvw}(f,x)-(f,x)\right| > r+\epsilon} \Big|L^{\beta}_{rst,uvw}(f,x) - (f,x)\Big|$$
$$\geq r+\epsilon \cdot \Big|\Big\{(m,n,k) \in I_{pqj} : \Big|L^{\beta}_{rst,uvw}(f,x) - (f,x)\Big| \geq r+\epsilon\Big\}\Big|.$$

Given $\delta > 0$,

$$\frac{1}{g(\lambda_{pqj})} \Big|\Big\{(m,n,k) \in I_{pqj} : \Big|L^{\beta}_{rst,uvw}(f,x) - (f,x)\Big| \geq r+\epsilon\Big\}\Big| \geq \delta$$
$$\Longrightarrow \frac{1}{g(\lambda_{pqj})} \sum_{(m,n,k)\in I_{pqj}} \Big|L^{\beta}_{rst,uvw}(f,x) - (f,x)\Big| \geq \epsilon\delta.$$

i.e., $$\Big\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} \Big|\Big\{(m,n,k) \in I_{pqj} : \Big|L^{\beta}_{rst,uvw}(f,x) - (f,x)\Big| \geq r+\epsilon\Big\}\Big| \geq \delta\Big\}$$
$$\subset \left\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})} \left\{\sum_{(m,n,k)\in I_{pqj}} \Big|L^{\beta}_{rst,uvw}(f,x) - (f,x)\Big|\right\} \geq \epsilon\delta.\right\}$$

since $L^{\beta}_{rst,uvw}(f,x) \to (f,x)[V,\lambda](I)^g$ and hence it follows that $L^{\beta}_{rst,uvw}(f,x) \to (f,x)(I_\lambda)^g$ is a proper ideal of I.

Now to prove that $(I_\lambda)^g \subsetneq [V,\lambda](I)^g$, take a fixed element $A \in I$. Define a triple sequence Bernstein Stancu Cheney and Sharma operators defined by

$$L^{\beta}_{rst,uvw}(f,x) = \begin{cases} \sqrt{g(\lambda_{pqj})}, & \begin{array}{r} \text{for } p - \left[\sqrt{g(\lambda_{pqj})}\right] + 1 \le m \le p, \\ q - \left[\sqrt{g(\lambda_{pqj})}\right] + 1 \le n \le q, \\ j - \left[\sqrt{g(\lambda_{pqj})}\right] + 1 \le k \le j; (p,q,j) \notin A \end{array} \\ \\ 0, & \begin{array}{r} \text{for } p - \left[\sqrt{g(\lambda_{pqj})}\right] + 1 \le m \le p, \\ q - \left[\sqrt{g(\lambda_{pqj})}\right] + 1 \le n \le q, \\ j - \left[\sqrt{g(\lambda_{pqj})}\right] + 1 \le k \le j; (p,q,j) \in A \end{array} \end{cases}$$

for every $\epsilon > 0$ $(0 < \epsilon < 1)$. Then

$$\frac{1}{g(\lambda_{pqj})}\left|\left\{(m,n,k) \in I_{pqj} : \left|L^{\beta}_{rst,uvw}(f,x)\right| \ge r + \epsilon\right\}\right| = \frac{\left[\sqrt{g(\lambda_{pqj})}\right]}{g(\lambda_{pqj})} \to 0$$

as $p,q,j \to \infty$ and $(p,q,j) \notin A$, for every $\delta > 0$,

$$\begin{aligned} \Bigg\{(p,q,j) \in \mathbb{N}^3 : \frac{1}{g(\lambda_{pqj})}\Bigg|\Bigg\{(m,n,k) \in I_{pqj} :& \\ \left|L^{\beta}_{rst,uvw}(f,x) - (f,x)\right| \ge r + \epsilon\Bigg\}\Bigg| \ge \delta\Bigg\}& \\ \subset A \bigcup \left\{(1,1,1),(1,2,1),\ldots,(u,v,w)\right\}& \end{aligned}$$

for some $(u,v,w) \in \mathbb{N}^3$. Since I is admissible of weight g, it follows that $L^{\beta}_{rst,uvw}(f,x) \to 0\,(I_\lambda)^g$. Hence $\frac{1}{g(\lambda_{pqj})}\sum_{(m,n,k)\in I_{pqj}}\left|L^{\beta}_{rst,uvw}(f,x)\right| \to \infty$ as $p,q,j \to \infty$, i.e. $L^{\beta}_{rst,uvw}(f,x) \not\to 0\,[V,\lambda](I)^g$, if $A \in I$ is infinite then $L^{\beta}_{rst,uvw}(f,x) \not\to \theta\,(I_\lambda)^g$. □

REFERENCES

[1] S. Aytar, *Rough statistical convergence*, Numer. Funct. Anal. Optim., **29** (2008), 291–303.

[2] S. Aytar, *The rough limit set and the core of a real sequence*, Numer. Funct. Anal. Optim., **29** (2008), 283–290.

[3] A. Esi, *On some triple almost lacunary sequence spaces defined by Orlicz functions*, Research and Reviews: Discrete Mathematical Structures, **1** (2014), 16–25.

[4] A. Esi and M. Necdet Catalbas, *Almost convergence of triple sequences*, Global Journal of Mathematical Analysis, **2** (2014), 6–10.

[5] A. Esi and E. Savas, *On lacunary statistically convergent triple sequences in probabilistic normed space*, Appl. Math. Inf. Sci., **9** (2015), 2529–2534.

[6] A. Esi, S. Araci and M. Acikgoz, *Statistical convergence of Bernstein operators*, Appl. Math. Inf. Sci., **10** (2016), 2083–2086.

[7] A. Esi, N. Subramanian and A. Esi, *On triple sequence space of Bernstein-Stancu operator of rough I_λ statistical convergence of weighted $g(A)$*, Ann. Fuzzy Math. Inform., **16** (2018), 337–361.

[8] S. Debnath, B. Sarma and B.C. Das, *Some generalized triple sequence spaces of real numbers*, J. Nonlinear Anal. Optim., **6** (2015), 71–79.

[9] A.J. Dutta, A. Esi and B.C. Tripathy, *Statistically convergent triple sequence spaces defined by Orlicz function*, J. Math. Anal., **4** (2013), 16–22.

[10] E. Dündar and C. Cakan, *Rough I-convergence*, Demonstr. Math., **47** (2014), 638–651.

[11] M. Jeyam Bharathi, S. Velmurugan, N. Subramanian and R. Srikanth, *On triple sequence space of Bernstein operator of rough I_λ-statistical convergence of weighted $g\,(A)$*, Journal of intelligent and fuzzy systems, **36** (2019), 13–27.

[12] H.X. Phu, *Rough convergence in normed linear spaces*, Numer. Funct. Anal. Optim., **22** (2001), 199–222.

[13] H.X. Phu, *Rough continuity of linear operators*, Numer. Funct. Anal. Optim., **23** (2002), 139–146.

[14] H.X. Phu, *Rough convergence in infinite dimensional normed spaces*, Numer. Funct. Anal. Optim., **24** (2003), 285–301.

[15] A. Sahiner, M. Gurdal and F.K. Duden, *Triple sequences and their statistical convergence*, Selçuk J. Appl. Math., **8** (2007), 49–55.

[16] A. Sahiner and B. C. Tripathy, *Some I-related properties of triple sequences*, Selçuk J. Appl. Math., **9** (2008), 9–18.

[17] N. Subramanian and A. Esi, *The generalized tripled difference of χ^3 sequence spaces*, Global Journal of Mathematical Analysis, **3** (2015), 54–60.

[18] N. Subramanian, A. Esi and A. Indumathi, *Stancu type of Cheney and Sharma operators of Pascal rough triple sequences*, Communications in Nonlinear Analysis, **6** (2019), 64–77.

Received May 13, 2019
Accepted November 7, 2019

SASTRA Deemed to be University
School of Arts Sciences and Humanities
Department of Mathematics
Thanjavur-613 401, India
E-mail: nsmaths@yahoo.com
https://orcid.org/0000-0003-3249-6525

Malatya Turgut Ozal University
Engineering Faculty
Department of Basic Engineering Sciences
44040 Malatya, Turkey
E-mail: aesi23@hotmail.com
https://orcid.org/0000-0003-3137-3865

SASTRA Deemed to be University
School of Arts Sciences and Humanities
Department of Mathematics
Thanjavur-613 401, India
E-mail: nsmaths@gmail.com
https://orcid.org/0000-0002-5895-673X