MATHEMATICA, 63 (86), N° 1, 2021, pp. 85–97

ON THE STABILITY OF 2D GENERAL ROESSER LYAPUNOV SYSTEMS

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Abstract. This paper addresses the problem of stability for general two-dimensional (2D) discrete-time and continuous-discrete time Lyapunov systems, where the linear matrix inequalities (LMI's) approach is applied to derive a new sufficient condition for the asymptotic stability.

MSC 2010. 31Axx, 34Dxx, 34Exx, 34G10, 34H05, 93-xx, 93D20.

Key words. 2D general systems, Lyapunov systems, Roesser system, stability, LMI.

1. INTRODUCTION

The most popular models of two-dimensional linear systems are the models introduced by Roesser [29], Fornasini-Marchecini [7, 8]. They have been generalized for singular 2D models by Kurek [24], Kaczorek in [15, 16, 17, 18, 20] and has many applications in control theory, modern circuit design and digital image processing, iterative learning, control synthesis or repetitive processes, image processing, seismological and geographical data processing, power transmission lines etc. The stability test is the most important and fundamental problem for design and analysis of systems. A number of stability test of 2D systems has been studied. Thus internal stability and asymptotic behavior of 2D linear models were investigated by Valcher [32] and asymptotic stability of 2D linear systems was investigated in [6, 12, 13, 19, 25, 30, 34]. An LMI approach to checking stability of 2D systems was proposed by Twardy [31], with generalizations to 2D positive systems by delays in Kaczorek [21]. In [34] sufficient LMI conditions for the internal stability of 2D singular linear systems

This paper presents research results of the ACSY-Team (Analysis & Control systems team) with Laboratory of Pure and Applied Mathematics (LMPA) and of the doctoral training on the Operational Research and Decision Support funded by the General Directorate for Scientific Research and Technological Development of Algeria (DGRSDT) and supported by University of Mostaganem Abdelhamid Ibn Badis (UMAB) and initiated by the concerted research project on Control and Systems theory (PRFU Project Code C00L03UN270120200003).

DOI: 10.24193/mathcluj.2021.1.08

with respect to acceptability and jump modes were given. Another LMI approach for the stability of the 2D state-space singular models was investigated by Bouagada and Van Dooren in [3, 4], and in [1, 2, 5, 11, 33, 34].

In the last few years, a new class of 1D and 2D discrete-time and continuous-time Lyapunov linear systems has been introduced. In the 2D Lyapunov systems described by the Roesser model, the independent variables are discrete and/or continuous and propagating in two different directions. Such models appear for example in circuit design, X-ray image enhancement. Thus, the controllability and observability of the Lyapunov systems were treated in Murty and Apparao [26]. The positive 1D and 2D discrete-time and continuous-time Lyapunov systems in [22, 27]. Discrete-time and continuous-time Lyapunov cone-systems were considered by [23] and [28].

In this work, the new general 2D discrete-time singular Lyapunov systems and also the 2D continuous-discrete-time singular Lyapunov systems are considered. The main purpose of this paper is to present a sufficient condition for the asymptotic stability test in terms of linear matrix inequalities (LMIs). An LMI approach is used to produce highly significant new results on the stability analysis of these processes and to design the control schemes for these models.

2. PRELIMINARIES

We denote by $\mathbb{R}^{m\times n}$, ($\mathbb{C}^{m\times n}$), the set of real (complex) matrices with m rows and n columns and by $\mathbb{R}^m$, ($\mathbb{C}^m$), the set of real (complex) vectors. Also, $\mathbb{Z}_+$ denotes the positive integers and $\mathbb{R}_+$ the positive real line. Some of the following definitions and results can be found in [15].

Definition 2.1. Let the matrices $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{q\times p}$ such that the Kronecker product $A \otimes B$ of matrices A and B is the block matrix

$$A \otimes B = [a_{ij}B] \in \mathbb{R}^{mq\times np}$$

for all $i = 1, ..., m$ and $j = 1, .., n$. The matrix $A \otimes B$ is $(mq \times np)$ matrix with (mn) blocks $[a_{ij}B]$ of order (pq).

Here, we will mention some properties and rules for the Kronecker product. Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{q\times p}$, than the following properties hold:

$$(A \otimes B)^T = A^T \otimes B^T \tag{1}$$

$$rank(A \otimes B) = rank(A) \cdot rank(B). \tag{2}$$

Definition 2.2. With each matrix $A = [a_{ij}] \in \mathbb{R}^{m\times n}$, we associate the vector $vec(A) \in \mathbb{R}^{mn}$ defined by

$$vec(A) = [a_{11}, \dots, a_{m1}, a_{12}, \dots, a_{m2}, \dots, a_{1n}, \dots, a_{mn}]^T$$

(a column vector of each sequential column is stacked on top of one other).

Theorem 2.3. *Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{q\times p}$, $C \in \mathbb{R}^{m\times p}$ be given and let $X \in \mathbb{R}^{n\times q}$ unknown. Then, using the Kronecker product of the matrices A* *and B^T, we can transform the matrix equation $AXB = C$ into linear system of nq equations in np unknowns given by*

$$\left(A \otimes B^T\right) vec(X) = vec(C) \tag{3}$$

We refer the reader to [14] for the proof.

3. THE GENERAL 2D DISCRETE-TIME LYAPUNOV SYSTEM

Now, we will introduce the definition of the model proposed for study and some results concerning those systems established in [3, 4, 31].

Consider now the system

$$E\begin{bmatrix} X^h(i+1,j) \\ X^v(i,j+1) \end{bmatrix} = A_0 \begin{bmatrix} X^h(i,j) \\ X^v(i,j) \end{bmatrix} + \begin{bmatrix} X^h(i,j) \\ X^v(i,j) \end{bmatrix} A_1 + BU(i,j)$$

$$Y(i,j) = CX(i,j) + DU(i,j), \text{ where } i,j \in \mathbb{Z}_+$$

and $X^h(i,j) \in \mathbb{R}^{n_1 \times n}$ and $X^v(i,j) \in \mathbb{R}^{n_2 \times n}$ represent the horizontal and the vertical state matrix at the point (i,j) with $n = n_1 + n_2$. $U(i,j) \in \mathbb{R}^{m \times n}$ is the input matrix and $Y(i,j) \in \mathbb{R}^{p \times n}$ is the output matrix at the point (i,j). $E \in \mathbb{R}^{n \times n}$, $A_0, A_1 \in \mathbb{R}^{n \times n}$ $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$ and $D \in \mathbb{R}^{p \times m}$. The system (3) is called the general 2D discrete-time Lyapunov system described by the Roesser model.

Applying Theorem 2.3 and property (1), then the general Lyapunov system 3 is transformed into the following equivalent 2D discrete-time general Roesser model (2D-GRM).

$$\hat{E}\begin{bmatrix} \hat{X}^h(i+1,j) \\ \hat{X}^v(i,j+1) \end{bmatrix} = \hat{A}\begin{bmatrix} \hat{X}^h(i,j) \\ \hat{X}^v(i,j) \end{bmatrix} + \hat{B}\hat{U}(i,j) \tag{4}$$

$$\hat{Y}(i,j) = \hat{C}\hat{X}(i,j) + \hat{D}\hat{U}(i,j), \; \textit{where } i,j \in \mathbb{Z}_+, \tag{5}$$

where $\hat{X}^h(i,j) \in \mathbb{R}^{n.n_1}$; $\hat{X}^v(i,j) \in \mathbb{R}^{n.n_2}$ are respectively the horizontal and vertical state vectors; $\hat{U}(i,j) \in \mathbb{R}^{m.n}$ is the input vector, $\hat{Y}(i,j) \in \mathbb{R}^{n.p}$ is the output vector and $\hat{E} \in \mathbb{R}^{(n^2)\times(n^2)}$; $\hat{A} \in \mathbb{R}^{(n^2)\times(n^2)}$; $\hat{B} \in \mathbb{R}^{(n^2)\times(m.n)}$; $\hat{C} \in \mathbb{R}^{(p.n)\times(n^2)}$; $\hat{D} \in \mathbb{R}^{(p.n)\times(m.n)}$, and defined by the following relations:

$$\begin{aligned} \hat{E} &= E \otimes I_n \\ \hat{A} &= (A_0 \otimes I_n) + (I_n \otimes A_1^T) \\ \hat{B} &= B \otimes I_n \\ \hat{C} &= C \otimes I_n \\ \hat{D} &= D \otimes I_n \end{aligned} \tag{6}$$

3.1. STABILITY OF GENERAL 2D LYAPUNOV SYSTEM

Let us first introduce the notion of asymptotic stability of the 2D Lyapunov-GRM and the general 2D Lyapunov system.

DEFINITION 3.1. Consider the system (4)–(5) with a zero input (i.e., $\hat{U}(i,j) = 0$ for $i \geq 0, j \geq 0$), so we call the autonomous system of the system (4)–(5) the following one

$$\hat{E}\begin{bmatrix} \hat{X}^h(i+1,j) \\ \hat{X}^v(i,j+1) \end{bmatrix} = \hat{A}\begin{bmatrix} \hat{X}^h(i,j) \\ \hat{X}^v(i,j) \end{bmatrix} \tag{7}$$

$$\hat{Y}(i,j) = \hat{C}\hat{X}(i,j). \tag{8}$$

DEFINITION 3.2. The general 2D GRM system (7) is asymptotically stable if the zero input response with any bounded boundary conditions satisfying $\sup_i \left\|\hat{X}(i,0)\right\| < \infty$ and $\sup_j \left\|\hat{X}(0,j)\right\| < \infty$, converges to zero, i.e., $\lim_{i,j\to\infty} \hat{X}(i,j) = 0$.

It is shown [33, 34] that the characteristic polynomial of the 2D-GRM (4) is defined by

$$\begin{aligned} H(z_1,z_2) &= \det[\hat{E}\ \mathrm{diag}(z_1 I_{n.n_1}, z_2 I_{n.n_2}) - \hat{A}] \\ &= \sum_{k=0}^{\bar{n}_1}\sum_{l=0}^{\bar{n}_2} a_{kl} z_1^k z_2^k, \end{aligned} \tag{9}$$

where $0 \leq \bar{n}_k \leq n.n_k$ for $k = 1,2$ and

$$\mathrm{diag}(z_1 I_{n.n_1}, z_2 I_{n.n_2}) = \begin{bmatrix} z_1 I_{n.n_1} & 0 \\ 0 & z_2 I_{n.n_2} \end{bmatrix}. \tag{10}$$

We assume that $a_{\bar{n}_1,\bar{n}_2} \neq 0$, which guaranties the acceptability of the system (4) (see [34]) and that the system (4) is free of jump [34] and causal, which is guaranteed by the following relation: $\deg\ \det[s\hat{E} - \hat{A}] = \mathrm{rank}\hat{E} = \mathrm{rank}(E \otimes I_n) = n\mathrm{rank}E$. Based on [4, 13, 30], we adapt the following necessary and sufficient conditions of stability for the 2D-GRM in terms of the characteristic polynomial in the following results.

THEOREM 3.3. *The general* 2*D Lyapunov system* (3) *is asymptotically stable if and only if* $H(z_1,z_2) \neq 0$ *for every pair* (z_1,z_2) *such that* $|z_1| \leq 1$ *and* $|z_2| \leq 1$*, where* $H(z_1,z_2)$ *is defined by the relations* (9)*.*

REMARK 3.4. Note that some authors prefer representing the delay operator of discrete-time systems by z^{-1} rather than z, which explains why different forms of conditions are found in the literature. Those results can easily been adapted when passing from one convention to another.

The condition of some theorems implies checking the non-singularity of a matrix of two variables in a connected 2D domain. A smart result provided first in [13] and [12], and proved later on in [6], shows that this can be reduced to testing two simpler conditions.

THEOREM 3.5. *The general* 2*D discrete system* (3) *is asymptotically stable if and only if*

$$H(z_1, 0) \neq 0 \, for \; |z_1| \leq 1, \tag{11}$$

$$H(z_1, z_2) \neq 0 \, for \; |z_1| = 1 \, and \; |z_2| \leq 1 \tag{12}$$

Proof. This theorem was incorrectly proved in [12], [13] and [30]. Corrected proofs appeared later in [6] and [25]. All proofs are based on the fact that the functions with z_1 and z_2 via $H(z_1, z_2) = 0$ are algebraic functions. □

In the next section we transform these conditions into equivalent LMI conditions, which can be checked in polynomial time.

3.2. LMI CONDITIONS FOR STABILITY TEST FOR 2D DISCRETE-TIME SYSTEMS

In order to reduce the above to an LMI formulation, we will need the following theorem, used in [4] to characterize positive polynomial matrices that depend on a real parameter ω on the unit circle.

THEOREM 3.6. *A hermitian polynomial matrix* $P(z) = \sum_{i=0}^{2} P_i z^i$ *with respect to* $P_{-i} = P_i^*$ *is positive definite on unit circle if and only if there exists a hermitian matrix* X *such that*

$$\begin{bmatrix} P_0 - X & P_1 \\ P_1^* & X \end{bmatrix} > 0.$$

The following main theorem gives a sufficient condition for the asymptotic stability of the general 2D Lyapunov system.

THEOREM 3.7. *The general* 2*D Lyapunov system* (3) *is asymptotically stable if there exist hermitian matrices* X_0, X_1, X_2 *with* $X_0 \geq 0, X_1 \geq 0$ *and* $X_2 \geq 0$ *satisfying the following LMI*

$$(A_0 \otimes I_n + I_n \otimes A_1^T)^T X_1 (A_0 \otimes I_n + I_n \otimes A_1^T) - \bar{E}_{1,0}^T X_1 \bar{E}_{1,0} > 0 \tag{13}$$

$$\begin{bmatrix} (A_0 \otimes I_n + I_n \otimes A_1^T)^T X_2 (A_0 \otimes I_n + I_n \otimes A_1^T) - X_0 \\ -\bar{E}_{1,0}^T X_2 (A_0 \otimes I_n + I_n \otimes A_1^T) \end{bmatrix.$$
$$\left. \begin{matrix} -(A_0 \otimes I_n + I_n \otimes A_1^T)^T X_2 \bar{E}_{1,0}) \\ X_0 + \bar{E}_{1,0}^T X_2 \bar{E}_{1,0} + \bar{E}_{0,1}^T X_2 \bar{E}_{0,1} \end{matrix} \right] > 0 \tag{14}$$

with

$$\bar{E}_{k,l} = (E \otimes I_n).\mathrm{diag}(k.I_{n.n_1}, l.I_{n.n_2}) \in \mathbb{R}^{n^2 \times n^2} \text{for } k, l = 0, 1. \tag{15}$$

Proof. The characteristic polynomial for the 2D-GRM is defined by

$$H(z_1, z_2) = \det\left[\hat{E}\ \mathrm{diag}(z_1 I_{n.n_1}, z_2 I_{n.n_2}) - \hat{A}\right]$$

with $\hat{E} = E \otimes I_n$ and $\hat{A} = A_0 \otimes I_n + I_n \otimes A_0^T$. Condition (11) on the characteristic polynomial leads to

$$H(z_1, 0) = \det\left[z_1 \hat{E}\ \mathrm{diag}(I_{n.n_1}, 0) - \hat{A}\right]\ for\,|z_1| \leq 1 \tag{16}$$

That is satisfied if and only if the following LMI yields

$$\hat{A}^T X_1 \hat{A} - \mathrm{diag}(I_{n.n_1}, 0)\hat{E}^T X_1 \hat{E}\ \mathrm{diag}(I_{n.n_1}, 0),\ X_1 > 0,\ X_1^* = X_1. \tag{17}$$

Substituting $\hat{A}$, $\hat{E}$ and $\bar{E}_{1,0}$ in the relation (17), we obtain the LMI (13). The condition (12) expresses that for all $\omega \in \mathbb{R}$ and $|z_2| \leq 1$ we have

$$H(e^{j\omega}, z_2) = \det[z_2 \hat{E}\ \mathrm{diag}(0, I_{n.n_2}) + e^{j\omega}\hat{E}\ \mathrm{diag}(I_{n.n_1}, 0) - \hat{A}] \tag{18}$$

for $|z_2| \leq 1$. This is equivalent to $\det(z_2 V - W) \neq 0$ with $V = \hat{E}\mathrm{diag}(0, I_{n.n_2})$ and $W = e^{j\omega}\hat{E}\ \mathrm{diag}(I_{n.n_1}, 0) - \hat{A}$ which holds if and only if the following LMI is feasible $W^* X_2 W - V^* X_2 V > 0,\ X_2^* = X_2,\ X_2 > 0$, where X_2 will depend on ω. If we impose that X_2 to be constant then the relation (18) is equivalent to the following $e^{j\omega}P_1 + e^{-j\omega}P_1^* + P_0 > 0$, where $P_1 = -\hat{A}X_2\hat{E}\ \mathrm{diag}(I_{n.n_1}0)$,

$$\begin{aligned} P_0 \quad = \quad & -\hat{A}X_2\hat{A} + \mathrm{diag}(I_{n.n_1}, 0)\hat{E}^T X_2\ \hat{E}\mathrm{diag}(I_{n.n_1}, 0) \\ & -\mathrm{diag}(0, I_{n.n_2})\hat{E}^T X_2\ \hat{E}\mathrm{diag}(0, I_{n.n_2}). \end{aligned}$$

Note that $P_j^* = P_{-j}$. Applying theorem (3.6) then yields the following condition: $\begin{bmatrix} P_0 - X & P_1 \\ P_1^* & X \end{bmatrix} > 0$, for some hermitian matrix X. Let us now define a new hermitian matrix X_0 by

$$\begin{aligned} X = X_0 + \mathrm{diag}(I_{n.n_1}, 0)\hat{E}^T X_2\ \hat{E}\mathrm{diag}(I_{n.n_1}, 0) \\ + \mathrm{diag}(0, I_{n.n_2})\hat{E}^T X_2\ \hat{E}\mathrm{diag}(0, I_{n.n_2}). \end{aligned}$$

So, it is clear that $X_0^* = X_0$ and that $P_0 - X = \hat{A}X_2\hat{A} - X_0$. Substituting $\hat{E}$, $\hat{A}$ and $\bar{E}_{k,l}$ we obtain LMI (14). □

4. THE GENERAL 2D CONTINUOUS-DISCRETE TIME LYAPUNOV SYSTEMS

4.1. STABILITY OF THE GENERAL 2D CONTINUOUS-DISCRETE TIME LYAPUNOV SYSTEM

Here we consider the general 2D continuous-discrete time Lyapunov system, defined with a case similar to that in the previous section.

Definition 4.1. The general 2D continuous-discrete time Lyapunov system described by the Roesser model is defined by the equations

$$E\begin{bmatrix}\dot{X}^h(t,i)\\ X^v(t,i+1)\end{bmatrix}=A_0\begin{bmatrix}X^h(t,i)\\ X^v(t,i)\end{bmatrix}+\begin{bmatrix}X^h(t,i)\\ X^v(t,i)\end{bmatrix}A_1+BU(t,i) \tag{19}$$

$$Y(t,i)=CX(t,i)+DU(t,i) \tag{20}$$

for $t\in\mathbb{R}$, $i\in\mathbb{Z}_+$ with $\dot{X}^h(t,i)=\frac{\partial X^h(t,i)}{\partial t}$. $X^h(t,i)\in\mathbb{R}^{n_1\times n}$ and $X^v(t,i)\in\mathbb{R}^{n_2\times n}$ represent the horizontal and the vertical state matrices at the point (t,i), and $n=n_1+n_2$. Matrices $E,\ A_0,\ A_1,\ B,\ C,\ D$ have the same dimensions as in the previous section.

By the same analogy shown previously, we transform the model (19)–(20) into its equivalent 2D-GRM.

For the general 2D Lyapunov system (19) the vectorization form yields the equivalent 2D-GRM

$$\hat{E}\begin{bmatrix}\dot{\hat{X}}^h(t,i)\\ \hat{X}^v(t,i+1)\end{bmatrix}=\hat{A}\begin{bmatrix}\hat{X}^h(t,i)\\ \hat{X}^v(t,i)\end{bmatrix}+\hat{B}\hat{U}(t,i) \tag{21}$$

$$\hat{Y}(t,i)=\hat{C}\hat{X}(t,i)+\hat{D}\hat{U}(t,i),$$

with $t\in\mathbb{R}$ and $i\in\mathbb{Z}_+$, and the matrices $\hat{E},\ \hat{A},\ \hat{B},\ \hat{C},\ \hat{D}$ have the same forms as in (6).

In this case, the characteristic polynomial for the system (21) is defined by the relation $H(s,z)=\det[\hat{E}\ \mathrm{diag}(sI_{n.n_1},zI_{n.n_2})-\hat{A}]=\sum_{k=0}^{\bar{n}_1}\sum_{l=0}^{\bar{n}_2}a_{kl}s^kz^l$, where $0\le\bar{n}_k\le n.n_k$ for $k=1,2$.

We also assume that $a_{\bar{n}_1,\bar{n}_2}\neq 0$ which guaranties the acceptability of the system (21) (see [34]), and that the system (21) is free of jump [34] and causal, which is guaranteed by the following relation: $\deg\det[s\hat{E}-\hat{A}]=\mathrm{rank}\hat{E}=\mathrm{rank}(E\otimes I_n)=n\mathrm{rank}(E)$.

Definition 4.2. The 2D-GRM (21) is called asymptotically stable if, for any bounded boundary conditions $\hat{X}(t,0)\in\mathbb{R}_+^{n^2}$, $t\in\mathbb{R}$, $\hat{X}(0,i)\in\mathbb{R}_+^{n^2}$, $i\in\mathbb{Z}_+$, we have $\lim_{t,i\to\infty}\hat{X}(t,i)=0$.

To introduce an LMI condition for the general 2D Lyapunov system (21), we have to apply the following result derived in [13] and [30].

Theorem 4.3. *The continuous-discrete* 2D*-GRM* (21) *is asymptotically stable if and only if*

$$H(s,0)\neq 0\ for\ \Re(s)\ge 0 \tag{22}$$

$$H(s,z)\neq 0\ for\ \Re(s)=0\ and\ |z|\le 1 \tag{23}$$

THEOREM 4.4 ([4]). *A hermitian polynomial matrix* $P(\omega) = \sum_{i=0}^{2} P_i\omega^i$ *with* $P_i = P_i^*$ *is positive definite on* $\omega \in \mathbb{R}$ *if and only if there exists a hermitian matrix* X *such that*

$$\begin{bmatrix} P_0 & (P_1 - jX)/2 \\ (P_1 + jX)/2 & P_2 \end{bmatrix} > 0.$$

4.2. LMI CONDITIONS FOR STABILITY TEST FOR 2D CONTINUOUS-DISCRETE TIME SYSTEMS

On the basis of the above definitions and theorems, we now propose the sufficient LMI conditions for the asymptotic stability of 2D models described in (19).

THEOREM 4.5. *The general* 2D *continous-discrete Lyapunov system* (19) *is asymptotically stable if there exist hermitian matrices* X_0, X_1, X_2 *with* $X_0 \geq 0, X_1 \geq 0,$ *and* $X_2 \geq 0$ *satisfying the following LMIs:*

$$\bar{E}_{1,0}^T X_1 (A_0 \otimes I_n + I_n \otimes A_1^T) + (A_0 \otimes I_n + I_n \otimes A_1^T)^T X_1 \bar{E}_{1,0} < 0 \tag{24}$$

$$\begin{bmatrix} (A_0 \otimes I_n + I_n \otimes A_1^T)^T X_2 (A_0 \otimes I_n \\ +I_n \otimes A_1^T) - \bar{E}_{0,1}^T X_2 \bar{E}_{1,0} \\ -(A_0 \otimes I_n + I_n \otimes A_1^T)^T X_2 \bar{E}_{1,0} + \bar{E}_{1,0}^T X_2 (A_0 \otimes I_n \\ +I_n \otimes A_1^T) + X_0 \\ \\ X_0 \\ \bar{E}_{1,0}^T X_2 \bar{E}_{1,0} \end{bmatrix} > 0 \tag{25}$$

where $\bar{E}_{k,l} = (E \otimes I_n).\mathrm{diag}(kI_{n.n_1}, lI_{n.n_2})$, $k, l = 0, 1$.

Proof. The characteristic polynomial for the system (21) is defined by

$$H(s,z) = \det \Big[\hat{E}\ \mathrm{diag}(sI_{n.n_1}, zI_{n.n_2}) - \hat{A}\Big],\ \Re(s) \geq 0. \tag{26}$$

Condition (22) implies $H(s,0) = \det \Big[s\hat{E}\ \mathrm{diag}(I_{n.n_1}, 0) - \hat{A}\Big],\ \Re(s) \geq 0.$

Applying the fact that for the continuous time we have $\det(sV - W) \neq 0$ is equivalent to $W^*XV + V^*XW > 0;\ X > 0,\ X^* = X$, so there exists $X_1 > 0$ with $X_1^* = X_1$ satisfying $\mathrm{diag}(I_{n.n_1}, 0)\hat{E}^T X_1 \hat{A} + \hat{A}^T X_1 \hat{E}\ \mathrm{diag}(I_{n.n_1}, 0) < 0$ and, by substituting $\hat{E}, \hat{A}$ and $\bar{E}_{1,0}$, the LMI (24) yields. Condition (23) leads

$$H(s,z) = \det \Big[\hat{E}\ \mathrm{diag}(sI_{n.n_1}, zI_{n.n_2}) - \hat{A}\Big],\ \Re(s) = 0,\ |z| \leq 1.$$

Putting $s = j\omega$ for $\omega \in \mathbb{R}$, we have $H(j\omega, z) \neq 0$, which is equivalent to

$$\det \Big[z\hat{E}\ \mathrm{diag}(0, I_{n.n_2}) + j\omega\hat{E}\ \mathrm{diag}(I_{n.n_1}, 0) - \hat{A}\Big] \neq 0,\ \omega \in \mathbb{R},\ |z| \leq 1. \tag{27}$$

Since that in the discrete time we have the following equivalence $\det(zV - W) \neq 0$ if and only if $W^*XW - V^*XV > 0;\ X > 0,\ X^* = X$ and putting $V = \hat{E}\ \mathrm{diag}(0, I_{n.n_2})$ and $W = \hat{A} - j\omega\hat{E}\ \mathrm{diag}(I_{n.n_1}, 0)$.

Then we have from (27) that there exists a hermitian matrix X_2 satisfying the following

$$\left[\hat{A}^T + j\omega \mathrm{diag}(I_{n.n_1}, 0)\hat{E}^T\right] X_2 \left[\hat{A} - j\omega\hat{E}\mathrm{diag}(I_{n.n_1}, 0)\right] - \left[\mathrm{diag}(0, I_{n.n_2})\hat{E}^T X_2 \hat{E}\mathrm{diag}(0, I_{n.n_1})\right] > 0.$$

So

$$\omega^2 \left[\mathrm{diag}(I_{n.n_1}, 0)\hat{E}^T X_2 \hat{E}\mathrm{diag}(I_{n.n_1}, 0)\right] + j\omega \left[\mathrm{diag}(I_{n.n_1}, 0)\hat{E}^T X_2 \hat{A} - \hat{A}^T X_2 \hat{E}\mathrm{diag}(I_{n.n_1}, 0)\right] + \left[\hat{A}^T X_2 \hat{A} - \mathrm{diag}(0, I_{n.n_2})\hat{E}^T X_2 \hat{E}\mathrm{diag}(0, I_{n.n_2})\right] > 0$$

which is of the form $P_2\omega^2 + P_1\omega + P_0 > 0$, where

$$\begin{aligned} P_2 &= \mathrm{diag}(I_{n.n_1}, 0)\hat{E}^T X_2 \hat{E}\mathrm{diag}(I_{n.n_1}, 0) \\ P_1 &= j\left[\mathrm{diag}(I_{n.n_1}, 0)\hat{E}^T X_2 \hat{A} - \hat{A}^T X_2 \hat{E}\mathrm{diag}(I_{n.n_1}, 0)\right] \\ P_0 &= \hat{A}^T X_2 \hat{A} - \mathrm{diag}(0, I_{n.n_2})\hat{E}^T X_2 \hat{E}\mathrm{diag}(0, I_{n.n_2}). \end{aligned}$$

Note that $P_k^* = P_k$ for all $k = 0, 1, 2$.

Applying Theorem 4.4 it yields that there exists a hermitian matrix X satisfying the following LMI:

$$\begin{bmatrix} P_0 & (P_1 - jX)/2 \\ (P_1 + jX)/2 & P_2 \end{bmatrix} > 0. \tag{28}$$

Let us now define a new hermitian matrix X_0 by

$$X = 2X_0 - \hat{A}^T X_2 \hat{E}\mathrm{diag}(I_{n.n_1}, 0) + \mathrm{diag}(I_{n.n_1}, 0)\hat{E}^T X_2 \hat{A}$$

(see that $X_0^* = X_0$). So, by replacing X and rewriting (28), we obtain

$$\begin{bmatrix} P_0 & -jX_0 \\ j[X_0 + \mathrm{diag}(I_{n.n_1}, 0)\hat{E}^T X_2 \hat{A} - \hat{A}^T X_2 \hat{E}\mathrm{diag}(I_{n.n_1}, 0)] & P_2 \end{bmatrix} > 0. \tag{29}$$

Multiplying the matrix in the relation left and right by a block diagonal congruence transformation matrix $\mathrm{diag}(I_{n.n_1}, jI_{n.n_2})^*$ and $\mathrm{diag}(I_{n.n_1}, jI_{n.n_2})$ and substituting the matrices $\hat{E}, \hat{A}$ and $\bar{E}_{0,1}$ by the equivalent matrices, we obtain the LMI (25). □

Remark 4.6. It is well known that for a matrix $M \in \mathbb{C}$, if M is positive definite, then for all $X \in \mathbb{C}$ the matrix X^*MX is positive semi-definite and if X is nonsingular, then X^*MX is positive definite. One needs only to observe the reversibility of the matrix $\mathrm{diag}(I_{n.n_1}, jI_{n.n_2})$.

Example 4.7. Let us consider the general 2D discrete-time Lyapunov system (3) with the following system of matrices:

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$A_0 = \begin{pmatrix} 0.0400 & -0.6800 & 0.7600 & -1.4480 & 2.2320 \\ 0.1160 & -0.7720 & 0.6840 & -0.6000 & 0.6800 \\ -0.4720 & 1.9440 & 0.5520 & -0.2800 & 0.6400 \\ -0.8400 & -0.9160 & 0.6800 & 0.1200 & 0.4000 \\ 0.4000 & 0.6000 & 0.5520 & 0.1200 & 0.6400 \end{pmatrix},$$

Using our results, we find that our LMIs, as in Theorem 3.7, are feasible, and the feasible solution can be found:

$$X_0 = \begin{pmatrix} 0.5246 & -0.0322 & -0.0172 & 0.1075 & 0.0737 \\ -0.0322 & 0.9091 & 0.0864 & 0.0563 & 0.0481 \\ -0.0172 & 0.0864 & 0.4289 & 0.0152 & 0.3407 \\ 0.1075 & 0.0563 & 0.0152 & 0.3629 & -0.2573 \\ 0.0737 & 0.0481 & 0.3407 & -0.2573 & 0.6691 \end{pmatrix},$$

$$X_1 = \begin{pmatrix} 0.3663 & -0.5618 & -0.1453 & 0.0913 & -0.0007 \\ -0.5618 & 2.0866 & 0.5943 & -0.4911 & -0.6537 \\ -0.1453 & 0.5943 & 1.1623 & -0.2827 & -0.9865 \\ 0.0913 & -0.4911 & -0.2827 & 0.5410 & 0.3693 \\ -0.0007 & -0.6537 & -0.9865 & 0.3693 & 1.9431 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0.4799 & -0.3758 & -0.0841 & -0.1598 & -0.1893 \\ -0.3758 & 1.0490 & 0.1765 & -0.0820 & -0.0601 \\ -0.0841 & 0.1765 & 0.6484 & 0.1128 & -0.4855 \\ -0.1598 & -0.0820 & 0.1128 & 0.8709 & 0.1149 \\ -0.1893 & -0.0601 & -0.4855 & 0.1149 & 1.8655 \end{pmatrix}.$$

We conclude that the system is asymptotically stable.

Example 4.8. Let us now consider the general 2D discrete-time Lyapunov system (3) with the given system matrices: $E=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_0=\begin{pmatrix} 0.2 & 0 \\ 0.1 & 0.1 \end{pmatrix}$, $A_1=\begin{pmatrix} 0.4 & 0 \\ 0.5 & 0.3 \end{pmatrix}$. Applying Theorem 3.7, we find that the $LMIs$ are feasible, and the considered system is asymptotically stable. A solution for the LMIs is as follows:

$$X_0 = \begin{pmatrix} 0.0204 & 0.0031 & 0.0705 & 0.0581 \\ 0.0031 & 0.0267 & -0.0124 & 0.0538 \\ 0.0705 & -0.0124 & 0.3117 & 0.2386 \\ 0.0581 & 0.0538 & 0.2386 & 0.4059 \end{pmatrix},$$

$$X_1 = \begin{pmatrix} 0.0000 & 0.0000 & -0.0000 & -0.0000 \\ 0.0000 & 0.0000 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & 1.3276 & -0.2345 \\ -0.0000 & -0.0000 & -0.2345 & 1.7255 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0.7807 & -0.0917 & 0.0361 & 0.0158 \\ -0.0917 & 0.9111 & 0.0076 & 0.0258 \\ 0.0361 & 0.0076 & 2.0335 & -0.3107 \\ 0.0158 & 0.0258 & -0.3107 & 2.3021 \end{pmatrix}.$$

EXAMPLE 4.9. In the following we consider the general 2D discrete-time Lyapunov system (3) with the system matrices given by:

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0.4719 & 0.1250 \\ 0.1250 & 0.4719 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.4743 & -0.0076 \\ -0.0076 & 0.4743 \end{pmatrix}.$$

In this case, and by the use of our method we find that the LMIs in Theorem 3.7 are feasible and a feasible solution can be found as follows:

$$X_0 = \begin{pmatrix} 0.0140 & 0.0001 & 0.1062 & 0.0002 \\ 0.0001 & 0.0140 & 0.0002 & 0.1062 \\ 0.1062 & 0.0002 & 0.8038 & -0.0051 \\ 0.0002 & 0.1062 & -0.0051 & 0.8038 \end{pmatrix},$$

$$X_1 = \begin{pmatrix} 0.0000 & -0.0000 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & -0.0000 & 0.0000 \\ 0.0000 & -0.0000 & 1.0459 & 0.0075 \\ -0.0000 & 0.0000 & 0.0075 & 1.0459 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0.5961 & 0.0043 & -0.0660 & -0.0014 \\ 0.0043 & 0.5960 & -0.0014 & -0.0660 \\ -0.0660 & -0.0014 & 1.6508 & 0.0138 \\ -0.0014 & -0.0660 & 0.0138 & 1.6508 \end{pmatrix},$$

which yields that the system is asymptotically stable.

5. CONCLUDING REMARKS

In this paper, sufficient conditions for 2D general Lyapunov, Roesser systems are derived to guarantee asymptotic stability. We have developed new tests of stability for 2D discrete and continuous-discrete systems. An LMI approach is then described. In this case, all obtained LMIs have at most the dimension $2n^2 \times 2n^2$.

REFERENCES

[1] M. Alfidi and A. Hmamed, *Control for stability and positivity of 2D linear discrete-time systems*, WSEAS Transactions Systems and Control, **2** (2007), 546–556.

[2] O. Bachelier, N. Yeganefar, D. Mehdi, and W. Paszke, *On stabilization of 2D Roesser models*, IEEE Trans. Automat. Control, **62** (2017), 2505–2511.

[3] D. Bouagada and P. Van Dooren, *LMI conditions for the stability of 2D state-space models*. In: *Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems*, MTNS, 2010.

[4] D. Bouagada and P. Van Dooren, *On the stability of 2D state-space models*, Numer. Linear Algebra Appl., **20** (2013), 198–207.

[5] M. Chadli and M. Darouach, *Robust admissibility of uncertain switched singular systems*, Internat. J. Control, **84** (2011), 1587–1600.

[6] D.L. Davis, *A Correct proof of Huang's theorem on stability*, IEEE Transactions on Acoustics, Speech, and Signal Processing, **24** (1976), 424–426.

[7] E. Fornasini and G. Marchesini, *State-space realization theory of two-dimensional filters*, IEEE Trans. Automat. Control, **21** (1976), 484–492.

[8] E. Fornasini and G. Marchesini, *Doubly indexed dynamical systems: state-space models and structural properties*, Mathematical Systems Theory, **12** (1978), 59–72.

[9] Y. Genin, Y. Hachez, Y. Nesterov, R. Stefan, P. Van Dooren and S. Xu, *Positivity and linear matrix inequalities*, Eur. J. Control, **8** (2002), 275–298.

[10] Y. Genin, Y. Hachez, Y. Nesterov and P. Van Dooren, *Optimization problems over positive pseudopolynomial matrices*, SIAM J. Matrix Anal. Appl., **25** (2003), 57–79.

[11] A. Hmamed, S. Kririm, A. Benzaouia and F. Tadeo, *Delay-dependent stability and stabilization of continuous 2D delayed systems with saturating control*, Internat. J. Systems Sci., **47** (2016), 3004–3015.

[12] T. Huang's, *Stability of two dimensional recursive filters*, IEEE Transactions on Audio and Electroacoustics, **20** (1972), 158–163.

[13] E.I. Jury, *Inners and stability of dynamic systems*, John Wiley sons, New York, 1973.

[14] T. Kaczorek, *Vectors and matrices in automation and electrotechnics*, WNT, Warsaw, 1998.

[15] T. Kaczorek, *Two Dimensional Linear Systems*, Springer Verlag, Berlin, 1985.

[16] T. Kaczorek, *The singular general model of 2D systems and its solutions*, IEEE Trans. Automat. Control, **33** (1988), 1060–1061.

[17] T. Kaczorek, *Equivalence of singular 2D linear models*, Bulletin of the Polish Academy of Sciences, Electronics and Electrotechnics, **37** (1989), 263–267.

[18] T. Kaczorek, *Positive 1D and 2D systems*, Springer Verlag, 2001.

[19] T. Kaczorek, *Asymptotic stability of positive 2D linear systems*, Proceedings of the 13th Scientific Conf. on Computer Applications in Electrical Engineering, Poznan, Poland, 2008.

[20] T. Kaczorek, *Canonical forms of singular 1D and 2D linear systems*, Int. J. Appl. Math. Comput. Sci., **13** (2003), 61–72.

[21] T. Kaczorek, *LMI approach to stability of 2D positive systems*, Multidimens. Syst. Signal Process, **20** (2008), 39–54.

[22] T. Kaczorek and P. Przyborowsky, *Reachability, controllability to zero and observability of the positive discrete-time Lyapunov systems*, Control Cybernet., **38** (2009), 529–541.

[23] T. Kaczorek and P. Przyborowskyi, *Continuous-time linear Lyapunov cone-systems*. In: *Proceedings of the 13th IEEE IFAC, Inter. Conf. Methods and Models in Automation and Robotics*, **1** (2007), 225–229.

[24] J. Kurek, *The general state-space model for a two dimensional linear digital system*, IEEE Trans. Automat. Control, **30** (1985), 600–602.
[25] J. Murray, *Another proof and sharpening of Huang's theorem*, IEEE Transactions on Acoustics, Speech, and Signal Processing, **25** (1977), 581–582.
[26] M.S.N. Murty and B.V. Apparao, *Controllability and observability of Lyapunov systems*, Ranchi University Mathematical Journal, **32** (2005), 55–65.
[27] P. Przyborowsky and T. Kaczorek, *Positive* 2*D discrete-time linear Lyapunov systems*, Int. J. Appl. Math. Comput. Sci., **19** (2009), 95–105.
[28] P. Przyborowski and T. Kaczorek, *Linear Lyapunov cone-systems*, in *Automation and Robotics*, J. I-Tech Education and Publishing, 2008, 169–190.
[29] R.P. Roesser, *A discrete state space model for linear image processing*, IEEE Trans. Automat. Control, **20** (1975), 1–10.
[30] D.D. Siljak, *Stability criteria for two-variable polynomials*, IEEE Trans. Circuits Syst., **22** (1975), 185–189.
[31] M. Twardy, *An LMI approach to checking stability of* 2*D positive system*, Bulletin of the Polish Academy of Sciences: Technical Sciences, **54** (2007), 385–395.
[32] M.E. Valcher, *On the internal stability and asymptotic behavior of* 2*D positive systems*, IEEE Trans. Circuits Syst., **44** (1997), 602–613.
[33] H. Xu and Y. Zou, H_∞ *control for* 2*D singular delayed systems*, Internat. J. Systems Sci., **42** (2011), 609–619.
[34] Y. Zou, H. Hu and W. Wang, *Stability for two-dimensional singular discrete systems described by general model*, Multidimens. Syst. Signal Process, **19** (2008), 219–229.
[35] Y. Zou and S.L. Campbell, *The jump behavior and stability analysis for* 2*D singular systems*, Multidimens. Syst. Signal Process, **11** (2000), 321–338.

Received November 30, 2018
Accepted July 11, 2020

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