MATHEMATICA, 63 (86), N° 1, 2021, pp. 77–84

ON A CLASS OF MEROMORPHIC FUNCTIONS DEFINED BY USING A FRACTIONAL OPERATOR

ESZTER GAVRIŞ

Abstract. We introduce a class of meromorphic functions $SD_{\lambda}^{\nu,n}(\alpha)$ using the fractional operator

$$\mathcal{D}_{\lambda}^{\nu,n} f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(\nu+1)_{k+1}}{(2-\lambda)_{k+1}} (k+2)^{n+1} a_k z^k,$$

$-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$. Some inclusion relations and other properties of the class are investigated.

MSC 2010. 30C45.
Key words. Meromorphic function, fractional operator, integral operator.

1. INTRODUCTION

Let Σ denote the class of functions of the form $f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k$, which are analytic in $\mathbb{U}^* = \{z : 0 < |z| < 1\}$.

Motivated by [5], we define the fractional operator $\mathcal{D}_{\lambda}^{\nu,n} : \Sigma \to \Sigma$, by

$$\mathcal{D}_{\lambda}^{\nu,n} f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(\nu+1)_{k+1}}{(2-\lambda)_{k+1}} (k+2)^{n+1} a_k z^k,$$

where $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0, z \in \mathbb{U}^*$ and the symbol $(\gamma)_k$ denotes the Pochhammer symbol, for $\gamma \in \mathbb{C}$, defined by

$$(\gamma)_k = \begin{cases} 1, k = 0 \\ \gamma(\gamma+1)...(\gamma+k-1), k \in \mathbb{N} \end{cases} = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}, \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-.$$

We note that the operator $\mathcal{D}_{0}^{0,n} f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} (k+2)^n a_k z^k$ was introduced and studied in [6].

REMARK 1.1. The operator $\mathcal{D}_{\lambda}^{\nu,n}$ satisfies the following identities:

$$\mathcal{D}_{\lambda}^{\nu,n+1} f(z) = 2\mathcal{D}_{\lambda}^{\nu,n} f(z) + z\big(\mathcal{D}_{\lambda}^{\nu,n} f(z)\big)', \tag{1}$$

$$\mathcal{D}_{\lambda}^{\nu+1,n} f(z) = \frac{\nu+2}{\nu+1}\mathcal{D}_{\lambda}^{\nu,n} f(z) + \frac{1}{\nu+1} z\big(\mathcal{D}_{\lambda}^{\nu,n} f(z)\big)', \tag{2}$$

The author thanks the referee for his helpful comments and suggestions.

DOI: 10.24193/mathcluj.2021.1.07

$$\mathcal{D}_{\lambda+1}^{\nu,n} f(z) = \frac{2-\lambda}{1-\lambda}\mathcal{D}_{\lambda}^{\nu,n} f(z) + \frac{1}{1-\lambda} z\big(\mathcal{D}_{\lambda}^{\nu,n} f(z)\big)', \tag{3}$$

where $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0$.

Definition 1.2. A function $f \in \Sigma$ is said to be in the class $SD_{\lambda}^{\nu,n}(\alpha)$ if it satisfies

$$\Re\left(\frac{\mathcal{D}_{\lambda}^{\nu,n+1} f(z)}{\mathcal{D}_{\lambda}^{\nu,n} f(z)} - 2\right) < -\alpha, z \in \mathbb{U}, \tag{4}$$

for some $\alpha (0 \leq \alpha < 1), -\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0$.

To prove our results, we need the followings.

Lemma 1.3 ([3]). *Let the function w be regular and nonconstant in $|z| < 1$, with $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then we have $z_0 w'(z_0) = k w(z_0)$, where k is a real number and $k \geq 1$.*

Lemma 1.4 ([4]). *Let $\phi(u, v)$ be a complex valued function, $\phi : D \to \mathbb{C}, D \subset \mathbb{C}^2$, and let $u = u_1 + \mathrm{i}u_2, v = v_1 + \mathrm{i}v_2$. Suppose that the function $\phi(u, v)$ satisfies the following conditions:*
(i) *$\phi(u, v)$ is continuous in D;*
(ii) *$(1, 0) \in D$ and $\Re(\phi(1, 0)) > 0$;*
(iii) *$\Re(\phi(\mathrm{i}u_2, v_1)) \leq 0$ for all $(\mathrm{i}u_2, v_1) \in D$ such that $v_1 \leq \frac{-(1+u_2^2)}{2}$.*
Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be regular in $\mathbb{U}$ such that $(p(z), zp'(z)) \in D$ for all $z \in \mathbb{U}$. If $\Re(\phi(p(z), zp'(z))) > 0, z \in \mathbb{U}$, then $\Re(p(z)) > 0, z \in \mathbb{U}$.

2. MAIN RESULTS

To prove our results, we use the methods used in [2, 6].

Theorem 2.1. $SD_{\lambda}^{\nu,n+1}(\alpha) \subset SD_{\lambda}^{\nu,n}(\alpha), n \in \mathbb{N}_0$.

Proof. Let $f \in SD_{\lambda}^{\nu,n+1}(\alpha)$. Therefore, we have

$$\Re\left(\frac{\mathcal{D}_{\lambda}^{\nu,n+2} f(z)}{\mathcal{D}_{\lambda}^{\nu,n+1} f(z)} - 2\right) < -\alpha, z \in \mathbb{U}. \tag{5}$$

Let w be a regular function in the unit disk $\mathbb{U}$, with $w(0) = 0$, defined by

$$\frac{\mathcal{D}_{\lambda}^{\nu,n+1} f(z)}{\mathcal{D}_{\lambda}^{\nu,n} f(z)} - 2 = -\frac{1 + (2\alpha - 1)w(z)}{1 + w(z)}. \tag{6}$$

The equality (6) may be written as

$$\frac{\mathcal{D}_{\lambda}^{\nu,n+1} f(z)}{\mathcal{D}_{\lambda}^{\nu,n} f(z)} = \frac{1 + (3 - 2\alpha)w(z)}{1 + w(z)}. \tag{7}$$

Differentiating (7) logarithmically and multiplying by z, we obtain

$$\frac{z(\mathcal{D}_{\lambda}^{\nu,n+1} f(z))'}{\mathcal{D}_{\lambda}^{\nu,n+1} f(z)} - \frac{z(\mathcal{D}_{\lambda}^{\nu,n} f(z))'}{\mathcal{D}_{\lambda}^{\nu,n} f(z)} = \frac{z(3 - 2\alpha)w'(z)}{1 + (3 - 2\alpha)w(z)} - \frac{zw'(z)}{1 + w(z)}. \tag{8}$$

Using (1) in (8) we get

$$\begin{aligned}(9)\qquad &\frac{\mathcal{D}_\lambda^{\nu,n+2}f(z)-2\mathcal{D}_\lambda^{\nu,n+1}f(z)}{\mathcal{D}_\lambda^{\nu,n+1}f(z)}-\frac{\mathcal{D}_\lambda^{\nu,n+1}f(z)-2\mathcal{D}_\lambda^{\nu,n}f(z)}{\mathcal{D}_\lambda^{\nu,n}f(z)}\\ &=\frac{z(3-2\alpha)w'(z)}{1+(3-2\alpha)w(z)}-\frac{zw'(z)}{1+w(z)}.\end{aligned}$$

Using (7) in (9) we get after some calculations the following

$$\frac{\frac{\mathcal{D}_\lambda^{\nu,n+2}f(z)}{\mathcal{D}_\lambda^{\nu,n+1}f(z)}-2+\alpha}{1-\alpha}=\frac{2zw'(z)}{(1+w(z))(1+(3-2\alpha)w(z))}-\frac{1-w(z)}{1+w(z)}.$$

We claim that $|w(z)|<1$ for $z\in\mathbb{U}$. Otherwise there exists a point $z_0\in\mathbb{U}$ such that $\max\limits_{|z|<|z_0|}|w(z)|=|w(z_0)|=1$. Using Lemma 1.3, we obtain

$$\frac{\frac{\mathcal{D}_\lambda^{\nu,n+2}f(z_0)}{\mathcal{D}_\lambda^{\nu,n+1}f(z_0)}-2+\alpha}{1-\alpha}=\frac{2kw(z_0)}{(1+w(z_0))(1+(3-2\alpha)w(z_0))}-\frac{1-w(z_0)}{1+w(z_0)}.$$

Thus

$$\Re\left(\frac{\frac{\mathcal{D}_\lambda^{\nu,n+2}f(z_0)}{\mathcal{D}_\lambda^{\nu,n+1}f(z_0)}-2+\alpha}{1-\alpha}\right)\geq\frac{1}{2(2-\alpha)}>0,$$

which contradicts (5). Hence $|w(z)|<1$ for $z\in\mathbb{U}$ and from (6) follows (4). Consequently, $f\in SD_\lambda^{\nu,n}(\alpha)$. $\square$

Remark 2.2. Taking $\lambda=0$ and $\nu=0$, we obtain Theorem 2.1 from [6].

Using Lemma 1.4 instead of Lemma 1.3 we will obtain an improvement of Theorem 2.1.

Theorem 2.3. *$SD_\lambda^{\nu,n+1}(\alpha)\subset SD_\lambda^{\nu,n}(\beta)$, for $n\in\mathbb{N}_0$, where*

$$(10)\qquad \beta=\frac{5+2\alpha-\sqrt{(3-2\alpha)^2+8}}{4},$$

and $\beta\in(\alpha,1)$.

Proof. Let $f\in SD_\lambda^{\nu,n+1}(\alpha)$, where $0\leq\alpha<1$, and let p be a function defined by

$$(11)\qquad \frac{\mathcal{D}_\lambda^{\nu,n+1}f(z)}{\mathcal{D}_\lambda^{\nu,n}f(z)}=\gamma+(1-\gamma)p(z),\gamma>1,z\in\mathbb{U},$$

where $\gamma=\frac{(3-2\alpha)+\sqrt{(3-2\alpha)^2+8}}{4}$. Then the function p is of the form $p(z)=1+p_1z+p_2z^2+...$ and analytic in $\mathbb{U}$. Differentiating logarithmically both sides of (11) and making use of the identity (1), we obtain

$$\frac{\mathcal{D}_\lambda^{\nu,n+2}f(z)}{\mathcal{D}_\lambda^{\nu,n+1}f(z)}=\gamma+(1-\gamma)p(z)+\frac{z(1-\gamma)p'(z)}{\gamma+(1-\gamma)p(z)},$$

or

$$-\Re\left(\frac{\mathcal{D}_\lambda^{\nu,n+2}f(z)}{\mathcal{D}_\lambda^{\nu,n+1}f(z)}-2\right)-\alpha$$

$$=\Re\Big(2-\alpha-\gamma-(1-\gamma)p(z)-\frac{(1-\gamma)zp'(z)}{\gamma+(1-\gamma)p(z)}\Big)>0, z\in\mathbb{U}.$$

We define the function ϕ by

$$\phi(u,v)=2-\alpha-\gamma-(1-\gamma)u-\frac{(1-\gamma)v}{\gamma+(1-\gamma)u}.$$

Then ϕ has the following properties:

(i) ϕ is continuous in $D=\big(\mathbb{C}-\{\frac{-\gamma}{1-\gamma}\}\big)\times\mathbb{C}$;

(ii) $(1,0)\in D$ and $\Re(\phi(1,0))=1-\alpha>0$;

(iii) for all $(\mathrm{i}u_2,v_1)\in D$ such that $v_1\le-\frac{1}{2}(1+u_2^2)$,

$$\begin{aligned}\Re\big(\phi(\mathrm{i}u_2,v_1)\big)&=2-\alpha-\gamma-\frac{\gamma(1-\gamma)v_1}{\gamma^2+(1-\gamma)^2u_2^2}\\&\le 2-\alpha-\gamma+\frac{\gamma(1-\gamma)(1+u_2^2)}{2\big(\gamma^2+(1-\gamma)^2u_2^2\big)}\\&=-\frac{(1-\gamma)(1-2\gamma)u_2^2}{2\gamma\big(\gamma^2+(1-\gamma)^2u_2^2\big)}\le 0.\end{aligned}$$

Therefore, by Lemma 1.4, we have $\Re p(z)>0$ in $\mathbb{U}$, hence $\Re\frac{\mathcal{D}_\lambda^{\nu,n+1}f(z)}{\mathcal{D}_\lambda^{\nu,n}f(z)}<\gamma, z\in\mathbb{U}$, or equivalently $\Re\left(\frac{\mathcal{D}_\lambda^{\nu,n+1}f(z)}{\mathcal{D}_\lambda^{\nu,n}f(z)}-2\right)<-\beta, z\in\mathbb{U}$, where β is given by (10). Consequently, $f\in SD_\lambda^{\nu,n}(\beta)$. □

REMARK 2.4. Taking $\lambda=0$ and $\nu=0$, we obtain a particular case of Theorem 2.5 from [1].

THEOREM 2.5. $SD_\lambda^{\nu+1,n}(\alpha)\subset SD_\lambda^{\nu,n}(\alpha), \nu>-1$.

Proof. Let $f\in SD_\lambda^{\nu+1,n}(\alpha)$. Therefore, we have

$$\Re\left(\frac{\mathcal{D}_\lambda^{\nu+1,n+1}f(z)}{\mathcal{D}_\lambda^{\nu+1,n}f(z)}-2\right)<-\alpha, z\in\mathbb{U}. \tag{12}$$

Let w be a regular function in the unit disk $\mathbb{U}$, with $w(0)=0$, defined by (6). Using (1) and (2), the equality (7) may be written as

$$\frac{\mathcal{D}_\lambda^{\nu+1,n}f(z)}{\mathcal{D}_\lambda^{\nu,n}f(z)}=\frac{\nu+1+(3-2\alpha+\nu)w(z)}{(\nu+1)(1+w(z))}. \tag{13}$$

Differentiating (13) logarithmically and multiplying by z, we obtain

$$\frac{z(\mathcal{D}_\lambda^{\nu+1,n}f(z))'}{\mathcal{D}_\lambda^{\nu+1,n}f(z)}-\frac{z(\mathcal{D}_\lambda^{\nu,n}f(z))'}{\mathcal{D}_\lambda^{\nu,n}f(z)}=\frac{(3-2\alpha+\nu)zw'(z)}{\nu+1+(3-2\alpha+\nu)w(z)}-\frac{zw'(z)}{1+w(z)}. \tag{14}$$

Using (1) in (14) we get

$$\frac{\mathcal{D}_{\lambda}^{\nu+1,n+1}f(z)-2\mathcal{D}_{\lambda}^{\nu+1,n}f(z)}{\mathcal{D}_{\lambda}^{\nu+1,n}f(z)}-\frac{\mathcal{D}_{\lambda}^{\nu,n+1}f(z)-2\mathcal{D}_{\lambda}^{\nu,n}f(z)}{\mathcal{D}_{\lambda}^{\nu,n}f(z)} = \frac{(3-2\alpha+\nu)zw'(z)}{\nu+1+(3-2\alpha+\nu)w(z)}-\frac{zw'(z)}{1+w(z)}. \tag{15}$$

Using (7) in (15) we get after some calculations the following

$$\frac{\frac{\mathcal{D}_{\lambda}^{\nu+1,n+1}f(z)}{\mathcal{D}_{\lambda}^{\nu+1,n}f(z)}-2+\alpha}{1-\alpha}=\frac{2zw'(z)}{(1+w(z))(\nu+1+(3-2\alpha+\nu)w(z))}-\frac{1-w(z)}{1+w(z)}.$$

We claim that $|w(z)|<1$ for $z\in\mathbb{U}$. Otherwise there exists a point $z_0\in\mathbb{U}$ such that $\max_{|z|<|z_0|}|w(z)|=|w(z_0)|=1$. Using Lemma 1.3, we obtain

$$\frac{\frac{\mathcal{D}_{\lambda}^{\nu+1,n+1}f(z_0)}{\mathcal{D}_{\lambda}^{\nu+1,n}f(z_0)}-2+\alpha}{1-\alpha}=\frac{2kw(z_0)}{(1+w(z_0))(\nu+1+(3-2\alpha+\nu)w(z_0))}-\frac{1-w(z_0)}{1+w(z_0)}.$$

Thus

$$\Re\left(\frac{\frac{\mathcal{D}_{\lambda}^{\nu+1,n+1}f(z_0)}{\mathcal{D}_{\lambda}^{\nu+1,n}f(z_0)}-2+\alpha}{1-\alpha}\right)\geq\frac{1}{2(2-\alpha+\nu)}>0,$$

which contradicts (12). Hence $|w(z)|<1$ for $z\in\mathbb{U}$ and from (6) follows (4). Consequently, $f\in SD_{\lambda}^{\nu,n}(\alpha)$. $\square$

Theorem 2.6. $SD_{\lambda+1}^{\nu,n}(\alpha)\subset SD_{\lambda}^{\nu,n}(\alpha), -\infty<\lambda<1$.

Proof. Let $f\in SD_{\lambda+1}^{\nu,n}(\alpha)$. Therefore, we have

$$\Re\left(\frac{\mathcal{D}_{\lambda+1}^{\nu,n+1}f(z)}{\mathcal{D}_{\lambda+1}^{\nu,n}f(z)}-2\right)<-\alpha, z\in\mathbb{U}. \tag{16}$$

Let w be a regular function in the unit disk $\mathbb{U}$, with $w(0)=0$, defined by (6). Using (1) and (3), the equality (7) may be written as

$$\frac{\mathcal{D}_{\lambda+1}^{\nu,n}f(z)}{\mathcal{D}_{\lambda}^{\nu,n}f(z)}=\frac{1-\lambda+(3-2\alpha-\lambda)w(z)}{(1-\lambda)(1+w(z))}. \tag{17}$$

Differentiating (17) logarithmically and multiplying by z, we obtain

$$\frac{z(\mathcal{D}_{\lambda+1}^{\nu,n}f(z))'}{\mathcal{D}_{\lambda+1}^{\nu,n}f(z)}-\frac{z(\mathcal{D}_{\lambda}^{\nu,n}f(z))'}{\mathcal{D}_{\lambda}^{\nu,n}f(z)}=\frac{(3-2\alpha-\lambda)zw'(z)}{1-\lambda+(3-2\alpha-\lambda)w(z)}-\frac{zw'(z)}{1+w(z)}. \tag{18}$$

Using (1) in (18) we get

$$\frac{\mathcal{D}_{\lambda+1}^{\nu,n+1}f(z)-2\mathcal{D}_{\lambda+1}^{\nu,n}f(z)}{\mathcal{D}_{\lambda+1}^{\nu,n}f(z)}-\frac{\mathcal{D}_{\lambda}^{\nu,n+1}f(z)-2\mathcal{D}_{\lambda}^{\nu,n}f(z)}{\mathcal{D}_{\lambda}^{\nu,n}f(z)}=$$

$$\frac{(3-2\alpha-\lambda)zw'(z)}{1-\lambda+(3-2\alpha-\lambda)w(z)}-\frac{zw'(z)}{1+w(z)}. \tag{19}$$

Using (7) in (19) we get after some calculations the following

$$\frac{\frac{\mathcal{D}_{\lambda+1}^{\nu,n+1}f(z)}{\mathcal{D}_{\lambda+1}^{\nu,n}f(z)}-2+\alpha}{1-\alpha}=\frac{2zw'(z)}{(1+w(z))(1-\lambda+(3-2\alpha-\lambda)w(z))}-\frac{1-w(z)}{1+w(z)}.$$

We claim that $|w(z)|<1$ for $z\in\mathbb{U}$. Otherwise there exists a point $z_0\in\mathbb{U}$ such that $\max\limits_{|z|<|z_0|}|w(z)|=|w(z_0)|=1$. Using Lemma 1.3, we obtain

$$\frac{\frac{\mathcal{D}_{\lambda+1}^{\nu,n+1}f(z_0)}{\mathcal{D}_{\lambda+1}^{\nu,n}f(z_0)}-2+\alpha}{1-\alpha}=\frac{2kw(z_0)}{(1+w(z_0))(1-\lambda+(3-2\alpha-\lambda)w(z_0))}-\frac{1-w(z_0)}{1+w(z_0)}.$$

Thus

$$\Re\left(\frac{\frac{\mathcal{D}_{\lambda+1}^{\nu,n+1}f(z_0)}{\mathcal{D}_{\lambda+1}^{\nu,n}f(z_0)}-2+\alpha}{1-\alpha}\right)\geq\frac{1}{2(2-\alpha-\lambda)}>0,$$

which contradicts (16). Hence $|w(z)|<1$ for $z\in\mathbb{U}$ and from (6) follows (4). Consequently, $f\in SD_\lambda^{\nu,n}(\alpha)$. □

THEOREM 2.7. *Let $f\in\Sigma$ satisfying the condition*

$$\Re\left(\frac{\mathcal{D}_{\lambda}^{\nu,n+1}f(z)}{\mathcal{D}_{\lambda}^{\nu,n}f(z)}-2\right)<-\alpha+\frac{1-\alpha}{2(1-\alpha+c)},z\in\mathbb{U},$$

$$n\in\mathbb{N}_0,-\infty<\lambda<2,\nu>-1,c>0, \tag{20}$$

then

$$F(z)=\frac{c}{z^{c+1}}\int_0^z t^c f(t)dt\in SD_\lambda^{\nu,n}(\alpha).$$

Proof. From the definition of F we have

$$z(\mathcal{D}_\lambda^{\nu,n}F(z))'=c\mathcal{D}_\lambda^{\nu,n}f(z)-(c+1)\mathcal{D}_\lambda^{\nu,n}F(z). \tag{21}$$

Using (21) and (1), the inequality (20) may be written as

$$\Re\left(\frac{\frac{\mathcal{D}_{\lambda}^{\nu,n+2}F(z)}{\mathcal{D}_{\lambda}^{\nu,n+1}F(z)}+(c-1)}{1+(c-1)\frac{\mathcal{D}_{\lambda}^{\nu,n}F(z)}{\mathcal{D}_{\lambda}^{\nu,n+1}F(z)}}-2\right)<-\alpha+\frac{1-\alpha}{2(1-\alpha+c)}.$$

Let w be a regular function in the unit disk $\mathbb{U}$, with $w(0)=0$, defined by

$$\frac{\mathcal{D}_{\lambda}^{\nu,n+1}F(z)}{\mathcal{D}_{\lambda}^{\nu,n}F(z)}-2=-\frac{1+(2\alpha-1)w(z)}{1+w(z)}.$$

This equality may be written as

$$\frac{\mathcal{D}_\lambda^{\nu,n+1}F(z)}{\mathcal{D}_\lambda^{\nu,n}F(z)}+c-1=\frac{c+(2-2\alpha+c)w(z)}{1+w(z)}. \tag{22}$$

Differentiating (22) logarithmically and simplifying we obtain

$$\frac{\frac{\mathcal{D}_\lambda^{\nu,n+2}F(z)}{\mathcal{D}_\lambda^{\nu,n+1}F(z)}+(c-1)}{1+(c-1)\frac{\mathcal{D}_\lambda^{\nu,n}F(z)}{\mathcal{D}_\lambda^{\nu,n+1}F(z)}}-2$$

$$=-\left(\alpha+(1-\alpha)\frac{1-w(z)}{1+w(z)}\right)+\frac{2(1-\alpha)zw'(z)}{(1+w(z))(c+(2-2\alpha+c)w(z))}.$$

The remaining part of the proof is similar to that of Theorem 2.1. □

REMARK 2.8. Taking $\lambda=0$ and $\nu=0$, we obtain Theorem 2.2 from [6].

THEOREM 2.9. *$f\in SD_\lambda^{\nu,n}(\alpha)$ if and only if the integral operator $F\in SD_\lambda^{\nu,n+1}(\alpha)$, where $F(z)=\frac{1}{z^2}\int\limits_0^z tf(t)dt$.*

Proof. From the definition of F we have

$$z(\mathcal{D}_\lambda^{\nu,n}F(z))'+2\mathcal{D}_\lambda^{\nu,n}F(z)=\mathcal{D}_\lambda^{\nu,n}f(z). \tag{23}$$

By using the relation (1), the equality (23) becomes $\mathcal{D}_\lambda^{\nu,n}f(z)=\mathcal{D}_\lambda^{\nu,n+1}F(z)$. Hence $\mathcal{D}_\lambda^{\nu,n+1}f(z)=\mathcal{D}_\lambda^{\nu,n+2}F(z)$. Therefore

$$\frac{\mathcal{D}_\lambda^{\nu,n+1}f(z)}{\mathcal{D}_\lambda^{\nu,n}f(z)}=\frac{\mathcal{D}_\lambda^{\nu,n+2}F(z)}{\mathcal{D}_\lambda^{\nu,n+1}F(z)}.$$

This completes the proof. □

REMARK 2.10. Taking $\lambda=0$ and $\nu=0$, we obtain Theorem 2.3 from [6].

THEOREM 2.11. *$f\in SD_\lambda^{\nu,n}(\alpha)$ if and only if the integral operator $F\in SD_\lambda^{\nu+1,n}(\alpha)$, where $F(z)=\frac{\nu+1}{z^{\nu+2}}\int\limits_0^z t^{\nu+1}f(t)dt$.*

Proof. From the definition of F we have

$$z(\mathcal{D}_\lambda^{\nu,n}F(z))'+(\nu+2)\mathcal{D}_\lambda^{\nu,n}F(z)=(\nu+1)\mathcal{D}_\lambda^{\nu,n}f(z). \tag{24}$$

By using the relation (2), the equality (24) becomes $\mathcal{D}_\lambda^{\nu,n}f(z)=\mathcal{D}_\lambda^{\nu+1,n}F(z)$. Hence $\mathcal{D}_\lambda^{\nu,n+1}f(z)=\mathcal{D}_\lambda^{\nu+1,n+1}F(z)$. Therefore

$$\frac{\mathcal{D}_\lambda^{\nu,n+1}f(z)}{\mathcal{D}_\lambda^{\nu,n}f(z)}=\frac{\mathcal{D}_\lambda^{\nu+1,n+1}F(z)}{\mathcal{D}_\lambda^{\nu+1,n}F(z)}.$$

This completes the proof. □

THEOREM 2.12. *$f \in SD_{\lambda}^{\nu,n}(\alpha)$ if and only if the integral operator $F \in SD_{\lambda+1}^{\nu,n}(\alpha)$, where $F(z) = \frac{1-\lambda}{z^{2-\lambda}} \int_0^z t^{1-\lambda} f(t)dt$.*

Proof. From the definition of F we have

$$z(\mathcal{D}_{\lambda}^{\nu,n}F(z))' + (2-\lambda)\mathcal{D}_{\lambda}^{\nu,n}F(z) = (1-\lambda)\mathcal{D}_{\lambda}^{\nu,n}f(z). \tag{25}$$

By using the relation (3), the equality (25) becomes $\mathcal{D}_{\lambda}^{\nu,n}f(z) = \mathcal{D}_{\lambda+1}^{\nu,n}F(z)$. Hence $\mathcal{D}_{\lambda}^{\nu,n+1}f(z) = \mathcal{D}_{\lambda+1}^{\nu,n+1}F(z)$. Therefore

$$\frac{\mathcal{D}_{\lambda}^{\nu,n+1}f(z)}{\mathcal{D}_{\lambda}^{\nu,n}f(z)} = \frac{\mathcal{D}_{\lambda+1}^{\nu,n+1}F(z)}{\mathcal{D}_{\lambda+1}^{\nu,n}F(z)}.$$

This completes the proof. □

REFERENCES

[1] F.M. Al-Oboudi and H.A. Al-Zkeri, *On some classes of meromorphic starlike functions defined by a differential operator*, Global Journal of Pure and Applied Mathematics, **3** (2007), 1–11.

[2] N.E. Cho and J.A. Kim, *On certain classes of meromorphically starlike functions*, Int. J. Math. Math. Sci., **18** (3) (1995), 463–468.

[3] I.S. Jack, *Functions starlike and convex of order α*, J. Lond. Math. Soc., **3** (1971), 469–474.

[4] S.S. Miller and P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., **65** (1978), 289–305.

[5] P. Sharma, R.K. Raina and G.Ş. Sălăgean, *Some geometric properties of analytic functions involving a new fractional operator*, Mediterr. J. Math., **13** (2016), 4591–4605.

[6] B. A. Uralegaddi and C. Somanatha, *New criteria for meromorphic starlike univalent functions*, Bull. Aust. Math. Soc., **43** (1991), 137–140.

Received August 1, 2019
Accepted December 4, 2019

Babeş-Bolyai University
Faculty of Mathematics and Computer Science
Department of Mathematics
Cluj-Napoca, Romania
E-mail: szatmari.eszter@math.ubbcluj.ro