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PARSEVAL'S EQUALITY IN FUZZY NORMED LINEAR SPACES

BAYAZ DARABY, FATANEH DELZENDEH, and ASGHAR RAHIMI

Abstract. In this paper, we investigate Parseval's equality and define the fuzzy frame on Felbin fuzzy Hilbert spaces. For showing the importance of defining fuzzy frame, we know that, in the classical Hilbert space, $C(\Omega)$ is not normable, but, in this manuscript, we prove that $C(\Omega)$ (the vector space of all continuous functions on Ω) is normable in a Felbin fuzzy Hilbert space and so the defining fuzzy frame on $C(\Omega)$ is possible. These consequences of the category of fuzzy frames in Felbin fuzzy Hilbert spaces are wider than the category of the frames in the classical Hilbert spaces.

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Key words. Fuzzy norm, fuzzy inner product space, fuzzy Hilbert space, fuzzy frame.

1. INTRODUCTION

The idea of fuzzy norms on a linear space was first introduced by Katsaras [17] in 1984. Later on, many authors, Felbin [15], Cheng, Mordeson [5], Bag and Samanta [2] etc. gave different definitions of fuzzy normed linear spaces. R. Biswas [4] and A. M. El-Abye and H. M. El-Hamouly [14] tried to give a meaningful definition of fuzzy inner product space and associated a fuzzy norm function with those definitions which are restricted to the real linear space only. P. Mazumder and S.K. Samanta introduced the definition of fuzzy inner product space from the point of veiw of Bag and Samanta fuzzy norm[2]. Recently, B. Daraby and et. al. [7] studied some properties of fuzzy Hilbert spaces and they showed that all results in classical Hilbert spaces are immediate consequences of the corresponding results for Felbin-fuzzy Hilbert spaces. Moreover, by an example, they showed that the spectrum of the category of Felbin- fuzzy Hilbert spaces is broader than the category of classical Hilbert spaces [8]. Also, in [20], M. Mursaleen and et. al. investigated the convergence of fuzzy number sequences in general form statistically.

One of the important concepts in the study of vector spaces is the basis, which allows every vector to be uniquely represented as a linear combination of the basis elements. The main feature of a basis $\{x_k\}$ in a Hilbert space H is

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that every $x \in H$ can be represented as a linear combinations of the elements x_k in the form:

$$x = \sum_{k=1}^{\infty} c_k(x) x_k. \tag{1}$$

The coefficients $c_k(x)$ are unique. However, the linear independence property for a basis, which implies the uniqueness of coefficients, is restrictive in applications; sometimes it is impossible to find vectors which both fulfill the basis requirements and also satisfy external conditions demanded by applied problems. For such purposes, a more flexible types of spanning sets is needed. Frames provide these alternatives. Frames are used in signal and image processing, non-harmonic Fourier series, data compression and sampling theory. Today, frame theory has ever increasing applications to problems in both pure and applied mathematics physics, engineering, computer science etc.

Many physical systems are inherently nonlinear functions and must be described by non-linear models. But some systems have of uncertain structured and it is not possible to provide an accurate mathematical model. Therefore, to these systems, the conventional control models can not be used, for solving this problems, we need to use a new concept namely fuzzy frames theory and fuzzy waveletes. Fuzzy frame and fuzzy wavelet inspired from frame theory, wavelet theory and fuzzy concepts. For achieveing approximation functions, control and identification of nonlinear systems are presented [3, 21] . It not only retains the frame and wavelet properties but also has advantages such as simple structure to approximation and good interperability approximation of non-linear functions.

We introduce the concept of a fuzzy inner product and show the Parseval's equality holds with Felbin-fuzzy norm. In this paper, we study the Parseval's equality in fuzzy normed linear spaces and define fuzzy frame in Felbin-fuzzy Hilbert spaces. For more illustration, we expressed some examples of fuzzy frames.

2. SOME PRELIMINARIES

In this section, some definitions and preliminary results are given which will be used in this paper. For details, we refer to [1, 9, 10, 16].

Definition 2.1 ([16]). A mapping $\eta : \mathbb{R} \longrightarrow [0,1]$ is called a fuzzy real number with α-level set $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies the following conditions:
$N1$) there exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.
$N2$) for each $\alpha \in (0,1]$, there exist real numbers $\eta_\alpha^- \leq \eta_\alpha^+$ such that the α-level set $[\eta]_\alpha$ is equal to the closed interval $[\eta_\alpha^-, \eta_\alpha^+]$.
The set of all fuzzy real numbers is denoted by $F(\mathbb{R})$. If $\eta \in F(\mathbb{R})$ and $\eta(t) = 0$ whenever $t < 0$, then it is called a non-negative fuzzy real number and $F^+(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers. Real number

$\eta^- \in F^+(\mathbb{R})$ for all and each $\alpha \in (0,1]$ is positive. The fuzzy real number defined by

$$\tilde{r}(t) = \begin{cases} 1 & , \quad t = r \\ 0 & , \quad t \neq r, \end{cases}$$

it follows that $\mathbb{R}$ can be embedded in $F(\mathbb{R})$.

Definition 2.2 ([16]). Let X be a vector space over $\mathbb{R}$. Assume the mappings

$$L, R : [0,1] \times [0,1] \longrightarrow [0,1]$$

are symmetric and non-decreasing in both arguments, and that $L(0,0) = 0$ and $R(1,1) = 1$.

Let $\|.\| : X \longrightarrow F^+(R)$. The quadruple $(X, \|.\|, L, R)$ is called a fuzzy normed linear space (briefly, FNS) with the fuzzy norm $\|.\|$, if the following conditions are satisfied:

(F1) if $x \neq 0$, then $\inf_{0<\alpha\leq 1} \|x\|_\alpha^- > 0$,
(F2) $\|x\| = \tilde{0}$ if and only if $x = 0$,
(F3) $\|rx\| = |\tilde{r}|\|x\|$ for $x \in X$ and $r \in R$,
(F4) for all $x, y \in X$,
(F4L) $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$ whenever $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s+t \leq \|x+y\|_1^-$,
(F4R) $\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$ whenever $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ and $s+t \geq \|x+y\|_1^-$.

Definition 2.3 ([1]). Let X be a vector space over $\mathbb{R}$. Suppose $\|.\| : X \longrightarrow F^+(\mathbb{R})$ is a mapping satisfying:

(i) $\|x\| = \tilde{0}$ iff $x = 0$,
(ii) $\|rx\| = \tilde{r} \odot \|x\|$,where $x \in X$ and $r \in \mathbb{R}$,
(iii) for all $x, y \in X$, $\|x+y\| \preccurlyeq \|x\| \oplus \|y\|$ and
(A') $x \neq 0$ then $\|x\|(t) = 0$, for all $t \leq 0$.

Then $(X, \|.\|)$ is called a fuzzy normed linear space and $\|.\|$ is called a fuzzy norm on X.

In the rest of this paper, we use this definition of fuzzy norm. We note that $\|.\|_\alpha^s$, $s = -, +$ are crisp norms on X where $[\|x\|]_\alpha = [\|x\|_\alpha^-, \|x\|_\alpha^+]$, $0 < \alpha \leq 1$.

Definition 2.4 ([16]). Let $(X, \|.\|)$ be a FNS.

i) A sequence $\{x_n\} \subseteq X$ is siad to converge to $x \in X$, if $\lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0$, for all $\alpha \in (0,1]$.
ii) A sequence $\{x_n\} \subseteq X$ is called Cauchy, if $\lim_{m,n\to\infty} \|x_n - x_m\|_\alpha^+ = 0$, for all $\alpha \in (0,1]$.

Definition 2.5 ([16]). Let $(X, \|.\|)$ be a FNS. A subset A of X is said to be complete, if every Cauchy sequence in A converges in A.

Definition 2.6 ([16]). Let X be a vector space over R. A Felbin-fuzzy inner product on X is a mapping $\langle .,. \rangle : X \times X \longrightarrow F(R)$ such that for all vectors $x, y, z \in X$ and all $r \in R$, we have:

(IP1) $\langle x + y, z\rangle = \langle x, z\rangle \oplus \langle y, z\rangle$;
(IP2) $\langle rx, y\rangle = \tilde{r}\langle x, y\rangle$;
(IP3) $\langle x, y\rangle = \langle y, x\rangle$;
(IP4) $\langle x, x\rangle \geq \tilde{0}$;
(IP5) $\inf_{\alpha\in(0,1]}\langle x, x\rangle^-_\alpha > 0$ if $x \neq 0$;
(IP6) $\langle x, x\rangle = \tilde{0}$ if and only if $x = 0$.

The vector space X equipped with a Felbin-fuzzy inner product is called a Felbin-fuzzy inner product space.

A Felbin-fuzzy inner product on X defines a fuzzy number

$$\|x\| = \sqrt{\langle x, x\rangle}, \forall x \in X. \tag{2}$$

Definition 2.7 ([16]). Let $(X^*, \|.\|^*)$ be a completion of a Felbin-fuzzy normed linear space $(X, \|.\|)$ and $x^*, y^* \in X^*$ with representatives $\{x_n\}$ and $\{y_n\}$, respectively. Suppose $\alpha \in (0, 1]$ and $\{\alpha_k\}$ is a strictly increasing sequence converging to α. Define

$$[\langle x^*, y^*\rangle]_\alpha = [\lim_{n,k\to\infty}\langle x_n, y_n\rangle^-_{\alpha_k}, \lim_{n,k\to\infty}\langle x_n, y_n\rangle^+_{\alpha_k}].$$

Lemma 2.8 ([16]). *A fuzzy inner product space X together with its corresponding norm $\|.\|$ satisfy the Schwarz inequality*

$$|\langle x, x\rangle| \leq \|x\|\|y\|, \forall x, y \in X.$$

Theorem 2.9 ([16]). *The function $\|.\|$ defined in Definition (2.6) is a fuzzy norm.*

A fuzzy Hilbert space is a complete Felbin-fuzzy inner product space with the Felbin-fuzzy norm defined by relation (2).

Lemma 2.10 ([8]). *Let $\gamma, \delta \in F(R)$ and $[\gamma]_\alpha = [\gamma^-_\alpha, \gamma^+_\alpha]$, $[\delta]_\alpha = [\delta^-_\alpha, \delta^+_\alpha]$. Then for all $\alpha \in (0, 1]$,*

$$\begin{aligned}
[\gamma \oplus \delta]_\alpha &= [\gamma^-_\alpha + \delta^-_\alpha, \gamma^+_\alpha + \delta^+_\alpha] \\
[\gamma \ominus \delta]_\alpha &= [\gamma^-_\alpha - \delta^-_\alpha, \gamma^+_\alpha - \delta^+_\alpha] \\
[\gamma \odot \delta]_\alpha &= [\gamma^-_\alpha \delta^-_\alpha, \gamma^+_\alpha \delta^+_\alpha] \\
[\frac{1}{\gamma}]_\alpha &= [\frac{1}{\gamma^+_\alpha}, \frac{1}{\gamma^-_\alpha}] \\
[|\gamma|]_\alpha &= [\max(0, \gamma^-_\alpha, \gamma^+_\alpha), \max(|\gamma^-_\alpha|, |\gamma^+_\alpha|].
\end{aligned}$$

Theorem 2.11 ([8, Bessel's inequality]). *Let H be a Felbin-fuzzy inner product space and for any $x \in H$ there exists $\{x_n\} \subset H$, $x_n \to x$, $\{e_n\}$ be a fuzzy orthonormal sequence in H, then*

$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \preccurlyeq \|x\|^2.$$

3. FUZZY FRAME

In this section, after a short introduction to history of frame, we define fuzzy frame and prove some new results.

Frames were introduced already in 1952 by Duffin and Schaeffer in the fundamental paper [13]; they used frames as a tool in the study of nonharmonic Fourier series, i.e., sequences of the type $\{e^{i\lambda_n}x\}_{n\in Z}$, where $\{\lambda_n\}_{n\in Z}$ is a family of real or complex numbers. Apparently, the importance of the concept was not realized by the mathematical community; at least it took almost 30 years before the next treatment appeared in print. Frames were presented in the abstract setting, and again used in the context of nonharmonic Fourier series. Then, in 1985, as the wavelet area began, Daubechies, Grossmann and Meyer [11, 12] observed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$ which are very similar to expansions using orthogonal bases.

Definition 3.1 (Fuzzy orthonormal set and sequences). A fuzzy orthogonal set M in a fuzzy inner product space X is a subset $M \subset X$ whose elemmaents are pairwise orthogonal. A fuzzy orthonormal set $M \subset X$ is a fuzzy orthogonal set in X whose elements have fuzzy norm $\tilde{1}$, that is, for all $x, y \in M$,

$$\langle x, y\rangle = \begin{cases} \tilde{1} & , \quad x = y \\ \tilde{0} & , \quad x \neq y. \end{cases}$$

If a fuzzy orthogonal or fuzzy orthonormal set M is countable, we can arrage it in a sequence $\{x_n\}$ and call it a fuzzy orthogonal or fuzzy orthonormal sequence, respectively. More generally, a family $\{x_i\}_{i\in I}$, is called fuzzy orthogonal if $x_i \perp x_j$ for all $i, j \in I, i \neq j$. The family is called fuzzy orthonormal if it is fuzzy orthogonal and all x_i have fuzzy norm $\tilde{1}$, so that for all $i, j \in I$. We have

$$\langle x_i, x_j\rangle = \tilde{\delta_{ij}} = \begin{cases} \tilde{1} & , \quad i = j \\ \tilde{0} & , \quad i \neq j. \end{cases}$$

A linear combination of fuzzy vectors $x_1, x_2, \cdots, x_n$ of a fuzzy vector space X is an expression of the form

$$\tilde{\alpha_1}x_1 + \tilde{\alpha_2}x_2 + \cdots + \tilde{\alpha_n}x_n,$$

where the coefficients $\tilde{\alpha_1}, \tilde{\alpha_1}, \cdots, \tilde{\alpha_n}$ are fuzzy numbers. For any nonempty subset $M \subset X$ the set of all linear combinations of fuzzy vectors of M is called the span of M. Obviously, this is a subspace Y of X, we say that Y is generated by M.

Definition 3.2. Linear independence of a given set M of fuzzy vectors $x_1, x_2, \cdots, x_n$, $n \geq 1$ in a fuzzy vector space X is defined by means of the equation

$$\tilde{\alpha_1}x_1 + \tilde{\alpha_2}x_2 + \cdots + \tilde{\alpha_n}x_n \tag{3}$$

where $\tilde{\alpha_1}, \tilde{\alpha_1}, \cdots, \tilde{\alpha_n}$ are fuzzy numbers. Equation (3) holds for $\tilde{\alpha_1} = \tilde{\alpha_1} = \cdots = \tilde{\alpha_n} = \tilde{0}$. If this is the only n-tuple of fuzzy numbers for which (3) holds, the set is said to be linearly independent.

THEOREM 3.3. *Let H be a fuzzy Hilbert space. If $\{e_n\}_{n=1}^{\infty}$ is a fuzzy orthonormal sequence in H, then the following statements are equivalent:*

i) *$\{e_n\}_{n=1}^{\infty}$ is complete fuzzy orthonormal.*
ii) *If $\langle x, e_n\rangle = \tilde{0}$ for $n = 1, 2, \cdots$, then $x = 0$.*
iii) *$x = \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n$, $\forall x \in H$.*
iv) *$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2$, For every $x \in H$.*

Proof. (i)$\Rightarrow$(ii) Let $\{e_i\}$ be a complete fuzzy orthonormal sequence and $\langle x, e_n\rangle = \tilde{0}$ for $n = 1, 2, \cdots$. Setting for a fixed i and choosing $e_i = \dfrac{x}{\|x\|}$, we have: $\|e_i\|^2 = \langle e_i, e_i\rangle = \tilde{1}$ and $\langle e_i, e_n\rangle = \tilde{0}$ for $n = 1, 2, \ldots$. Therefore we get a fuzzy orthonormal sequence, that is a contraction to completeness. Therfore $e_i = 0$ and it follows that $x = 0$.

(ii)$\Rightarrow$(iii) This is obvious.

(iii)$\Rightarrow$(iv) Suppose that $x = \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n$ and let

$$[\langle x, e_n\rangle]_\alpha = \left[\langle x, e_n\rangle_\alpha^-, \langle x, e_n\rangle_\alpha^+\right].$$

Then

$$\begin{aligned}
\|x\|_\alpha^{+^2} &= \left\langle \sum_{n=1}^{\infty}\langle x, e_n\rangle_\alpha^+ e_n, \sum_{n=1}^{\infty}\langle x, e_n\rangle_\alpha^+ e_n \right\rangle \\
&= \sum_{n=1}^{\infty}\langle x, e_n\rangle_\alpha^+ \overline{\langle x, e_n\rangle_\alpha^+}\langle e_n, e_n\rangle = \sum_{n=1}^{\infty} |\langle x, e_n\rangle_\alpha^+|^2.
\end{aligned}$$

Similarly, $\|x\|_\alpha^{-^2} = \sum_{n=1}^{\infty} |\langle x, e_n\rangle_\alpha^-|^2$. □

REMARK 3.4. The mentioned equality in Theorem 3.3 (iv), is called Parseval's equation.

Let $X \neq \{0\}$ be a finite-dimensional fuzzy vector space. We will assume that X is equipped with a Felbin-fuzzy inner product $\langle ., .\rangle$, which we choose to be linear in the first entry. A sequence in X is a basis for X if th following conditions are satisfied:

i) $X = span\{e_k\}_{k=1}^{n}$,
ii) $\{e_k\}_{k=1}^{n}$ is linearly independent, i.e., if $x = \sum_{k=1}^{n} \tilde{\alpha_k} e_k = \tilde{0}$ for some scalar coefficients $\{\tilde{\alpha_k}\}_{k=1}^{n}$, then $\tilde{\alpha_k} = \tilde{0}$ for all $k = 1, ..., n$.

As a consequence of this definition, every $x \in X$ has a unique representation in terms of the elements in the basis, i.e., there exist unique coefficients $\{\tilde{\alpha_k}\}_{k=1}^{n}$ such that

$$x = \sum_{k=1}^{n} \tilde{\alpha_k} e_k. \tag{4}$$

Sometimes, in particular in high-dimensional vector spaces, it is cumbersome to find the coefficients $\{\tilde{\alpha_k}\}_{k=1}^{n}$. But if $\{e_k\}_{k=1}^{n}$ is a fuzzy orthonormal basis, i.e., a basis for which

$$\langle e_k, e_j\rangle = \tilde{\delta_{ij}} = \begin{cases} \tilde{1} & , \quad k = j \\ \tilde{0} & , \quad k \neq j, \end{cases}$$

then the coefficients $\{\tilde{\alpha_k}\}_{k=1}^{n}$ are easy to find. Taking the fuzzy inner product of x in (4) and let $[\langle x, e_j\rangle]_\alpha = [\langle x, e_j\rangle_\alpha^-, \langle x, e_j\rangle_\alpha^+]$ with an arbitrary e_j gives:

$$\langle x, e_j\rangle_\alpha^- = \langle \sum_{k=1}^{n} \tilde{\alpha_k} e_k, e_j\rangle_\alpha^- = \sum_{k=1}^{n} \tilde{\alpha_k}\langle e_k, e_j\rangle_\alpha^- = \tilde{\alpha_j},$$

and

$$\langle x, e_j\rangle_\alpha^+ = \langle \sum_{k=1}^{n} \tilde{\alpha_k} e_k, e_j\rangle_\alpha^+ = \sum_{k=1}^{n} \tilde{\alpha_k}\langle e_k, e_j\rangle_\alpha^+ = \tilde{\alpha_j}.$$

So

$$x = \sum_{k=1}^{n} \langle x, e_k\rangle e_k. \tag{5}$$

We now introduce fuzzy frames.

Definition 3.5. A countable family of elements $\{x_n\}_{n\in I}$ in finite-dimensional fuzzy Hilbert space X is a fuzzy frame for X if there exist fuzzy numbers $\eta, \mu \in F(R)$ and $\eta, \mu \succ \tilde{0}$ such that

$$\eta\|x\|^2 \preccurlyeq \sum_{n\in I} |\langle x, x_n\rangle|^2 \preccurlyeq \mu\|x\|^2, \forall x \in X. \tag{6}$$

The numbers η, μ are called fuzzy frame bounds. Fuzzy frame bounds are not unique. The optimal lower fuzzy frame bound is supremum over all upper fuzzy frame bounds, and the optimal upper fuzzy frame bound is the infimum over all upper fuzzy frame bounds. Note that the optimal fuzzy frame bounds actually are fuzzy frame bounds. The fuzzy frame is normalized if $\|x_n\| = \tilde{1}$ for all $n \in I$. A tight fuzzy frame is fuzzy frame with equal fuzzy frame bounds and in case $\eta = \mu = \tilde{1}$, we call Parseval fuzzy frame. In case the upper Inequality in 6 satisfy, $\{x_n\}_{n\in I}$ is called fuzzy Bessel sequence. It follows from the definition that if $\{x_n\}_{n\in I}$ is a fuzzy frame for X, then $\overline{span}\{x_n\}_{n\in I} = X$.

Theorem 3.6. *Let X be a fuzzy Hilbert space and $\{e_n\}_{n\in I}$ be a fuzzy orthonormal sequence of X. Then for every $x \in X$,*

$$\sum_{n\in I} |\langle x, x_n\rangle|^2 \preccurlyeq \mu\|x\|^2.$$

Proof. By Bessel's inequality in crisp inner product, we have

$$\sum_{n\in I} |\langle x, x_n\rangle|^2 \preccurlyeq \mu\|x\|^2.$$

Since $\|.\|_\alpha^s$ where $s = -, +$ is classic norms on X we have:

$$\sum_{n\in I} |\langle x, x_n\rangle_\alpha^-|^2 \preccurlyeq \mu(\|x\|_\alpha^-)^2,$$

and

$$\sum_{n\in I} |\langle x, x_n\rangle_\alpha^+|^2 \preccurlyeq \mu(\|x\|_\alpha^+)^2.$$

It follows that for every $x \in X$, $\sum_{n\in I} |\langle x, x_n\rangle|^2 \preccurlyeq \mu\|x\|^2$. □

Example 3.7. Let $\{e_n\}_{n=1}^2$ be a fuzzy orthonormal basis for a two-dimensional fuzzy vector space X with fuzzy inner product μ. Let

$$x_1 = e_1, x_2 = e_1 - e_2, x_3 = e_1 + e_2.$$

Then $\{x_n\}_{n=1}^3$ is a fuzzy frame for V. Since $[\langle x, x_n\rangle]_\alpha = [\langle x, x_n\rangle_\alpha^-, \langle x, x_n\rangle_\alpha^+]$

$$\begin{aligned}
\sum_{k=1}^{3} |\langle x, x_n\rangle_\alpha^-|^2 &= |\langle x, x_1\rangle_\alpha^-|^2 + |\langle x, x_2\rangle_\alpha^-|^2 + |\langle x, x_3\rangle_\alpha^-|^2 \\
&= |\langle x, e_1\rangle_\alpha^-|^2 + |\langle x, e_1 - e_2\rangle_\alpha^-|^2 + |\langle x, e_1 + e_2\rangle_\alpha^-|^2 \\
&= |\langle x, e_1\rangle_\alpha^-|^2 + |\langle x, e_1\rangle_\alpha^- - \langle x, e_2\rangle_\alpha^-|^2 + |\langle x, e_1\rangle_\alpha^- + \langle x, e_2\rangle_\alpha^-|^2 \\
&\preccurlyeq \tilde{3}(\|x\|_\alpha^-)^2
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=1}^{3} |\langle x, x_n\rangle_\alpha^+|^2 &= |\langle x, x_1\rangle_\alpha^+|^2 + |\langle x, x_2\rangle_\alpha^+|^2 + |\langle x, x_3\rangle_\alpha^+|^2 \\
&= |\langle x, e_1\rangle_\alpha^+|^2 + |\langle x, e_1 - e_2\rangle_\alpha^+|^2 + |\langle x, e_1 + e_2\rangle_\alpha^+|^2 \\
&= |\langle x, e_1\rangle_\alpha^+|^2 + |\langle x, e_1\rangle_\alpha^+ - \langle x, e_2\rangle_\alpha^+|^2 + |\langle x, e_1\rangle_\alpha^+ + \langle x, e_2\rangle_\alpha^+|^2 \\
&\preccurlyeq \tilde{3}(\|x\|_\alpha^+)^2
\end{aligned}$$

thus $\mu = \tilde{3}$ is an upper fuzzy frame bound, i.e., $\sum_{k=1}^3 |\langle x, x_n\rangle|^2 \preccurlyeq \tilde{3}\|x\|^2$. Similarly,

$$\textstyle\sum_{k=1}^3 |\langle x, x_n\rangle_\alpha^\pm|^2 \succcurlyeq \tilde{2}(\|x\|_\alpha^\pm)^2.$$

Thus $\sum_{k=1}^3 |\langle x, x_n\rangle|^2 \succcurlyeq \tilde{2}\|x\|^2$. And $\eta = \tilde{2}$ is a lower fuzzy frame bound.

Example 3.8. Let $\{x_n\}_{n=1}^\infty := \{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \cdots\}$; that is, $\{x_n\}_{n=1}^\infty$ is the sequence where each vector $\dfrac{1}{\sqrt{k}}e_k$, $k \in N$, is repeated k times. Then, for each $x \in X$,

$$\begin{aligned}
\sum_{n=1}^{\infty} |\langle x, x_n\rangle_\alpha^\pm|^2 &= \sum_{k=1}^{\infty} \tilde{k}|\langle x, \frac{1}{\sqrt{k}}x_k\rangle_\alpha^\pm|^2 \\
&= \sum_{k=1}^{\infty} \tilde{k}|\langle \frac{1}{\sqrt{k}}x_k, x\rangle_\alpha^\pm|^2 \\
&= (\|x\|_\alpha^\pm)^2.
\end{aligned}$$

therefore $\sum_{n=1}^\infty |\langle x, x_n\rangle|^2 = \|x\|^2$ and $\{x_n\}_{n=1}^\infty$ is a fuzzy frame for X with fuzzy frame bound $\eta = \mu = \tilde{1}$.

From [8], we know that each classical Hilbert space is a Felbin fuzzy Hilbert space, and all the results and theorems in the classical Hilbert space are true for the Felbin fuzzy Hilbert spaces. Therefore, each frame in the classical space is a fuzzy frame for the Felbin fuzzy Hilbert space. In the following, we give an example that it is not frame in the classical Hilbert space, but in the Felbin fuzzy Hilbert space, it is a fuzzy frame, and this is the reason for studying fuzzy frames in the Felbin fuzzy Hilbert space.

In the following example, we give a frame on $C(\Omega)$.

EXAMPLE 3.9. Suppose that $\{g_k(.)\}_{k\in I}$ is a sequence on $C(\Omega)$ (the vector space of all continuous functions on Ω) where $g_k(x) = sin\dfrac{x}{k}$ and $0 < x \leq 1$. We show that $\{g_k(.)\}_{k\in I}$ is fuzzy frame on $C(\Omega)$.
If $\alpha \in (0, 1]$, then there exists n such that $\dfrac{1}{n+1} < 1 - \alpha \leq \dfrac{1}{n}$. Consider $k_n = [\dfrac{1}{n}, 1 - \dfrac{1}{n}]$, then $\Omega = \bigcup_{n\in I} k_n$.

For $f \in C(\Omega)$ and let $[\langle f, g_k\rangle]_\alpha = [\langle f, g_k\rangle_\alpha^-, \langle f, g_k\rangle_\alpha^+]$ we have:

$$\begin{aligned} \sum_{k\in I} |\langle f, g_k\rangle_\alpha^-|^2 &= \sum_{k\in I} \left(\int_{k_n} f g_k \mathrm{d}\mu\right)^2 \\ &\preccurlyeq \sum_{k\in I} \left(\int_{k_n} f^2 \mathrm{d}\mu \int_{k_n} g_k^2 \mathrm{d}\mu\right) \\ &= (\|f\|_\alpha^-)^2 \mu(k_n) \sum_{k=1}^{\infty} \frac{1}{k^2} \\ &= \mu(\|f\|_\alpha^-)^2, \end{aligned}$$

and

$$\begin{aligned} \sum_{k\in I} |\langle f, g_k\rangle_\alpha^+|^2 &= \sum_{k\in I} \left(\int_{k_n} f g_k \mathrm{d}\mu\right)^2 \\ &\preccurlyeq \sum_{k\in I} \left(\int_{k_n} f^2 \mathrm{d}\mu \int_{k_n} g_k^2 \mathrm{d}\mu\right) \\ &= (\|f\|_\alpha^+)^2 \mu(k_n) \sum_{k=1}^{\infty} \frac{1}{k^2} \\ &= \mu(\|f\|_\alpha^+)^2. \end{aligned}$$

Thus

$$\textstyle\sum_{k\in I} |\langle f, g_k\rangle|^2 \preccurlyeq \mu\|f\|^2,$$

and the sequence $\{g_k(.)\}_{k\in I}$ is a fuzzy Bessel sequence on $C(\Omega)$.

For the lower boundedness, we consider

$$\eta := \textstyle\sum_{k\in I} |\langle f, g_k\rangle_\alpha^-|^2 = \bigwedge \left\{\sum_{k\in I} |\langle f, g_k\rangle_\alpha^-|^2 : f \in C(\Omega), \|f\|_\alpha^- = 1\right\}.$$

It is clear that $\eta > 0$. Now given $f \in C(\Omega)$ and $f \neq 0$, we have

$$\sum_{k\in I} |\langle f, g_k\rangle_\alpha^-|^2 = \sum_{k\in I} \left(\int_{k_n} f g_k \mathrm{d}\mu\right)^2$$

$$
\begin{aligned}
&= \sum_{k\in I}\left(\int_{k_n}\frac{f}{\|f\|_\alpha^-}g_k\mathrm{d}\mu\right)^2(\|f\|_\alpha^-)^2\\
&\succcurlyeq \eta(\|f\|_\alpha^-)^2,
\end{aligned}
$$

for similarly we consider

$$\eta := \sum_{k\in I}|\langle f, g_k\rangle_\alpha^+|^2 = \bigwedge\left\{\sum_{k\in I}|\langle f, g_k\rangle_\alpha^+|^2 : f\in C(\Omega), \|f\|_\alpha^+ = 1\right\}.$$

It is clear that $\eta > 0$. Now given $f\in C(\Omega)$ and $f\neq 0$, we have

$$
\begin{aligned}
\sum_{k\in I}|\langle f, g_k\rangle_\alpha^+|^2 &= \sum_{k\in I}\left(\int_{k_n} f g_k\mathrm{d}\mu\right)^2\\
&= \sum_{k\in I}\left(\int_{k_n}\frac{f}{\|f\|_\alpha^+}g_k\mathrm{d}\mu\right)^2(\|f\|_\alpha^+)^2\\
&\succcurlyeq \eta(\|f\|_\alpha^+)^2,
\end{aligned}
$$

Hence

$$\sum_{k\in I}|\langle f, g_k\rangle|^2 \succcurlyeq \eta\|f\|^2,$$

and $\{g_k(.)\}_{k\in I}$ is a fuzzy frame on $C(\Omega)$.

4. CONCLUSION

We know that $C(\Omega)$ was not normable on the classical Hilbert spaces, but in this paper we have shown $C(\Omega)$ is normable over fuzzy Hilbert spaces and by an example we defined fuzzy frame on $C(\Omega)$. In the above discussion, a frame in the classical Hilbert space is a fuzzy frame in the Felbin fuzzy Hilbert space. Therefore, all the frame theorems in the classical state have equivalent results in the Felbin's fuzzy Hilbret space.

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University of Maragheh
Department of Mathematics
Maragheh, Iran
E-mail: `bdaraby@maragheh.ac.ir`
E-mail: `fatenehdelzende@yahoo.com`
E-mail: `rahimi@maragheh.ac.ir`