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EXISTENCE OF SOLUTIONS FOR SOME DEGENERATE SEMILINEAR ELLIPTIC EQUATIONS WITH MEASURE DATA

ARUN KUMAR BADAJENA and SHESADEV PRADHAN

Abstract. We study the existence of a weak solution for the degenerate semilinear elliptic problem

$$-\sum_{i,j=1}^{n} \mathrm{D}_j(a_{ij}(x)\mathrm{D}_i u(x)) - \lambda g(x)u(x) = -f(x,u(x)) + \mu \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$ and λ is a real parameter. Here $g : \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are functions satisfying suitable hypotheses and μ is a Radon measure.

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1. INTRODUCTION

In this work we study the existence of (weak) solutions in the weighted Sobolev spaces $W_0^{1,2}(\Omega,\omega)$ for the Dirichlet problem

$$\begin{aligned} \mathrm{L}u(x) - \lambda g(x)u(x) &= -f(x,u(x)) + \mu \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where L is the partial differential operator $\mathrm{L}u(x) = -\sum_{i,j=1}^{n} \mathrm{D}_j(a_{ij}(x)\mathrm{D}_i u(x))$ with $\mathrm{D}_j = \frac{\partial}{\partial x_j}(j = 1; \cdots ; n), \lambda \in \mathbb{R}$, μ is a Radon measure, Ω is a bounded open subset of $\mathbb{R}^N$. The coefficients a_{ij} are measurable, real-valued functions and the coefficient matrix $A = (a_{ij})$ is symmetric and satisfies the degenerate elliptic condition

$$\alpha|\xi|^2\omega(x) \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \beta|\xi|^2\omega(x). \tag{2}$$

for all $\xi \in \mathbb{R}^N$ and for almost every $x \in \Omega$, ω is a weight function (i.e., a locally integrable function on $\mathbb{R}^N$) and α, β are positive constants.

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In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various kinds of singularities in the coefficients, it is natural to look for solutions in the weighted Sobolev spaces (see [2–5]). A class of weights which is well understood is the class of A_p weights (Muckenhoupt class) introduced by B. Muckenhoupt (see [10]). For more interesting examples of weights (p-admissible weights) we refer to [8].

The problem (1) with $\mu = 0$ is studied by Cavalheiro in [2]. In this work, we extend this result when $\mu \neq 0$ is a Radon measure. We note that the idea of the proof when μ is a compactly supported smooth function is same as in [2]. To study the existence of solutions for (1), the main tool we used is a compact embedding theorem obtained in [1]. The study is inspired by a problem in bounded domain given in the book by Zeidler [14].

This paper is organized as follows. Section 2 deals with preliminaries and weak formulation of the problem. Section 3 concerns with the main result, namely the existence of a weak solution of (1).

2. PRELIMINARIES

Let ω be a locally integrable nonnegative function on $\mathbb{R}^N$ and assume that $0 < \omega < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p weight, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|}\int_B \omega(x)\,\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B \omega^{\frac{1}{1-p}}(x)\,\mathrm{d}x\right)^{p-1} \leq C$$

for all ball $B \subset \mathbb{R}^N$, where $|\cdot|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^N$. If $1 < q \leq p$, then $A_q \subset A_p$ (we refer to [7, 8] or [11] for more information about A_p weights).

As an example of A_p weight, the function $\omega(x) = |x|^\delta,\ x \in \mathbb{R}^N$, is in A_p if and only if $-N < \delta < N(p-1)$ (see Corollary 4.4, Chapter IX in [11]). Any weight function $\omega(x)$ defines a measure on the measurable subsets of $\mathbb{R}^N$ denoted by ω. Let $A \subset \mathbb{R}^N$ be any measurable set. Then $\omega(A) = \int_A \omega(x)\,\mathrm{d}x$. We denote by $\mathcal{M}(\Omega)$ the set of all Radon measures on Ω.

Definition 2.1. Let ω be a weight, and let $\Omega \subset \mathbb{R}^N$ be open. For $0 < p < 1$ we define $L^p(\Omega,\omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)\,\mathrm{d}x\right)^{1/p} < \infty.$$

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega,\omega) \subset L^1_{loc}$ for every open set Ω (see Remark 1.2.4 in [12]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

DEFINITION 2.2. Let $\omega \in A_p$, $\Omega \subset \mathbb{R}^N$ be open and k be a nonnegative integer. We define the weighted Sobolev space $W^{k,p}(\Omega,\omega)$ as the set of functions $u \in L^p(\Omega,\omega)$ such that its weak derivatives $\mathrm{D}^\alpha u \in L^p(\Omega,\omega)$, $1 \le |\alpha| \le k$. The norm in $W^{k,p}(\Omega,\omega)$ is defined as

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \Big(\int_\Omega |u(x)|^p \omega(x)\,\mathrm{d}x + \sum_{1\le|\alpha|\le k} \int_\Omega |\mathrm{D}^\alpha u(x)|^p \omega(x)\,\mathrm{d}x \Big)^{1/p}. \tag{3}$$

We also define $W_0^{k,p}(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{k,p}(\Omega,\omega)} = \Big(\sum_{1\le|\alpha|\le k} \int_\Omega |\mathrm{D}^\alpha u(x)|^p \omega(x)\,\mathrm{d}x \Big)^{1/p}.$$

In the sequel, we denote by $H = W_0^{1,2}(\Omega,\omega)$ and the norm in H by

$$\|u\|_H = \|u\|_{W_0^{1,2}(\Omega,\omega)}.$$

If $\omega \in A_p$, then $W^{k,p}(\Omega,\omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm in (3). The spaces $W^{1,2}(\Omega,\omega)$ and $W_0^{1,2}(\Omega,\omega)$ are Hilbert spaces and hence reflexive. For more information about weighted Sobolev spaces $W^{k,p}(\Omega,\omega)$ with $\omega \in A_p$ (we refer to [8, 11, 12]). For information about weighted Sobolev spaces with other weights (we refer to [14]).

We need the following two results in the proof of the theorem.

THEOREM 2.3. *Let $\Omega \subset \mathbb{R}^N$ be open bounded and let $\omega \in A_p$, $1 < p < \infty$. Let $u_n \to u$ in $L^p(\Omega,\omega)$ then there exists a subsequence $\{u_{n_k}\}$ and a function $\Phi(x) \in L^p(\Omega,\omega)$ such that*

(1) *$u_{n_k}(x) \to u(x)$ ω- a.e in Ω as $n_k \to \infty$.*
(2) *$u_{n_k}(x) \le \Phi(x)$ ω- a.e in Ω.*

THEOREM 2.4. *Let $\Omega \subset \mathbb{R}^N$ be open bounded and let $\omega \in A_p$, $1 < p < \infty$. There exists constants c and η such that for all $u \in C_0^\infty(\Omega)$ and all k satisfying $1 \le k \le n/(n-1) + \eta$, we have*

$$\|u\|_{L^{kp}(\Omega,\omega)} \le c\|\nabla u\|_{L^p(\Omega,\omega)}$$

The proof of Theorem 2.3 follows the same lines as in [14] and for the proof of Theorem 2.4 we refer to [5].

REMARK 2.5. Let $\omega \in A_2$. Then

$$\|u\|_{L^2(\Omega,\omega)} \le C\|\nabla u\|_{L^2(\Omega,\omega)} = C\|u\|_H, \tag{4}$$

that is, the embedding $H \hookrightarrow L^2(\Omega,\omega)$ is continuous.

We next define the weak solution of the problem (1).

Definition 2.6. A function $u \in H$ is said to be the weak solution of the problem (1) if u satisfies

$$\begin{aligned}\int_\Omega a_{ij}(x)\mathrm{D}_i u(x)\mathrm{D}_j\phi(x)\,\mathrm{d}x - \lambda\int_\Omega g(x)u(x)\phi(x)\,\mathrm{d}x \\ = -\int_\Omega f(x,u(x))\phi(x)\,\mathrm{d}x + \int_\Omega \phi(x)\,\mathrm{d}\mu,\end{aligned} \tag{5}$$

for all $\phi \in H$.

Let Y^* denote the dual of the real Banach space Y and let $\|.\|$ denote the norm on real Banach space Y. Let $x \in Y$, $f \in Y^*$ and $\langle f|x\rangle_Y$ denotes the evaluation of linear functional f at x. From [14], we quote :

Definition 2.7. Let $\mathrm{A}{:}Y \to Y^*$ be an operator on a real Banach space Y.

(i) A is *monotone* if and only if $\langle \mathrm{A}u - \mathrm{A}v|u-v\rangle_Y \geq 0$ for all $u, v \in Y$.
(ii) A is *hemi-continuous* if and only if $t \mapsto \langle \mathrm{A}(u+tv), w\rangle_Y$ is continuous on $[0,1]$ for all $u, v, w \in Y$.
(iii) A is *angle-bounded* if and only if A is linear, monotone and there exists a constant $C \geq 0$ such that

$$|\langle \mathrm{A}u, v\rangle_Y - \langle \mathrm{A}v, u\rangle_Y|^2 \leq C\langle \mathrm{A}u, u\rangle_Y\langle \mathrm{A}v, v\rangle_Y$$

for all $u, v \in Y$.

In Section 3, we need the following result.

Theorem 2.8. *Let* $\mathrm{K} : X \to X^*$ *and* $\mathrm{F} : X^* \to X$ *be operators on the real separable Banach space* X*. Assume that*

(i) *the operator* $\mathrm{K} : X \to X^*$ *is linear, monotone and angle-bounded;*
(ii) *the operator* $\mathrm{F} : X^* \to X$ *is monotone and hemicontinuous.*

Then the operator equation $u + \mathrm{KF}u = 0$ *has exactly one solution* $u \in X^*$.

The proof of the Theorem 2.8 is found in [14, Theorem 28.A]. We need the following hypotheses for further study.

(F_1) Let the weight function $\omega \in A_p \cap C^1(\Omega)$ and suppose that ω can be expressed as $\omega(x) = f(r(x))$, where $r(x)$ is the distance $r(x) = dist(x, \partial\Omega)$ and $f \in C^1(\Omega)$ is positive, non-decreasing, has bounded derivative f' and satisfies $\lim_{r\to 0^+} f(r) = 0$.
(F_2) Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition, that is $x \mapsto f(x,t)$ is measurable on Ω for all t in $\mathbb{R}$ and $t \mapsto f(x,t)$ is continuous on $\mathbb{R}$ for almost all x in Ω;
(F_3) There exists two nonnegative functions g_1 and g_2 with $g_1 \in L^2(\Omega,\omega) \cap L^2(\Omega,\omega^{-1})$, $g_2 \in L^\infty(\Omega)$ and $g_2/\omega \in L^\infty(\Omega)$ such that

$$|f(x,t)| \leq g_1(x) + g_2(x)|t|;$$

(F_4) The map $t \mapsto f(x,t)$ is monotonically increasing on $\mathbb{R}$ for almost all x in Ω;
(F_5) The function $g/\omega \in L^\infty(\Omega)$.

3. THE MAIN RESULT

In this section, we establish the existence of a weak solution of (1).

THEOREM 3.1. *Assume that the hypotheses* (F_1)–(F_5) *hold. Suppose that* $\lambda C\|g/\omega\|_\infty < \alpha,\ \lambda > 0$, *where the constants* C *and* α *arise in the inequalities* (4) *and* (2), *respectively. Then the problem* (1) *has a unique weak solution* $u \in H$ *for every Radon measure* μ. *Moreover, if*

$$C(\lambda\|g/\omega\|_\infty + \|g_2/\omega\|_\infty) < \alpha,$$

then

$$\|u\|_H \le C_2. \tag{6}$$

Proof. The basic idea is to transform the problem (1) into an operator equation $u + \mathrm{KF}u = 0$ and then to use Theorem 2.8. We first prove the conclusion of Theorem 3.1 for μ a compactly supported smooth function and then we prove for any Radon measure μ.

Case I: $\mu \in C_c^\infty(\Omega)$.
We define the operator $\mathrm{B} : H \times H \to \mathbb{R}$ by

$$\mathrm{B}(u,\phi) = \int_\Omega a_{ij}(x)\mathrm{D}_i u(x)\mathrm{D}_j\phi(x)\,\mathrm{d}x - \lambda\int_\Omega g(x)u(x)\phi(x)\,\mathrm{d}x$$

for all $u, \phi \in H$. A function $u \in H$ is a weak solution of (1) if and only if

$$\mathrm{B}(u,\phi) = -\int_\Omega f(x,u(x))\phi(x)\,\mathrm{d}x, \qquad \text{for all } \phi \in H. \tag{7}$$

For convenience, we divide the proof into four steps.

Step 1: We define the operator $\mathrm{F} : L^2(\Omega,\omega) \to L^2(\Omega,\omega)$ by

$$(\mathrm{F}u)(x) = f(x,u(x)) - \mu(x).$$

By [2] the operator F is bounded, continuous and monotone.
Claim: B is bilinear, bounded and strongly positive. By Remark 2.5 and (F_5), we have

$$\begin{aligned}
|\mathrm{B}(u,\phi)| \le& \int_\Omega |a_{ij}(x)\mathrm{D}_i u(x)\mathrm{D}_j\phi(x)|\,\mathrm{d}x + \lambda\int_\Omega |g(x)||u(x)||\phi(x)|\,\mathrm{d}x\\
\le& \int_\Omega \beta|\mathrm{D}_i u(x)|\,|\mathrm{D}_j\phi(x)|\omega\,\mathrm{d}x + \lambda\|g/\omega\|_\infty\int_\Omega |u(x)||\phi(x)|\omega\,\mathrm{d}x\\
\le& \beta\Big(\int_\Omega |\mathrm{D}_i u(x)|^2\omega\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega |\mathrm{D}_j\phi(x)|^2\omega\,\mathrm{d}x\Big)^{1/2}\\
&\qquad + \lambda\|g/\omega\|_\infty\Big(\int_\Omega |u(x)|^2\omega\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega |\phi(x)|^2\omega\,\mathrm{d}x\Big)^{1/2}\\
\le& (\beta + \lambda C\|g/\omega\|_\infty)\|u\|_H\|\phi\|_H,
\end{aligned}$$

for all $u, \phi \in H$. Hence, the bilinear form B is bounded.

Since $\lambda C\|g/\omega\|_\infty < \alpha$, by Remark 2.5, and (F_5), we have

$$\begin{aligned}
\mathrm{B}(u,u) &= \int_\Omega a_{ij}(x)\mathrm{D}_i u(x)\mathrm{D}_j u(x)\,\mathrm{d}x - \lambda\int_\Omega g u^2\,\mathrm{d}x\\
&\geq \alpha\int_\Omega |\mathrm{D}_i u(x)|^2\,\omega\,\mathrm{d}x - \lambda\|g/\omega\|_\infty\int_\Omega u^2\omega\,\mathrm{d}x\\
&\geq (\alpha - \lambda C\|g/\omega\|_\infty)\|u\|_H^2
\end{aligned}$$

for all $u \in H$. Consequently, B is strongly positive.

Step 2: We consider the following linear boundary value problem

$$\begin{aligned}
-\mathrm{L}u(x) - \lambda g(x)u(x) &= h(x) \quad \text{in } \Omega\\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}\tag{8}$$

where $h : \Omega \to \mathbb{R}$ be defined by $h(x) := -f(x, v(x)) + \mu(x), v \in H$. Then, by [14, Theorem 22.C] there exists an unique $u \in H \subset L^2(\Omega,\omega)$ such that

$$\mathrm{B}(u,\phi) = \int_\Omega h(x)\phi(x)\,\mathrm{d}x.$$

We set $u = \mathrm{K}h$. By [14, Corollary 22.20] the solution operator $\mathrm{K} : L^2(\Omega,\omega) \to L^2(\Omega,\omega)$ is linear, monotone, compact and angle-bounded. Thus the problem (1) is equivalent to the operator equation

$$u + \mathrm{KF}u = 0, \qquad u \in L^2(\Omega,\omega).\tag{9}$$

Hence, by Theorem 2.8 there exists a unique solution to (9).

Step 3: In particular, taking $\phi = u \in W_0^{1,2}(\Omega,\omega)$, we have $\mathrm{B}(u,u) = -\int_\Omega f(x,u)u\,\mathrm{d}x + \int_\Omega \mu(x)u(x)\,\mathrm{d}x$. From Step 1, we obtain $\mathrm{B}(u,u) \geq (\alpha - \lambda C\|g/\omega\|_\infty)\|u\|_H^2$. Also, by (F_3) and Remark 2.5, we note that,

$$\begin{aligned}
&\left|-\int_\Omega f(x,u)u\,\mathrm{d}x + \int_\Omega \mu(x)u(x)\,\mathrm{d}x\right|\\
&\leq \int_\Omega |f(x,u)|\,|u|\,\mathrm{d}x + \int_\Omega |\mu(x)|\,|u(x)|\omega^{-1/2}\omega^{1/2}\,\mathrm{d}x\\
&\leq \int_\Omega (g_1 + g_2|u|)|u|\mathrm{d}x + \Big(\int_\Omega |\mu(x)|^2\,|u(x)|^2\omega\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega (\omega^{-1/2})^2\,\mathrm{d}x\Big)^{1/2}\\
&\leq \|g_1/\omega\|_2\Big(\int_\Omega |u(x)|^2\omega\,\mathrm{d}x\Big)^{1/2} + \|g_2/\omega\|_\infty\int_\Omega |u(x)|^2\omega\,\mathrm{d}x\\
&\qquad\qquad + \|\mu\|_\infty\Big(\int_\Omega |u(x)|^2\omega\,\mathrm{d}x\Big)^{1/2}\Big(\int_A \frac{1}{\omega}\,\mathrm{d}x\Big)^{1/2}\\
&\leq C(\|g_1/\omega\|_2 + \|\mu\|_\infty\|\omega^{-1}\|_1^{1/2})\|u\|_H + C\|g_2/\omega\|_\infty\|u\|_H^2,
\end{aligned}$$

where $A = \mathrm{supp}(\mu)$, is compact. Hence,

$$(\alpha - \lambda C\|g/\omega\|_\infty)\,\|u\|_H^2 \leq C(\|g_1/\omega\|_2 + \|\mu\|_\infty\|\omega^{-1}\|_1^{1/2})\|u\|_H + C\|g_2/\omega\|_\infty\|u\|_H^2$$

or,

$$\|u\|_H \leq C(\|g_1/\omega\|_2 + \|\mu\|_\infty \|\omega^{-1}\|_1^{1/2}) / \left(1 - C(\lambda\|g/\omega\|_\infty + \|g_2/\omega\|_\infty)\right) = C_2,$$

if

$$0 < \lambda < (\alpha - C\|g_2/\omega\|_\infty)/C\|g/\omega\|_\infty.$$

Case II: $\mu \in \mathcal{M}(\Omega)$.

For $\mu \in \mathcal{M}(\Omega)$, we can find a sequence $\{\mu_n\} \subset C_c^\infty(\Omega)$ such that

$$\int_\Omega \mu_n(x)\phi(x)\,\mathrm{d}x \to \int_\Omega \phi(x)\,\mathrm{d}\mu, \text{ for all } \phi \in C_0^\infty(\bar{\Omega}).$$

By the density of $C_0^\infty(\bar{\Omega})$ in H we also have

$$\int_\Omega \mu_n(x)\phi(x)\,\mathrm{d}x \to \int_\Omega \phi(x)\,\mathrm{d}\mu, \text{ for all } \phi \in H. \tag{10}$$

Now, from Case I we know that for each μ_n there exists a $u_n \in H$ such that u_n satisfies (1) with $\mu = \mu_n$ and

$$\|u_n\|_H \leq C.$$

Thus, the sequence $\{u_n\}$ is bounded in H and since H is reflexive there exists a subsequence, again denoted by, $\{u_n\}$ which converges weakly to some u in H.

We wish to show that u satisfies (1). Since ω satisfies (F_1), by [1, Lemma 5.2] the embedding

$$H \hookrightarrow L^2(\Omega, \omega)$$

is compact. Thus, $u_n \to u$ in $L^2(\Omega, \omega)$. Hence, by Theorem 3.2 there exists a subsequence, again denoted by, $\{u_n\}$ and a function $\Phi \in L^2(\Omega, \omega)$ such that $u_n \to u$, $\omega-$a.e. in Ω and $|u_n(x)| \leq \Phi(x)$, $\omega-$a.e. in Ω. Hence by (F_2)

$$f(x, u_n(x)) \to f(x, u(x)),\ \omega - a.e. \text{ in } \Omega.$$

And so,

$$f(x, u_n(x))\phi(x) \to f(x, u(x))\phi(x),\ \omega - a.e. \text{ in } \Omega,\ \forall \phi \in H.$$

Now, by $(H3)$ we have

$$\begin{aligned}\int_\Omega |f(x, u_n(x))\phi(x)|\,\mathrm{d}x &\leq \int_\Omega (g_1 + g_2|u_n|)|\phi|\,\mathrm{d}x \\ &\leq \|g_1/\omega\|_2\|\phi\|_{L^2(\Omega,\omega)} + \|g_2\|_\infty \int_\Omega \Phi(x)|\phi(x)\,\mathrm{d}x < \infty,\end{aligned}$$

since $\Phi \in L^2(\Omega, \omega)$ and $\phi \in H$. Thus, by Dominated convergence theorem,

$$\int_\Omega f(x, u_n(x))\phi(x)\,\mathrm{d}x \to \int_\Omega f(x, u(x))\phi(x)\,\mathrm{d}x. \tag{11}$$

Similarly, since $g/\omega \in L^\infty$ by (F_2), we have

$$\int_\Omega g(x)u_n(x)\phi(x)\,\mathrm{d}x \to \int_\Omega g(x)u(x)\phi(x)\,\mathrm{d}x. \tag{12}$$

Next, we need to show that

$$\int_\Omega a_{ij}(x)\mathrm{D}_i u_n(x)\mathrm{D}_j\phi(x)\,\mathrm{d}x \to \int_\Omega a_{ij}(x)\mathrm{D}_i u(x)\mathrm{D}_j\phi(x)\,\mathrm{d}x,\ \forall\phi\in H. \tag{13}$$

Now, for each $\phi\in H$, since the map $u\mapsto\int_\Omega a_{ij}\mathrm{D}_i u\,\mathrm{D}_j\phi$ is a continuous linear map on H and since $u_n\rightharpoonup u$ in H, we have

$$\int_\Omega a_{ij}(x)\mathrm{D}_i u_n(x)\mathrm{D}_j\phi(x)\,\mathrm{d}x \to \int_\Omega a_{ij}(x)\mathrm{D}_i u(x)\mathrm{D}_j\phi(x)\,\mathrm{d}x,\ \text{for each } \phi\in H. \tag{14}$$

That is, (13) holds. Now, combining (10), (11), (12) and (13) we have that u satisfies (5) that is, u is a weak solution of (1). Since, for each n, u_n satisfies (6), u also satisfies (6). This completes the proof of the theorem. □

In the following two results, we consider the cases $\lambda<0$, $\lambda>0$ and relax the hypothesis $\lambda\|g/\omega\|_\infty C<\alpha$ under the restriction that the function g does not change sign. The proof is similar to the Theorem 3.1; we restrict ourselves to sketch the deviations wherever needed.

Theorem 3.2. *Assume that the hypotheses* (F_1)–(F_5) *hold. Suppose that* $g\geq 0,\ \ \lambda<0$. *Then the BVP* (1) *has exactly one weak solution* $u\in H$.

Proof. We present a brief sketch of the proof: we note that

$$\begin{aligned}|\mathrm{B}(u,\phi)| &\leq \int_\Omega |a_{ij}(x)|\,|\mathrm{D}_i u(x)|\,|\mathrm{D}_j\phi(x)|\,\mathrm{d}x + |\lambda|\int_\Omega |g(x)|\,|u(x)|\,|\phi(x)|\,dx\\ &\leq \big(1+C|\lambda|\|g\|_\infty\big)\|u\|_H\|\phi\|_H.\end{aligned}$$

Consequently, B is bounded. Since $g\geq 0$, $\lambda<0$, as in Step 2 of Theorem 3.1 we note that

$$\begin{aligned}\langle \mathrm{B}u,u\rangle &= \int_\Omega a_{ij}(x)\mathrm{D}_i u(x)\mathrm{D}_j u(x)\,\mathrm{d}x - \lambda\int_\Omega g u^2\,\mathrm{d}x\\ &\geq \int_\Omega a_{ij}(x)\mathrm{D}_i u(x)\mathrm{D}_j u(x)\,\mathrm{d}x = \|u\|_H^2\end{aligned}$$

for all $u\in H$, which shows that, B is strongly positive. Since B is bilinear, bounded and strongly positive, as in Step 2 of Theorem 3.1, equation (8) has a weak solution, say $u\in H$. Also, the solution operator K is linear, monotone, compact and angle-bounded. Hence, the operator equation (9) has exactly one solution. □

Similarly, we have the following result.

Theorem 3.3. *Assume the hypotheses* (F_1)–(F_5). *Suppose that* $g\leq 0$, $\lambda>0$. *Then* (1) *has exactly one weak solution* $u\in H$.

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National Institute of Technology Rourkela
Department of Mathematics
Odisha, India
E-mail: `ons.arun93@gmail.com`
`https://orcid.org/0000-0001-5637-8510`
E-mail: `shesadevpradhan@gmail.com`
`https://orcid.org/0000-0002-1476-6301`