MATHEMATICA, Tome 62 (85), N° 2, 2020, pp. 179–188

ON DISTRIBUTIVE LATTICES OF LEFT k-ARCHIMEDEAN SEMIRINGS

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Abstract. Here we introduce the notion of left k-Archimedean semirings which generalize the notion of k-Archimedean semirings [1], and characterize the semirings which are distributive lattices (chains) of left k-Archimedean semirings. A semiring S is a left k-Archimedean semiring if for all $a, b \in S,\ b \in \sqrt{Sa}$, the k-radical of Sa. A semiring S is a distributive lattice of left k-Archimedean semirings if and only if for all $a, b \in S,\ ab \in \sqrt{Sa}$ and S is a chain of left k-Archimedean semirings if and only if $\sqrt{L}$ is a completely prime k-ideal, for every left k-ideal L of S.

MSC 2010. 16Y60.

Key words. k-radical, left k-Archimedean semiring, completely prime k-ideal, semiprimary k-ideal, congruence, distributive lattice congruence.

1. INTRODUCTION

In 1941, A. H. Clifford [4] first introduced and studied the semilattice decompositions of semigroups. The idea consists of decomposing a given semigroup S into component subsemigroups which are of simpler structure, through a congruence η on S such that the quotient semigroup S/η is the greatest semilattice homomorphic image of S and each η-class is a component subsemigroup. This well known result has since been generalized by M. S. Putcha, S. Bogdanović, M. Ćirić, F. Kmet and many others [3], [7], [8].

Both the greatest semilattice decomposition of semigroups and the greatest distributive lattice decomposition of semirings evolve out of the divisibility relation. In an additive idempotent semiring S, we define $a \longrightarrow b$ if $a \mid b^n$ for some $n \in \mathbb{N}$. The binary relation $\longrightarrow$ is neither symmetric nor transitive in general, which allows us to find the least distributive lattice congruence as the least congruence from $\longrightarrow$ in several ways. For example, symmetric opening of the transitive closure and the transitive closure of the symmetric opening of $\longrightarrow$ give us different description of the least distributive lattice congruence on S. Such variations in the description of the least distributive

The authors thank the editor for communicating the paper and the referee for his careful reading of the manuscript and helpful suggestions.

DOI: 10.24193/mathcluj.2020.2.07

lattice congruence lead us to introduce some new kinds of semirings such as k-Archimedean semirings. Considering left divisibility relation on a semiring as a generalization of the divisibility relation gives us the idea of left k-Archimedean semirings.

This article is a continuation of [1] where we introduced k-Archimedean semirings and studied the semirings which are distributive lattices of k-Archimedean semirings. Here we introduce the left k-Archimedean semirings and characterize the semirings which are distributive lattices (chains) of left k-Archimedean semirings. The left k-ideals play a crucial role in characterizing such semirings. A necessary and sufficient condition for a semiring S to be a distributive lattice of left k-Archimedean semirings is that $\sqrt{L}$ is a k-ideal of S, for every left k-ideal L of S.

The preliminaries and prerequisites we need are discussed in section 2. In section 3, several equivalent characterizations are made for the semirings which are distributive lattices of left k-Archimedean semirings, which is the main theorem of this article. In section 4, the semirings which are chains of left k-Archimedean semirings are characterized. A semiring S is a chain of left k-Archimedean semirings if and only if k-radical of every left k-ideal of S is a completely prime k-ideal.

2. PRELIMINARIES

A *semiring* $(S, +, \cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that both the *additive reduct* $(S, +)$ and the *multiplicative reduct* $(S, \cdot)$ are semigroups and such that the following distributive laws hold:

$$x.(y + z) = x.y + x.z \text{ and } (x + y).z = x.z + y.z.$$

Every distributive lattice D can be regarded as a semiring $(D, +, \cdot)$ such that both the additive reduct $(D, +)$ and the multiplicative reduct $(D, \cdot)$ are semilattices(that is, commutative and idempotent) together with the absorptive law:

$$x + x.y = x \text{ for all } x, y \in S.$$

Now onwards, we write xy for $x.y$ for $x, y \in S$. Thus a semiring is regarded as a common generalization of both rings and distributive lattices. By $\mathbb{SL}^+$ we denote the variety of all semirings $(S, +, \cdot)$ with $(S, +)$ is a semilattice. Throughout this paper, unless otherwise stated, S is always a semiring in $\mathbb{SL}^+$. Let A be a nonempty subset of S. Then the *k-closure* of A in S is defined by

$$\overline{A} = \{x \in S \mid x + a_1 = a_2 \text{ for some } a_1, a_2 \in A\}.$$

We have, $A \subseteq \overline{A}$ and if $(A, +)$ is a subsemigroup of $(S, +)$ then $\overline{A} = \{x \in S \mid \exists\ a \in A \text{ such that } x + a \in A\}$ and $\overline{\overline{A}} = \overline{A}$, since $(S, +)$ is a semilattice. A is called a k-set if $\overline{A} \subseteq A$. An ideal (left, right) K of S is called a *k-ideal* *(left, right)* of S if it is a k-set. The principal left k-ideal generated by $a \in S$ is denoted by $L_k(a)$ and is given by,

$$L_k(a) = \{x \in S \mid x + s_1a + a = s_2a + a, \text{ for some } s_1, s_2 \in S\}.$$

A nonempty subset A of S is called *completely prime (resp. semiprimary)* if for $x, y \in S$, $xy \in A$ implies $x \in A$ or $y \in A$ (resp. $x^n \in L$ or $y^n \in L$, for some $n \in \mathbb{N}$). Let F be a subsemiring of S. F is called a *left (right) filter* of S if: (i) for any $a, b \in S$, $ab \in F \Rightarrow b \in F$ $(a \in F)$; and (ii) for any $a \in F$, $b \in S$, $a + b = b \Rightarrow b \in F$. F is a *filter* of S if it is both a left and a right filter of S. The least filter of S containing a is denoted by $N(a)$. Let $\mathcal{N}$ be the equivalence relation on S defined by

$$\mathcal{N} = \{(x, y) \in S \times S \mid N(x) = N(y)\}.$$

An equivalence relation ρ on a semiring S is called a *congruence* on S if for $a, b, c, d \in S$,

$$a\rho b \text{ and } c\rho d \text{ implies } (a + c)\rho(b + d) \text{ and } ac\rho bd,$$

or, equivalently,

$$a\rho b \text{ implies } (a + c)\rho(b + c), (c + a)\rho(c + b), ac\rho bc, ca\rho cb.$$

A congruence relation ρ on S is called a *distributive lattice congruence* on S if the quotient semiring S/ρ is a distributive lattice. If $\mathcal{C}$ is a class of semirings we refer to semirings in $\mathcal{C}$ as $\mathcal{C}$-semirings. A semiring S is called a distributive lattice(resp. chain) of $\mathcal{C}$-semirings if there exists a congruence ρ on S such that S/ρ is a distributive lattice(resp. chain) and each ρ-class is a semiring in $\mathcal{C}$[1], [2].

We refer to [6] for the information we need concerning semigroup theory and to [2], [5] for notions concerning semiring theory.

Lemma 2.1. *Let S be a semiring.*

(a) *For $a, b \in S$ the following statements are equivalent:*

(i) *there are $s_i \in S$ such that $b + s_1a = s_2a$;*

(ii) *there are $s \in S$ such that $b + sa = sa$.*

(b) *If $a, b, c, d \in S$ are such that $c + xa = xa$ and $d + yb = yb$ for some $x, y \in S$, then there is some $z \in S$ such that $c + za = za$ and $d + zb = zb$.*

Proof. (a) Since (ii)⇒(i) is clear, we assume (i). For $x = s_1 + s_2$ one gets $b + s_1a + xa = s_2a + xa$ since $(S, +)$ is a semilattice. Hence (i) implies (ii).

(b) Clearly, $z = x + y$ is such an element. □

In view of this lemma, it follows that for $a \in S$, we have

$$L_k(a) = \{x \in S \mid x + sa + a = sa + a, \text{ for some } s \in S\}.$$

It is interesting to note that $\overline{Sa} = \{x \in S \mid x + sa = sa, \text{ for some } s \in S\}$ is a left k-ideal of S but may not contain a. Let A be a non-empty subset of a semiring S. Then we define the *k-radical* of A in S by

$$\sqrt{A} = \{x \in S \mid (\exists\, n \in \mathbb{N})\ x^n \in \overline{A}\}.$$

The notion of *k-Archimedean semiring* was introduced in [1]. Here we introduce *left k-Archimedean semiring.*

DEFINITION 2.2. A semiring S is called *left k-Archimedean* if for all $a \in S$, $S = \sqrt{Sa}$.

Right k-Archimedean semiring can be defined dually. Now by Lemma 2.1, a semiring S is left k-Archimedean if and only if for all $a, b \in S$ there exist $n \in \mathbb{N}$ and $x \in S$ such that $b^n + xa = xa$.

EXAMPLE 2.3. Let $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$, define '+' and '·' on $S = A \times A$ by: for all $(a, b), (c, d) \in S$,

$$(a, b) + (c, d) = (max\{a, c\}, max\{b, d\}), \ (a, b) \cdot (c, d) = (ac, b).$$

Then $(S, +, \cdot)$ is a left k-Archimedean semiring. Now let $(a, \frac{1}{2}), (c, \frac{1}{3}) \in S$. If possible, let there exist $n \in \mathbb{N}$ and $(x, y) \in S$ satisfying $(a, \frac{1}{2})^n + (c, \frac{1}{3})(x, y) = (c, \frac{1}{3})(x, y)$. This implies $(a^n, \frac{1}{2}) + (cx, \frac{1}{3}) = (cx, \frac{1}{3})$, which gives $max\{a^n, cx\} = cx$, $max\{\frac{1}{2}, \frac{1}{3}\} = \frac{1}{3}$. But the last equality is absurd. Consequently, $(S, +, \cdot)$ is not a right k-Archimedean semiring.

A semiring S is called a distributive lattice (chain) of left k-Archimedean semirings if there exists a congruence ρ on S such that S/ρ is a distributive lattice (chain) and each ρ-class is a left k-Archimedean semiring.

3. DISTRIBUTIVE LATTICE OF LEFT K-ARCHIMEDEAN SEMIRINGS

In this section, we characterize the semirings which are distributive lattices of left k-Archimedean semirings. In the subsequent proofs we will use that from $b + c = c$ for $b, c \in S$ in any semiring S it follows that $b^n + c^n = c^n$ for every $n \in \mathbb{N}$. This claim can be proved by induction. Since the case $n = 1$ is given, we may assume $b^n + c^n = c^n$ for some $n \in \mathbb{N}$. Then $b^{n+1} + c^n b = c^n b$ and by adding c^{n+1} on both sides we get $b^{n+1} + c^n(b + c) = c^n(b + c)$, and hence $b^{n+1} + c^{n+1} = c^{n+1}$.

LEMMA 3.1. *Let S be a semiring such that for all $a, b \in S$, $ab \in \sqrt{Sa}$. Then the following statements hold.*

(1) *For all $a, b \in S$, $\sqrt{Sab} = \sqrt{Sa} \cap \sqrt{Sb}$.*
(2) *For all $a, b \in S$, $\sqrt{Sab} = \sqrt{Sba}$.*
(3) *For all $a, b \in S$, $b \in \overline{Sa}$ implies that $b \in \sqrt{Sa^{2r}}$ for every $r \in \mathbb{N}$.*
(4) *For all $a, b \in S$, $a \in \sqrt{Sb}$ implies that $\sqrt{Sa} \subseteq \sqrt{Sb}$.*
(5) *The least distributive lattice congruence η on S is given by: for all $a, \ b \in S$,*

$$a\eta b \Leftrightarrow \ a \in \sqrt{Sb} \text{ and } b \in \sqrt{Sa}.$$

Proof. (1) For any $x \in \sqrt{Sab}$ we have $x \in \sqrt{Sb}$, and there are $n \in N$ and $y \in S$ such that $x^n + yab = yab$. Again, $(yab)^m \in \overline{Sya} \subseteq \overline{Sa}$ for some $m \in N$. Then $x^{nm} + (yab)^m = (yab)^m$ implies that $x^{nm} \in \overline{Sa}$, that is, $x \in \sqrt{Sa}$. Thus

$\sqrt{Sab} \subseteq \sqrt{Sa} \cap \sqrt{Sb}$. Conversely, for $x \in \sqrt{Sa} \cap \sqrt{Sb}$ there exist $n \in \mathbb{N}$ and $s \in S$ such that $x^n + sa = sa$ and $x^n + sb = sb$, and we get $x^{2n} + sasb = sasb$. Also there are $m \in \mathbb{N}$ and $u \in S$ such that $(bsas)^m + ubsa = ubsa$. Now we have $x^{2n(m+1)} + sas(bsas)^m b = sas(bsas)^m b$, that is, $x^{2n(m+1)} + sasubsab = sasubsab$, which yields $x \in \sqrt{Sab}$. Thus $\sqrt{Sa} \cap \sqrt{Sb} \subseteq \sqrt{Sab}$. Hence the result follows.

(2) Follows from (1).

(3) Let $b \in \overline{Sa}$. Then $b + sa = sa$, for some $s \in S$. Also there are $n \in \mathbb{N}$ and $t \in S$ such that $(as)^n + ta = ta$. Now $b^{n+1} + (sa)^{n+1} = (sa)^{n+1}$ gives $b^{n+1} + sta^2 = sta^2$. This yields $b \in \sqrt{Sa^2}$. So the result is true for $r = 1$. Let $b \in \sqrt{Sa^{2^k}}$, for some $k \in \mathbb{N}$. Then there exist $n \in \mathbb{N}$ and $s \in S$ such that $b^n + sa^{2^k} = sa^{2^k}$. Also $(a^{2^k}s)^m + ta^{2^k} = ta^{2^k}$, for some $m \in \mathbb{N}$ and $t \in S$. Then we have $b^{n(m+1)} + s(a^{2^k}s)^m a^{2^k} = s(a^{2^k}s)^m a^{2^k}$, that is, $b^{n(m+1)} + sta^{2^{k+1}} = sta^{2^{k+1}}$ which gives $b \in \sqrt{Sa^{2^{k+1}}}$. Hence by the principle of mathematical induction, $b \in \sqrt{Sa^{2^r}}$ for all $r \in \mathbb{N}$.

(4) For $a \in \sqrt{Sb}$ we have $m \in \mathbb{N}$ and $s \in S$ such that $a^m + sb = sb$. Let $x \in \sqrt{Sa}$. Then there is $n \in \mathbb{N}$ such that $x^n \in \overline{Sa}$. Suppose $r \in \mathbb{N}$ such that $2^r > m$. By (3), $x^n \in \sqrt{Sa^{2^r}}$ so that there are $p \in \mathbb{N}$ and $u \in S$ such that $x^{np} + ua^{2^r} = ua^{2^r}$ which gives $x^{np} + ua^{2^r - m}sb = ua^{2^r - m}sb$, that is, $x \in \sqrt{Sb}$. Hence the result.

(5) From [1, Theorem 3.4], we have the least distributive lattice congruence η on S as follows:

$$\eta = \rho \cap \rho^{-1}, \text{ where } \rho = \sigma^* \text{ and } a\sigma b \Leftrightarrow b \in \sqrt{SaS}.$$

Let ξ be the binary relation on S defined by: for all $a,\ b \in S$,

$$a\eta b \Leftrightarrow\ a \in \sqrt{Sb} \text{ and } b \in \sqrt{Sa}.$$

We will show $\xi = \eta$. Clearly $\sqrt{Sa} \subseteq \sqrt{SaS}$. Now let $x \in \sqrt{SaS}$. Then there are $n \in \mathbb{N}$ and $s \in S$ such that $x^n + sas = sas$. Again, $sas \in \sqrt{Ssa} \subseteq \sqrt{Sa}$, which implies that $(sas)^m + ta = ta$ for some $m \in \mathbb{N}$ and $t \in S$. Then $x^{nm} \in \overline{Sa}$, i.e. $x \in \sqrt{Sa}$. Thus $\sqrt{SaS} = \sqrt{Sa}$. Now $a\eta b$ implies that there are $c_1, c_2, ...c_n, d_1, d_2, ...d_m \in S$ such that $a\sigma c_1, c_1\sigma c_2, ..., c_{n-1}\sigma c_n, c_n\sigma b$ and $b\sigma d_1, d_1\sigma d_2, ..., d_{m-1}\sigma d_m, d_m\sigma a$. Then $c_1 \in \sqrt{Sa}, c_2 \in \sqrt{Sc_1}, ..., b \in \sqrt{Sc_n}$ and $d_1 \in \sqrt{Sa}, d_2 \in \sqrt{Sd_1}, ..., b \in \sqrt{Sd_m}$ so that $b \in \sqrt{Sa}$ and $a \in \sqrt{Sb}$. Thus $a\xi b$. Again $a\xi b$ implies $b \in \sqrt{Sa}$ and $a \in \sqrt{Sb}$ which yields $a\sigma b$ and $b\sigma a$, that is, $a\eta b$. Thus $\xi = \eta$. □

Remark 3.2. Let S be a semiring and $a \in S$. Then $\overline{Sa} \subseteq L_k(a)$ and usually this inclusion is proper. But, it is interesting to note that $\sqrt{Sa} = \sqrt{L_k(a)}$. Thus it follows that if for all $a, b \in S,\ ab \in \sqrt{Sa}$ then the least distributive lattice congruence η on S is given by: for all $a,\ b \in S$,

$$a\eta b \Leftrightarrow\ a \in \sqrt{L_k(b)} \text{ and } b \in \sqrt{L_k(a)}.$$

Now we prove the main theorem of this article.

THEOREM 3.3. *The following conditions on a semiring S are equivalent:*

(1) *S is a distributive lattice of left k-Archimedean semirings;*
(2) *for all $a, b \in S$, $b \in \overline{SaS}$ implies that $b \in \sqrt{Sa}$;*
(3) *for all $a, b \in S$, $ab \in \sqrt{Sa}$;*
(4) *for all $a, b \in S$, $ab \in \sqrt{L_k(a)}$;*
(5) *for all $a \in S$, $\sqrt{L_k(a)}$ is a k-ideal;*
(6) *$\sqrt{L}$ is a k-ideal of S, for every left k-ideal L of S;*
(7) *$\sqrt{Sa}$ is a k-ideal of S, for all $a \in S$;*
(8) *$N(a) = \{x \in S \mid a \in \sqrt{Sx}\}$, for all $a \in S$;*
(9) *for all $a, b \in S$, $\sqrt{Sab} = \sqrt{Sa} \cap \sqrt{Sb}$.*

Proof. Scheme of the proof: $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$, $(3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (3)$, $(3) \Leftrightarrow (8)$, $(3) \Leftrightarrow (9)$.

$(1) \Rightarrow (2)$: Let S be a distributive lattice $D = S/\rho$ of left k-Archimedean semirings $L_\alpha = a\rho$, $\alpha \in D$ and $a \in S$. Let a, $b \in S$ be such that $b \in \overline{SaS}$. Then $b + xax = xax$ for some $x \in S$. Now $xax\rho xa$ implies that xax, $xa \in L_\alpha$, for some $\alpha \in D$. Since L_α is a left k-Archimedean semiring, there exist $n \in \mathbb{N}$ and $y \in L_\alpha$ such that $(xax)^n + yxa = yxa$. Now $b^n + (xax)^n = (xax)^n$ implies that $b^n + yxa = yxa$, and so $b \in \sqrt{Sa}$.

$(2) \Rightarrow (3)$: This follows from $(ab)^2 \in \overline{SaS}$ and by (2).

$(3) \Rightarrow (1)$: By Lemma 3.1, the least distributive lattice congruence η on S is given by: for a, $b \in S$,

$$a\eta b \Leftrightarrow a \in \sqrt{Sb} \text{ and } b \in \sqrt{Sa}.$$

Let L be an η class. Then L is a subsemiring of S, since η is a distributive lattice congruence. Let $a, b \in L$. Then there exist $n \in \mathbb{N}$ and $x \in S$ such that $a^n + xb = xb$. Again $axb \in \sqrt{Sax}$ implies that there are $m \in \mathbb{N}$ and $y \in S$ such that $(axb)^m + yax = yax$. Now we have $a^{n+1} + axb = axb$ which yields $a^{m(n+1)} + yax = yax$ so that $a \in \sqrt{Sax}$. Also $ax \in \sqrt{Sa}$. Thus $a\eta ax$ which implies that $ax \in L$. Hence $a^{n+1} + axb = axb$ shows that $a \in \sqrt{Lb}$. Thus L is a left k-Archimedean semiring.

$(3) \Rightarrow (4)$: Let $a, b \in S$. Then $\overline{Sa} \subseteq L_k(a)$ implies that $ab \in \sqrt{L_k(a)}$.

$(4) \Rightarrow (5)$: Let $a, c \in S$ and $u \in \sqrt{L_k(a)}$. Then $uc \in \sqrt{L_k(a)}$, by (4) of Lemma 3.1 and Remark 3.2. Also there exist $n \in \mathbb{N}$ and $s \in S$ such that $u^n + sa + a = sa + a$. Let $r \in \mathbb{N}$ be such that $2^r > n$. Now $cu \in \overline{Su}$ implies that $cu \in \sqrt{Su^{2^r}}$, whence there exist $p \in \mathbb{N}$ and $y \in S$ such that $(cu)^p + yu^{2^r} = yu^{2^r}$. Then we have $(cu)^p + (yu^{2^r-n}s + yu^{2^r-n})a = (yu^{2^r-n}s + yu^{2^r-n})a$ to get $cu \in \sqrt{L_k(a)}$. Let $u, v \in \sqrt{L_k(a)}$. Then there exist $n \in \mathbb{N}$ and $t \in S$ such that $u^n + ta = ta$ and $v^n + ta = ta$. Now we can write $(u + a)^n + sas + sa + as = u^n + sas + sa + as$, for some $s \in S$. Then, for $x = (u + a)s + s(u + a) + (u + a)t + u + a$ we have $(u + a)^{n+2} + xax = xax$ which implies that $u + v \in \sqrt{L_k(ax)}$. Again $ax \in \sqrt{L_k(a)}$ implies that

$\sqrt{L_k(ax)} \subseteq \sqrt{L_k(a)}$, by Lemma 3.1. Thus $u + a \in \sqrt{L_k(a)}$ which again implies that $\sqrt{L_k(u+a)} \subseteq \sqrt{L_k(a)}$. Arguing in a similar way, we have, $(u+v)^n + sus + su + us = v^n + sus + su + us$ for some $s \in S$, which implies that $(u+v)^{n+2} + w(u+a)w = w(u+a)w$, where $w = (u+v)s + s(u+v) + (u+v)t + u + v$. Thus $u+v \in \sqrt{L_k((u+a)w)} \subseteq \sqrt{L_k(u+a)} \subseteq \sqrt{L_k(a)}$ i.e. $u+v \in \sqrt{L_k(a)}$. Thus $\sqrt{L_k(a)}$ is an ideal of S. Let $s \in S$ and $l \in \sqrt{L_k(a)}$ be such that $s+l = l$. Then there exist $n \in \mathbb{N}$ and $t \in S$ such that $l^n + ta + a = ta + a$. Then $s^n + l^n = l^n$ implies that $s^n + ta + a = ta + a$, that is, $s \in \sqrt{L_k(a)}$. Thus $\sqrt{L_k(a)}$ is a k-ideal of S.

(5) $\Rightarrow$ (6) : Let L be a left k-ideal of S. Let $u, v \in \sqrt{L}$ and $s \in S$. Then there exist $n \in \mathbb{N}$ and $l_1, l_2 \in L$ such that $u^n + l_1 = l_1$ and $v^n + l_2 = l_2$. This implies that $u^n + l = l$ and $v^n + l = l$, where $l = l_1 + l_2 \in L$. Now (5) shows that $su, us, u+v \in \sqrt{L_k(l)} \subseteq \sqrt{L}$. Thus $\sqrt{L}$ is an ideal of S. Similarly as in (4) $\Rightarrow$ (5), it can be proved that $\sqrt{L}$ is a k-ideal of S.

(6) $\Rightarrow$ (7) : Let $a \in S$. Then $\overline{Sa}$ is a left k-ideal of S. Thus it follows that $\sqrt{Sa}$ is a k-ideal of S.

(7) $\Rightarrow$ (3) : Let $a, b \in S$. Then $a \in \sqrt{Sa}$ and $\sqrt{Sa}$ is a k-ideal of S. Thus $ab \in \sqrt{Sa}$.

(3) $\Rightarrow$ (8) : Let $a \in S$, $F = \{x \in S \mid a \in \sqrt{Sx}\}$ and $y, z \in F$. Then there exist $n \in \mathbb{N}$ and $u \in S$ such that $a^n + uy = uy$ and $a^n + uz = uz$. Then $a^n + u(y+z) = u(y+z)$ shows that $y + z \in F$. Also $a \in \sqrt{Sy} \cap \sqrt{Sz} = \sqrt{Syz}$, by Lemma 3.1, which implies that $yz \in F$. Thus F is a subsemiring of S. Now let $c, d \in S$ be such that $cd \in F$. Then $a \in \sqrt{Scd} = \sqrt{Sc} \cap \sqrt{Sd}$. Hence $c \in F$ and $d \in F$. Let $y \in F$ and $c \in S$ be such that $y + c = c$. Then there exist $n \in \mathbb{N}$ and $t \in S$ such that $a^n + ty = ty$, which implies $a^n + tc = tc$. Hence $c \in F$. Thus F is a filter of S containing a.

Let T be a filter of S containing a. Let $x \in F$. Then $a^n + ux = ux$, for some $n \in \mathbb{N}$, $u \in S$. Then $a^n \in T$ implies that $ux \in T$, and so $x \in T$. Thus $N(a) = F = \{x \in S \mid a \in \sqrt{Sx}\}$.

(8) $\Rightarrow$ (3) : Let $a, b \in S$. Then $ab \in N(ab)$ implies that $a \in N(ab) = \{x \in S \mid ab \in \sqrt{Sx}\}$. Hence $ab \in \sqrt{Sa}$.

(3) $\Rightarrow$ (9) : Follows from the Lemma 3.1.

(9) $\Rightarrow$ (3) : Let $a, b \in S$. Then $ab \in \sqrt{Sab} = \sqrt{Sa} \cap \sqrt{Sb} \subseteq \sqrt{Sa}$ implies that $ab \in \sqrt{Sa}$. □

4. CHAIN OF LEFT K-ARCHIMEDEAN SEMIRINGS

In this section we characterize the semirings which are chains of left k-Archimedean semirings. Let $(T, +, \cdot)$ be a distributive lattice with the partial order defined by $a \leq b \Leftrightarrow a + b = b$ for all $a, b \in S$. It is well known that $(T, \leq)$ is a chain if and only if $ab = b$ or $ab = a$ for all $a, b \in T$.

Theorem 4.1. *The following conditions on a semiring S are equivalent:*

(1) *S is a chain of left k-Archimedean semirings;*

(2) *S is a distributive lattice of left k-Archimedean semirings such that for all* $a, b \in S$,
$$b \in \sqrt{Sa} \text{ or } a \in \sqrt{Sb};$$
(3) *S is a distributive lattice of left k-Archimedean semirings such that for all* $a, b \in S$,
$$b \in \sqrt{L_k(a)} \text{ or } a \in \sqrt{L_k(b)};$$
(4) $N(a) = \{x \in S \mid a \in \sqrt{Sx}\}$, *and* $N(ab) = N(a) \cup N(b)$, *for all* $a, b \in S$;
(5) $\eta = \mathcal{N}$ *is the least chain congruence on S such that each of its congruence classes is a left k-Archimedean semiring.*

Proof. Scheme of the proof: (1) ⇒ (2) ⇒ (4) ⇒ (5) ⇒ (1), (2) ⇔ (3).

(1) ⇒ (2) : Let S be a chain $\mathcal{C}$ of left k-Archimedean semirings $\{S_\alpha \mid \alpha \in \mathcal{C}\}$. Then S is the distributive lattice $\mathcal{C}$ of left k-Archimedean semirings $\{S_\alpha \mid \alpha \in \mathcal{C}\}$. Let $a, b \in S$. Then there exist $\alpha, \beta \in \mathcal{C}$ such that $a \in S_\alpha$ and $b \in S_\beta$. Then $\alpha\beta = \alpha$ or $\alpha\beta = \beta$, since $\mathcal{C}$ is a chain. If $\alpha\beta = \alpha$ then $a, ab \in S_\alpha$. So $a \in \sqrt{Sab} \subseteq \sqrt{Sb}$. If $\alpha\beta = \beta$, then $b, ba \in S_\beta$ and hence $b \in \sqrt{Sba} \subseteq \sqrt{Sa}$.

(2) ⇒ (4) : For all $a \in S$, we have $N(a) = \{x \in S \mid a \in \sqrt{Sx}\}$, by Theorem 3.3.

Let $a, b \in S$. Then $a \in \sqrt{Sb}$ or $b \in \sqrt{Sa}$. If $a \in \sqrt{Sb}$, then there exist $m \in \mathbb{N}$ and $x \in S$ such that $a^m + xb = xb$. Again there exist $n \in \mathbb{N}$ and $y \in S$ such that $(bax)^n + yba = yba$, by Theorem 3.3. Then $a^m + xb = xb$ implies that $a^{(m+1)(n+1)} + axybab = axybab$ so that $ab \in N(a)$. Thus $N(ab) \subseteq N(a)$. If $b \in \sqrt{Sa}$, then there exist $p \in \mathbb{N}$ and $z \in S$ such that $b^p + za = za$ which implies that $b^{p+1} + zab = zab$. Thus $ab \in N(b)$, and so $N(ab) \subseteq N(b)$. Hence $N(ab) \subseteq N(a) \cup N(b)$. Again $ab \in N(ab)$ implies that $a \in N(ab)$ and $b \in N(ab)$ which implies that $N(a) \cup N(b) \subseteq N(ab)$. Thus $N(ab) = N(a) \cup N(b)$.

(4) ⇒ (5) : It follows from Lemma 3.1 and Theorem 3.3 that the least distributive lattice congruence η on S is given by: for $a, b \in S$, $a\eta b \Leftrightarrow a \in \sqrt{Sb}$ and $b \in \sqrt{Sa}$, and each η-class is a left k-Archimedean semiring. Then we have
$$\begin{aligned}\eta &= \{(x, y) \in S \times S \mid x \in \sqrt{Sy} \text{ and } y \in \sqrt{Sx}\}\\ &= \{(x, y) \in S \times S \mid N(x) = N(y)\} = \mathcal{N}.\end{aligned}$$
Again for all $a, b \in S$, we have $ab \in N(ab) = N(a) \cup N(b)$. This implies that $ab \in N(a)$ or $ab \in N(b)$, that is, $N(ab) \subseteq N(a) \subseteq N(a) \cup N(b) = N(ab)$ or $N(ab) \subseteq N(b) \subseteq N(a) \cup N(b) = N(ab)$. This gives $ab\mathcal{N}a$ or $ab\mathcal{N}b$, and thus $\mathcal{N}$ is a chain congruence.

(5) ⇒ (1) : Follows directly.

(2) ⇔ (3) : Follows from the Remark 3.2. □

Finally, we show that a necessary and sufficient condition for a semiring S being a chain of left k-Archimedean semirings is that for every left k-ideal L of S, $\sqrt{L}$ is a completely prime k-ideal of S..

THEOREM 4.2. *The following conditions on a semiring S are equivalent:*

(1) *S is a chain of left k-Archimedean semirings;*
(2) *$\sqrt{L}$ is a completely prime k-ideal of S for every left k-ideal L of S;*
(3) *$\sqrt{L_k(a)}$ is a completely prime k-ideal of S for every $a \in S$;*
(4) *for all $a, b \in S$, $\sqrt{L_k(ab)} = \sqrt{L_k(a)} \cap \sqrt{L_k(b)}$ and every left k-ideal of S is semiprimary.*

Proof. (1) $\Rightarrow$ (2) : Let S be a chain $\mathcal{C}$ of left k-Archimedean semirings $\{S_\alpha \mid \alpha \in \mathcal{C}\}$. Consider a left k-ideal L of S. Then $\sqrt{L}$ is a k-ideal of S, by Theorem 3.3. Let $x, y \in S$ such that $xy \in \sqrt{L}$. Then there exists $m \in N$ such that $u = (xy)^m \in \overline{L} = L$. Suppose $\alpha, \beta \in \mathcal{C}$ be such that $x \in S_\alpha$ and $y \in S_\beta$. Then $\alpha\beta = \alpha$ or $\alpha\beta = \beta$, since $\mathcal{C}$ is a chain. If $\alpha\beta = \alpha$, then $x, u \in S_\alpha$ implies that $x \in \sqrt{Su} \subseteq \sqrt{L}$ and so $x \in \sqrt{L}$. If $\alpha\beta = \beta$, then similarly we have $y \in \sqrt{L}$. Thus $\sqrt{L}$ is a completely prime k-ideal of S.

(2) $\Rightarrow$ (3) : Obvious.

(3) $\Rightarrow$ (4) : Let $a, b \in S$. Then $\sqrt{L_k(a)}, \sqrt{L_k(b)}$ and $\sqrt{L_k(ab)}$ are completely prime k-ideals of S. Let $x \in \sqrt{L_k(ab)}$. Then there exist $n \in \mathbb{N}$ and $s \in S$ such that $x^n + sab = sab$. Again, since $L_k(a)$ is a k-ideal, $sab \in L_k(a)$ and so $x^n \in L_k(a)$, which implies that $x \in \sqrt{L_k(a)}$. Thus $\sqrt{L_k(ab)} \subseteq \sqrt{L_k(a)}$. Similarly, $\sqrt{L_k(ab)} \subseteq \sqrt{L_k(b)}$. Thus $\sqrt{L_k(ab)} \subseteq \sqrt{L_k(a)} \cap \sqrt{L_k(b)}$. Let $z \in \sqrt{L_k(a)} \cap \sqrt{L_k(b)}$. Then there exist $n \in \mathbb{N}$ and $s \in S$ such that $z^n + sa = sa$ and $z^n + sb = sb$. Now $sabs \in \sqrt{L_k(ab)}$ implies that $sa \in \sqrt{L_k(ab)}$ or $bs \in \sqrt{L_k(ab)}$. If $sa \in \sqrt{L_k(ab)}$, then there exist $r \in \mathbb{N}$, $v \in S$ such that $(sa)^r + vab = vab$. Then $z^{nr} + (sa)^r = (sa)^r$ implies that $z \in \sqrt{L_k(ab)}$, and so $\sqrt{L_k(a)} \cap \sqrt{L_k(b)} \subseteq \sqrt{L_k(ab)}$. If $bs \in \sqrt{L_k(ab)}$, then similarly we have $\sqrt{L_k(b)} \subseteq \sqrt{L_k(ab)}$. Hence $\sqrt{L_k(a)} \cap \sqrt{L_k(b)} \subseteq \sqrt{L_k(ab)}$. Thus $\sqrt{L_k(ab)} = \sqrt{L_k(a)} \cap \sqrt{L_k(b)}$.

Let L be a left k-ideal of S and $a, b \in S$ be such that $ab \in L$. Then $L_k(ab) \subseteq L$. Now $ab \in \sqrt{L_k(ab)}$ implies $a^n \in L_k(ab)$ or $b^n \in L_k(ab)$, for some $n \in \mathbb{N}$ so that $a^n \in L$ or $b^n \in L$. Thus L is semiprimary.

(4) $\Rightarrow$ (1) : Let $a, b \in S$. Then $\sqrt{L_k(ab)} \subseteq \sqrt{L_k(a)}$ implies that $ab \in \sqrt{L_k(a)}$. Then by Lemma 3.1, Remark 3.2 and Theorem 3.3, it follows that the least distributive lattice congruence η on S is given by : for all $a, b \in S$, $a\eta b \Leftrightarrow a \in \sqrt{L_k(b)}$ and $b \in \sqrt{L_k(a)}$, and each η-class is a left k-Archimedean semiring. Now $ab \in \sqrt{L_k(ab)} = \sqrt{L_k(a)} \cap \sqrt{L_k(b)}$ implies that $ab \in \sqrt{L_k(a)}$ and $ab \in \sqrt{L_k(b)}$. Again $ab \in L_k(ab)$ implies that $a^m \in L_k(ab)$ or $b^m \in L_k(ab)$, for some $m \in \mathbb{N}$. Thus either $ab \in \sqrt{L_k(a)}$ and $a \in \sqrt{L_k(ab)}$, or $ab \in \sqrt{L_k(b)}$ and $b \in \sqrt{L_k(ab)}$, whence $a\eta ab$ or $b\eta ab$. Thus S/η is a chain. $\square$

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Received May 5, 2019
Accepted November 3, 2019

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