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A SHORT NOTE ON HARMONIC FUNCTIONS ON SELF-SIMILAR STRUCTURES

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Abstract. A sufficient condition is given concerning the harmonic structure on a post critically finite self-similar structure K that ensures that harmonic functions are not zero divisors in the algebra of real-valued continuous functions on K.

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1. INTRODUCTION

In the last four decades, it turned out that fractals are appropriate mathematical concepts to model many objects and phenomena both from nature and from various domains of science (e.g., physics, mechanics, chemistry, biology, economics, etc.). This motivated the efforts done by many mathematicians to develop an *analysis on fractals*. An overview of the foundations and of different approaches that characterize this new domain can be found, for instance, in the introductions of the monographs [10] and [13].

Important contributions in developing a suitable framework for the study of PDEs on fractals, i.e., in defining Sobolev-type spaces and Laplace-type operators on fractals, have been brought by J. Kigami. In the pioneering paper [8], Kigami founded his theory in case of the Sierpinski gasket (SG for short) in $\mathbb{R}^n$. This paper originated many subsequent studies devoted to elliptic equations on the SG. The papers [3-7], [11], [12] are only few examples in this sense.

Later (in [9] and [10]), Kigami generalized his approach to so-called *post critically finite self-similar structures*. A central concept in Kigami's theory is that of *harmonic structure*, and the related concept of *harmonic function*. Harmonic functions have finite energy (and hence belong to the Soboev-type space H^1) and can be constructed by an inductive procedure involving the harmonic structure. The starting point for the present note was [2], where it is shown that harmonic functions defined on the SG in $\mathbb{R}^n$ are not zero divisors in the algebra of real-valued continuous functions on the SG (as it is pointed out in [2], this property has connections with aspects studied in [3]). More exactly, taking into account that the SG is a typical example of a

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post critically finite (p. c. f. for short) self-similar structure, the question that naturally arises is whether a result similar to that mentioned before for the SG is valid in the more general setting of p. c. f. self-similar structures. The main tool used in [2] is the *harmonic extension procedure* on the SG. In the general setting of p. c. f. self-similar structures, the harmonic extension procedure is included in the harmonic structure. Thus the above mentioned question can be formulated in a more precise manner as

(Q) *What property of the harmonic structure on a p. c. f. self-similar structure K is responsible for the fact that harmonic functions on K are not zero divisors in the algebra of real-valued continuous functions on K?*

The aim of the present paper is to give an answer to (Q). More exactly, in Theorem 3.4 below there is given a sufficient condition on the harmonic structure that *prevents* harmonic functions from being zero divisors. As it is pointed out in the last section, this condition is fulfilled in case of the SG, thus the main result of [2] can be obtained from Theorem 3.4.

Notations. We denote by $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, \dots\}$ and by $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ the set of positive naturals. Furthermore, if K is a nonempty compact topological space, we denote by $C(K)$ the (real) algebra of real-valued continuous functions on K.

2. PRELIMINARIES

In order to make the paper self-contained, we recall some basic facts from [10] concerning the framework we are working in.

Throughout the paper, S is a non-empty finite set and t is a real number in the interval $(0, 1)$. First, we introduce the following notations:

- for $m \in \mathbb{N}^*$, let $W_m := S^m$ be the set of words of length m with symbols from S;
- for $m = 0$, let $W_0 := \{\emptyset\}$ and call $\emptyset$ the empty word (of length 0);
- $W_* := \bigcup_{m\in\mathbb{N}} W_m$;
- $\Sigma := S^{\mathbb{N}^*}$ is the set of sequences with elements in S;
- for $i \in S$, define $\sigma_i \colon \Sigma \to \Sigma$ by $\sigma_i(\omega_1, \omega_2, \dots) = (i, \omega_1, \omega_2, \dots)$, for all $(\omega_1, \omega_2, \dots) \in \Sigma$;
- $\sigma \colon \Sigma \to \Sigma$ stands for the left-shift operator, i.e., $\sigma(\omega_1, \omega_2, \dots) = (\omega_2, \omega_3, \dots)$, for all $(\omega_1, \omega_2, \dots) \in \Sigma$;
- for $\omega = (\omega_1, \omega_2, \dots)$, $\tau = (\tau_1, \tau_2, \dots) \in \Sigma$ with $\omega \neq \tau$, put

$$s(\omega, \tau) := \min\{m \in \mathbb{N}^* \mid \omega_m \neq \tau_m\} - 1;$$

- $d_t \colon \Sigma \times \Sigma \to \mathbb{R}$ is defined by

$$d_t(\omega, \tau) = \begin{cases} t^{s(\omega,\tau)}, & \text{if } \omega \neq \tau \\ 0, & \text{if } \omega = \tau. \end{cases} \tag{2.1}$$

It is known (see, e.g., [10, Theorem 1.2.2]) that (Σ, d_t) is a compact metric space. In the sequel, Σ is considered to be endowed with the metric d_t.

DEFINITION 2.1. Let K be a nonempty compact metric space and let $\{f_i \colon K \to K\}_{i\in S}$ be a family of continuous injections. Then $(K, S, \{f_i\}_{i\in S})$ is called a *self-similar structure* if there exists a continuous surjection $\pi \colon \Sigma \to K$ such that

$$f_i \circ \pi = \pi \circ \sigma_i, \ \forall i \in S. \tag{2.2}$$

In this case, for simplicity, often K itself is called a *self-similar structure*.

If $\mathcal{L} := (K, S, \{f_i\}_{i\in S})$ is a self-similar structure, we set

$$C_{\mathcal{L},K} := \bigcup_{\substack{i,j\in S\\ i\neq j}} (f_i(K) \cap f_j(K)), \quad \mathcal{C}_{\mathcal{L}} := \pi^{-1}(C_{\mathcal{L},K}), \quad \mathcal{P}_{\mathcal{L}} := \bigcup_{n\in\mathbb{N}^*} \sigma^n(\mathcal{C}_{\mathcal{L}}).$$

DEFINITION 2.2. The self-similar structure $\mathcal{L} = (K, S, \{f_i\}_{i\in S})$ is called *post critically finite* (p. c. f. for short) if the set $\mathcal{P}_{\mathcal{L}}$ is finite.

In what follows, $\mathcal{L} = (K, S, \{f_i\}_{i\in S})$ will stand for a p. c. f. self-similar structure, and K is assumed to be connected. (Characterizations for the connectivity of K are presented in [10, section 1.6].)

We introduce further notations:

- for $w \in W_*$, let $f_w \colon K \to K$ be defined by
$$f_w = \begin{cases} \mathrm{id}_K, & \text{if } w = \emptyset \\ f_{w_1} \circ \cdots \circ f_{w_m}, & \text{if } w = (w_1, \ldots, w_m) \in W_m, \text{ with } m \in \mathbb{N}^*; \end{cases}$$
- $V_0 := \pi(\mathcal{P}_{\mathcal{L}})$, $V_m := \bigcup_{w\in W_m} f_w(V_0)$, for all $m \in \mathbb{N}^*$.

REMARK 2.3. Since $\mathcal{L}$ is p. c. f., the sets V_m, $m \in \mathbb{N}$, are finite.

We assume throughout the paper that $V_0 \neq \emptyset$. We also recall from [10, Lemma 1.3.11] that

$$V_m \subseteq V_{m+1}, \ \forall\, m \in \mathbb{N}, \tag{2.3}$$

and that the set

$$V_* := \bigcup_{m\in\mathbb{N}} V_m \tag{2.4}$$

is dense in K.

Without entering into the (quite technical details) of the notion of *harmonic structure* on the p. c. f. self-similar structure $\mathcal{L}$, we assume further that $\mathcal{H}_{\mathcal{L}}$ is a harmonic structure on $\mathcal{L}$. It is known from [10, section 3.1] that this harmonic structure naturally induces a sequence $(H_m)_{m\in\mathbb{N}}$ of Laplacians, where H_m is a Laplacian on V_m, for every $m \in \mathbb{N}$. (For the definition of a Laplacian on a finite set we refer, e.g., to [10, Definition 2.1.2].)

The following result, in whose statement we use the notations from above, is a consequence of [10, Proposition 3.2.1 and Theorem 3.2.4].

THEOREM 2.4. *For any $\rho\colon V_0 \to \mathbb{R}$ there exists a unique $u \in C(K)$ such that*

$$\begin{cases} (H_m u)|_{V_m \setminus V_0} = 0, \ \forall\, m \geq 1, \\ u|_{V_0} = \rho. \end{cases}$$

The function $u\colon K \to \mathbb{R}$ in the above theorem is the *harmonic function with boundary value* ρ. Every function in $C(K)$, obtained according to Theorem 2.4 from a real-valued function on V_0, is called a *harmonic function* on K. We denote by $\mathcal{H}(K)$ the set of harmonic functions on K.

Notation. For a nonempty finite set X, we denote by $\ell(X)$ the (real) vector space of real-valued functions defined on X.

Finally, we introduce certain operators on $\ell(V_0)$ which will play an important role in the next section. For this, we denote first, for every $i \in S$ and every $v \in \ell(V_0)$, by $\tilde{v} \in \ell(V_1)$ the (unique) map (actually an extension of v) satisfying the conditions

$$\tilde{v}|_{V_0} = v \text{ and } (H_1\tilde{v})|_{V_1 \setminus V_0} = 0.$$

In order to define the above mentioned operators on $\ell(V_0)$, we identify, for every $w \in W_*$, the sets $\ell(V_0)$ and $\ell(f_w(V_0))$ through the injective map f_w (i.e., $u\colon V_0 \to \mathbb{R}$ will be identified with $\bar{u}\colon f_w(V_0) \to \mathbb{R}$, where $\bar{u}(f_w(p)) = u(p)$, for every $p \in V_0$). For every $i \in S$, let now $A_i\colon \ell(V_0) \to \ell(V_0)$ be the linear operator defined by

$$A_i v = \tilde{v}|_{f_i(V_0)}, \ \forall\, v \in \ell(V_0). \tag{2.5}$$

The proof of [10, Theorem 3.2.4] yields in particular the following result concerning harmonic functions. Again, we keep in the statement of this result the notations introduced above.

THEOREM 2.5. *Let $\rho \in \ell(V_0)$ and let $u \in \mathcal{H}(K)$ be the harmonic function with boundary value ρ. If $m \in \mathbb{N}^*$ and $w = (w_1, \ldots, w_m) \in W_m$, then the following equality holds*

$$u|_{f_w(V_0)} = A_{w_m} A_{w_{m-1}} \ldots A_{w_1} \rho.$$

3. MAIN RESULTS

We keep the notations from the previous section for a p. c. f. self-similar structure $\mathcal{L} = (K, S, \{f_i\}_{i\in S})$ and for a harmonic structure $\mathcal{H}_{\mathcal{L}}$ on $\mathcal{L}$.

DEFINITION 3.1. The harmonic structure $\mathcal{H}_{\mathcal{L}}$ induces the *h-recoverability property on the boundary of* K if the operators A_i, $i \in S$, defined in (2.5), are injective (hence invertible).

REMARK 3.2. We have chosen the term *h-recoverable*, since in case the operators A_i, $i \in S$, are invertible, one can obtain (recover), according to Theorem 2.5, the restriction to V_0 of a harmonic function $u \in \mathcal{H}(K)$ from the restriction of u to any $f_w(V_0)$, where $w \in W_*$. In the framework of self-similar structures, the set $f_w(V_0)$, with $w \in W_*$, is called the *boundary of the* *cell* $f_w(K)$ of K. Thus, in case of h-recoverability, the restriction of every harmonic function $u \in \mathcal{H}(K)$ to the boundary V_0 of the self-similar structure K can be recovered from the restriction of u to the boundary of any cell of K. A consequence of this fact is pointed out in the next result.

LEMMA 3.3. *If $\mathcal{H}_{\mathcal{L}}$ induces the h-recoverability property on the boundary of K, then the following statements are equivalent for $u \in \mathcal{H}(K)$.*

(i) $u|_{V_0} = 0$.
(ii) $u = 0$.
(iii) *There exists $m \in \mathbb{N}^*$ and $w \in W_m$ such that $u|_{f_w(V_0)} = 0$.*

Proof. The implication (i)⇒(ii) follows from Theorem 2.4, (ii)⇒(iii) is obvious, and (iii)⇒(i) is a consequence of Theorem 2.5 and the h-recoverability property. □

THEOREM 3.4. *Assume that $\mathcal{H}_{\mathcal{L}}$ induces the h-recoverability property on the boundary of the p. c. f. self-similar structure K. Then, for $u \in \mathcal{H}(K)$ and $v \in C(K)$, there are equivalent:*
1° $u \cdot v = 0$.
2° $u = 0$ *or* $v = 0$.

Proof. The implication $2^\circ \Rightarrow 1^\circ$ is obvious. We prove now that $1^\circ \Rightarrow 2^\circ$. Assume that $u \neq 0$. We show that $v = 0$. For this, let $m \in \mathbb{N}^*$ and $w = (w_1, \ldots, w_m) \in W_m$ be arbitrary. We are going to prove that

$$v|_{f_w(V_0)} = 0. \tag{3.1}$$

Let $p \in V_0$ be arbitrary. Since π is surjective, we can choose $\tau = (\tau_1, \tau_2, \ldots) \in \Sigma$ such that $p = \pi(\tau)$. According to Lemma 3.3, for every $n \in \mathbb{N}^*$, there exists $p_n \in V_0$ with the property that

$$u(f_w \circ f_{(\tau_1,\ldots,\tau_n)}(p_n)) \neq 0, \ \forall n \in \mathbb{N}^*.$$

By the hypothesis 1°, the above relation implies that

$$v(f_w \circ f_{(\tau_1,\ldots,\tau_n)}(p_n)) = 0, \ \forall n \in \mathbb{N}^*. \tag{3.2}$$

We show now that

$$\lim_{n\to\infty} f_{(\tau_1,\ldots,\tau_n)}(p_n) = p. \tag{3.3}$$

For this, choose, for every $n \in \mathbb{N}^*$, an element $\omega^n \in \Sigma$ such that $p_n = \pi(\omega^n)$. Using (2.2), we get

$$f_{(\tau_1,\ldots,\tau_n)}(p_n) = f_{(\tau_1,\ldots,\tau_n)}(\pi(\omega^n)) = \pi(\sigma_{\tau_1} \circ \cdots \circ \sigma_{\tau_n}(\omega^n)). \tag{3.4}$$

The definition (2.1) of the metric d_t yields that

$$d_t(\sigma_{\tau_1} \circ \cdots \circ \sigma_{\tau_n}(\omega^n), \tau) \leq t^n, \ \forall n \in \mathbb{N}^*,$$

which implies

$$\lim_{n\to\infty} \sigma_{\tau_1} \circ \cdots \circ \sigma_{\tau_n}(\omega^n) = \tau. \tag{3.5}$$

The relations (3.4) and (3.5), the continuity of π, and the equality $\pi(\tau) = p$ imply now (3.3). From (3.3) we obtain, according to the continuity of f_w, that

$$\lim_{n\to\infty} f_w \circ f_{(\tau_1,\dots,\tau_n)}(p_n) = f_w(p).$$

The above equality, together with (3.2) and the continuity of v, yield that $v(f_w(p)) = 0$. Since $p \in V_0$ was arbitrarily chosen, the equality (3.1) follows. But $w \in W_m$ was also arbitrary, so, taking into account that, by definition, $V_m := \bigcup_{w\in W_m} f_w(V_0)$, we get from (3.1) that $v|_{V_m} = 0$. Since $m \in \mathbb{N}^*$ was arbitrary, we obtain $v|_{V_m} = 0$, $\forall m \in \mathbb{N}^*$. The above relations lead, in connection with $V_0 \subseteq V_1$ (which follows from (2.3)) and with (2.4), to $v|_{V_*} = 0$. Since V_* is dense in K (as noted in the previous section), we conclude, by the continuity of v, that $v = 0$. □

Remark 3.5. Theorem 3.4 shows that the h-recoverability property is a sufficient condition that leads to the fact that harmonic functions on K are not zero divisors in the algebra $C(K)$. We do not know at the moment whether this condition is also necessary. Furthermore, Theorem 3.4 offers an answer to the question (Q) raised in the Introduction.

4. APPLICATIONS AND CONCLUSIONS

We denote by $|\cdot|$ the Euclidean norm on the spaces $\mathbb{R}^n$, $n \in \mathbb{N}^*$. In what follows, the spaces $\mathbb{R}^n$ are endowed with the topology induced by $|\cdot|$.

First we recall the definition of the Sierpinski gasket (SG) in $\mathbb{R}^n$. Let $n \in \mathbb{N}^*$ and $p_1, \dots, p_{n+1} \in \mathbb{R}^n$ with $|p_i - p_j| = 1$ for $i \neq j$. For every $i \in \{1, \dots, n+1\}$, define $f_i \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$f_i(x) = \frac{1}{2}\,x + \frac{1}{2}\,p_i,, \ \forall x \in \mathbb{R}^n.$$

The SG in $\mathbb{R}^n$ is the self-similar set with respect to $\{f_1, \dots, f_{n+1}\}$, i.e., the unique non-empty compact subset V of $\mathbb{R}^n$ that satisfies the equality

$$V = f_1(V) \cup \cdots \cup f_{n+1}(V).$$

The existence of such a set is a consequence of the Banach fixed-point theorem.

A straightforward computation yields that $(SG, \{1, \dots, n+1\}, \{f_i\}_{i=\overline{1,n+1}})$ is a p. c. f. self-similar structure. The case $n = 2$ can be found in [10, Examples 1.2.8 and 1.3.15]. We note that in this case the SG becomes the *Sierpinski triangle*, one of the most well-known fractals, constructed in 1915 by the Polish mathematician W. Sierpinski.

The harmonic extension procedure on the SG in $\mathbb{R}^n$ (briefly described in [2, section 2], and presented in detail in [1, section 3]) actually corresponds to a harmonic structure on $(SG, \{1, \dots, n+1\}, \{f_i\}_{i=\overline{1,n+1}})$ that induces the h-recoverability property. This follows from the formulas in [2, section 2] (resp., more detailed, from the formulas in the proof of [1, Theorem 3.1]). More exactly, it can be readily seen that the $n+1$ real matrices of dimension $(n+1)\times(n+1)$ that correspond in this case to the operators A_i, $i \in \{1, \dots, n+1\}$, defined according to (2.5), are invertible. For simplicity, we present here the case $n = 2$, that can be found in [10, Examples 3.1.5 and 3.2.6]. In this case the three matrices, denoted with the same letters as the operators, are

$$A_1 = \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \quad A_2 = \frac{1}{5}\begin{pmatrix} 2 & 2 & 1 \\ 0 & 5 & 0 \\ 1 & 2 & 2 \end{pmatrix}, \quad A_3 = \frac{1}{5}\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 5 \end{pmatrix},$$

hence they are invertible. Thus, [2, Theorem 3.3] is a consequence of Theorem 3.4.

As a conclusion, Theorem 3.4 gives a new insight on the results in [2], and it emphasizes the close relationship between harmonic functions on p. c. f. self-similar structures and the harmonic structure. Moreover, the property of harmonic functions pointed out by Theorem 3.4 has consequences on the energy defined on these structures, a fact that will be needed for further researches in this area (see also the footnote in the introduction).

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